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Clean classical rings of quotients of commutative rings, with applications to $C(X)$

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Commutative clean rings and related rings have received much recent attention. A ring R is clean if each $r \in R$ can be written $r = u + e$, where u is a unit and e an idempotent. This article deals mostly with the question: When is the classical ring of quotients of a commutative ring clean? After some general results, the article focuses on $C(X)$ to characterize spaces X when $Q_{\mathrm{cl}}(X)$ is clean. Such spaces include cozero complemented, strongly 0-dimensional and more spaces. Along the way, other extensions of rings are studied: directed limits and extensions by idempotents.

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1. Introduction

1.1. *Terminology and notation.*

Throughout, *all rings are assumed to be commutative with 1.* The set of *non-zero divisors* (also called *regular elements*) of R will be denoted $\mathfrak{R}(R)$, the set of zero-divisors is $\mathfrak{Z}(R)$, the set of units is $\mathfrak{U}(R)$ and the set of idempotents $\mathbf{B}(R)$. The Jacobson radical of R is written $\mathbf{J}(R)$. A ring R is called *indecomposable* if $\mathbf{B}(R) = \{0, 1\}$.

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A ring is called *clean* if every element is the sum of a unit and an idempotent, and it is called *almost clean* if each element is the sum of a non zero-divisor and an idempotent. The term *local ring* will refer to *any* ring with a unique maximal ideal. A *pm ring* (or *Gelfand ring*) is a ring such that each prime ideal is contained in a unique maximal ideal. A ring extension $R \subseteq S$ is called an *essential ring extension* if for each ideal $\mathbf{0} \neq I$ of S, $I \cap R \neq \mathbf{0}$.

In Section 2, on rings of continuous functions, the conventions of [10] are followed. In particular, a topological space X will always be assumed to be *completely regular*. A *clopen* subset of X will be one which is both closed and open.

1.2. *History and Outline.*

"Clean", for not necessarily commutative rings, was defined by Nicholson in [23] and, for rings where all idempotents are central, "clean" and "exchange" coincide (e.g., [5, Proposition 1.2 and Theorem 3.4]; see [23] for a discussion of exchange rings). A history of clean rings is found in McGovern [20], to which the reader is referred. Almost clean rings were defined by McGovern in [21] and further studied by Ahn and Anderson ([1]). There have been many subsequent papers devoted to clean and related rings; more references will appear in the body of this note. The structure of (commutative) clean and almost clean rings in terms of their Pierce sheaves is examined in [4].

We recall that an indecomposable ring R is clean if and only if it is local, and that all clean rings are pm rings (e.g., [1, Theorem 3 and Corollary 4]), but not all pm rings are clean. Section 1 of this note first looks at the *classical ring of quotients* (or *total ring of fractions*), $Q_{\mathrm{cl}}(R)$ of a ring R, to ask when it is clean. At first glance it seems plausible that when R is almost clean then $Q_{\mathrm{cl}}(R)$ should be clean; however, this is far from the case. Theorem 2.1 characterizes when $Q_{\mathrm{cl}}(R)$ is clean in the case when $Q_{\mathrm{cl}}(R)$ is a finite product of indecomposable rings (e.g., when R is noetherian). However, Example 2.2 shows that even when R is noetherian local, $Q_{\mathrm{cl}}(R)$ need not be clean. Moreover, there are reduced (almost) clean rings R with $Q_{\mathrm{cl}}(R)$ not clean (Example 2.3).

We then look at directed limits of clean and almost clean rings. Directed limits of clean rings are always clean, while directed limits of almost clean require additional hypotheses in order for the limit to be almost clean (Propositions 2.4 and 2.6).

Extensions of clean and almost clean rings, such as polynomial, power series and group rings have been looked at by various authors. We look at extensions by idempotents. If idempotents are adjoined to a clean ring the result is always clean; however, "almost clean" is not always preserved (Example 2.10).

Section 2 is devoted to rings of the form $C(X)$, the ring of continuous real valued functions on a topological space X. It is known that $C(X)$ is clean if and only if X is strongly 0-dimensional, that is, if the Čech-Stone compactification is 0-dimensional. (See the notes at the start of Section 2 for references.) We look at $Q_{\mathrm{cl}}(C(X))$, denoted $Q_{\mathrm{cl}}(X)$ as in [9]. It is first seen that, unlike the case of general rings, $C(X)$ clean implies $Q_{\mathrm{cl}}(X)$ clean (Proposition 3.1). The converse is false but it is true for F-spaces (Proposition 3.3).

The class of spaces for which $Q_{\mathrm{cl}}(X)$ is von Neumann regular (and, hence, clean) is known to be that of *cozero complemented spaces* ([13]). This is an extensive family of spaces which includes all metric spaces (see [13] for a long list of such spaces as well as a list of spaces which are not cozero complemented). The spaces for which $Q_{\mathrm{cl}}(X)$ is clean are characterized in Theorem 3.5, they are called *WZD-spaces* (for *weakly zero-dimensional*) and they include the cozero complemented and the strongly 0-dimensional spaces as well as others which have neither of these properties. However, not all spaces are WZD-spaces as examples show.

The note concludes with examples of WZD-spaces and spaces which are not WZD. It is also shown that if a WZD-space X is z-embedded in an ambient space T then T is a WZD-space.

There is on-going work on some related topics by Knox, Levy, McGovern and Shapiro ([17]).

2. Observations on $Q_{\mathrm{cl}}(R)$ and on extensions of clean and almost clean rings.

Because of the role played by units in clean rings and non zero-divisors in almost clean rings, the most natural ring of quotients to study is the classical (or total) ring of quotients. However, a word should be said about the *maximal (or complete) ring of quotients* of R, $Q_{\max}(R)$ (e.g., [18, §13]). When, $Q_{\max}(R)$ is self-injective then it is clean, since for a self-injective ring S, $S/\mathbf{J}(S)$ is von Neumann regular, and, hence, clean and moreover, idempotents lift modulo $\mathbf{J}(S)$; this shows that S is clean (e.g., [12, Proposition 6]). This occurs, in particular, if R is semiprime in which case $Q_{\max}(R)$ is von Neumann regular (e.g., [18, §13, Exercise 17]). All the rings $C(X)$ of Section 2 are semiprime. However, there are rings where $Q_{\max}(R)$ is not clean. The construction in [18, (8.30)], using, for example $R = \mathbb{Z}$, yields Kasch rings which are not almost clean. Kasch rings, however, coincide with their maximal rings of quotients.

2.1. *When is $Q_{cl}(R)$ clean?*

If R is noetherian and the associated primes of $\mathbf{0}$ are all "isolated" (i.e., minimal primes) then $Q_{\mathrm{cl}}(R)$ is artinian (see, for example, [18, Proposition 12.22]). When this occurs, $Q_{\mathrm{cl}}(R)$ is clean because it is a finite product of local rings. More generally we have the following which is a version of [1, Theorem 2.5] for $Q_{\mathrm{cl}}(R)$. It is not enough, here, to assume that R is a finite product of indecomposable rings since that condition does not imply that $Q_{\mathrm{cl}}(R)$ is a finite product of indecomposable rings as will be seen below. We do this for more general rings of fractions.

Theorem 2.1. *Let R be a ring and $S \subseteq \mathfrak{R}(R)$ a multiplicatively closed subset such that RS^{-1} is a finite product of indecomposable rings (e.g., if R is noetherian).*

Then, RS^{-1} is clean if and only if given $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ with the properties that $\mathfrak{p}, \mathfrak{q} \subseteq \mathfrak{Z}(R)$ with $(\mathfrak{p}+\mathfrak{q}) \cap S \neq \emptyset$ there exist $r \in \mathfrak{p}$ and $r' \in \mathfrak{q}$ with $r + r' \in S$ and $rr' = 0$.

Proof. (1) Suppose RS^{-1} is almost clean. The criterion of [1, Theorem 2.5] applies. Suppose we have primes $\mathfrak{p}, \mathfrak{q} \subseteq \mathfrak{Z}(R)$ with $(\mathfrak{p}+\mathfrak{q}) \cap S \neq \emptyset$. Then, $\mathfrak{p}S^{-1} + \mathfrak{q}S^{-1} = RS^{-1}$ and there exists $e = e^2 \in RS^{-1}$ with $e \in \mathfrak{p}S^{-1}$ and $1 - e \in \mathfrak{q}S^{-1}$. We write $e = rs^{-1}$ and $1 - e = (s - r)s^{-1}$ for some $r \in \mathfrak{p}$, $r' = s - r \in \mathfrak{q}$, $s \in S$. Then $r + r' = s \in S$ and $rr' = 0$.

(2) We assume the condition on R. Suppose $\mathfrak{p}', \mathfrak{q}' \in \operatorname{Spec} RS^{-1}$ with $\mathfrak{p}', \mathfrak{q}' \subseteq \mathfrak{Z}(RS^{-1})$ and $\mathfrak{p}' + \mathfrak{q}' = RS^{-1}$. Then there are $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p}' = \mathfrak{p}S^{-1}$ and $\mathfrak{q}' = \mathfrak{q}S^{-1}$; moreover, $\mathfrak{p}, \mathfrak{q} \subseteq \mathfrak{Z}(R)$ with $(\mathfrak{p}+\mathfrak{q}) \cap S \neq \emptyset$. We find $r \in \mathfrak{p}$, $r' \in \mathfrak{q}$ with $r + r' \in S$ and $rr' = 0$. With $s = r + r'$, $rs^{-1} + r's^{-1} = 1$ and then $e = rs^{-1}$ is the idempotent needed for the criterion of [1, Theorem 2.5]. □

Example 2.2. There is an indecomposable noetherian almost clean ring R such that $Q_{\mathrm{cl}}(R)$ is indecomposable and not clean. The example can be localized at a maximal ideal $\mathfrak{m}$ to give $R_{\mathfrak{m}}$ clean but $Q_{\mathrm{cl}}(R_{\mathfrak{m}})$ not clean.

Proof. Let K be a field and $R = K[W, X, Y, Z]/I$ where I is generated by $\{WX, X^2, XY, Y^2, YZ\}$. Write w for $W + I$, x for $X + I$, etc. The local version will use the maximal ideal $\mathfrak{m} = (w, x, y, z)$ and the same arithmetic will apply to $R_{\mathfrak{m}}$. Elements not in $\mathfrak{m}$ are non zero-divisors.

A typical element of (w, x, y, z) can be written $r = f_1(w) + yf_2(w) + a_1x + a_2y + f_3(z) + xf_4(z) + f_5(w, z)$, where f_1, f_2, f_3, f_4 and f_5 all have zero constant term, terms of f_5 all have the form $a_{ij}w^iz^j$, $a_{ij} \in K$ and $i, j \geq 1$, and $a_1, a_2 \in K$. If $s \in (w, x, y, z)$ is written similarly using $b_i \in K$ and g_j,$1 \leq j \leq 5$, then $rs = h_1 + yh_2 + c_1x + c_2y + h_3 + xh_4 + h_5$, where $h_1 = f_1g_1$, $h_2 = b_2f_1 + a_2g_1 + f_2g_1 + f_1g_2$, $c_1 = 0 = c_2$, $h_3 = f_3g_3$, $h_4 = b_1f_3 + a_1g_3 + f_4g_3 + f_3g_4$, $h_5 = f_1g_3 + f_3g_1 + f_1g_5 + f_5g_1 + f_3g_5 + f_5g_3 + f_5g_5$.

Theorem 2.1 is applied to R (or $R_{\mathfrak{m}}$). Note that the prime ideals (w, x, y) and (x, y, z) of R consist of zero-divisors but $r = w + z \in \mathfrak{R}(R)$. To see this, take, in the above calculation, $r = w + z$ and s; if $rs = 0$ one sees that $s = 0$. The next step is to show that if $r \in (w, x, y)$, $s \in (x, y, z)$ and $rs = 0$ then, $r + s \notin \mathfrak{R}(R)$.

The notation of the second paragraph is used again. We have $f_3 = 0 = g_1$. If $f_1 = 0$ then $r + s \in (x, y, z) \subseteq \mathfrak{Z}(R)$ and if $g_3 = 0$, $r + s \in (w, x, y) \subseteq \mathfrak{Z}(R)$. Hence, we may assume $f_1, g_3 \neq 0$. In the computation of rs, $h_5 = f_1g_3 + f_1g_5 + f_5g_3 + f_5g_5 = (f_1 + f_5)(g_3 + g_5) = 0$. Since there are no relations involving only w and z, this only occurs if $f_1 + f_5 = 0$ or $g_3 + g_5 = 0$, neither of which is possible because of the form of these elements. □

To see that R need not be almost clean in Theorem 2.1, we look at $R = K[X, Y]/(X(X-1)Y)$. This is not almost clean ([1, Example 2.9]) but since R is semiprime, $Q_{\mathrm{cl}}(R)$ is a finite product of fields.

We next look at an example of a *reduced* almost clean ring whose classical ring of quotients is not clean. Such an example cannot have the ACC on annihilator ideals.

Example 2.3. For $n \in 3\mathbb{N}$ let $R_n = (\mathbb{Z}/3\mathbb{Z})[X]$, for $n \in 3\mathbb{N}+1$ let $R_n = \mathbb{Z}$ and for $n \in 3\mathbb{N}+2$, say $n = 3k+2$, let $R_n = (\mathbb{Z}/p_k\mathbb{Z})[X]$, where p_k is the kth prime. Let R be the ring of sequences $(a_1, a_2, \ldots)$ with $a_n \in R_n$ for all $n \in \mathbb{N}$ with the proviso that there is some $f \in \mathbb{Z}[X]$ such that for some $m = 3k$, $a_m = \bar{f}, a_{m+1} = f(0), a_{m+2} = \bar{f}$, with the pattern repeated for all triples beyond m. Then, R is almost clean and reduced but $Q_{\mathrm{cl}}(R)$ is not clean.

Proof. The ring R is clearly reduced since all the components are in domains. We use the Pierce sheaf characterization of an almost clean ring found in [4, Theorem 2.4], which requires that for each $r \in R$ and $x \in \operatorname{Spec}\mathbf{B}(R)$, r is a non zero-divisor on a neighbourhood of x, or $r-1$ is a non zero-divisor on an neighbourhood of x. This is called the NZDC. Here $\operatorname{Spec} R = \mathbb{N} \cup \{\infty\}$ and, for $n \in \mathbb{N}$, the stalk is R_n, while $R_\infty = \mathbb{Z}[X]$. Notice that the terms in the positions $3\mathbb{N}+2$ completely determine the coefficients of f when r is "eventually" $f \in \mathbb{Z}[X]$ since the coefficients of f will be reduced modulo infinitely many primes.

A non zero-divisor must be non-zero in all components because all sequences with only finitely many non-zero components are in R. Let $r \in R$ be "eventually" $f \in \mathbb{Z}[X]$. In order to have all the components of r non-zero we must have, in particular, $f(0) \neq 0$ <u>and</u> some coefficient of f not divisible by 3. Suppose $f(0) = 0$ <u>or</u> all coefficients of f are divisible by 3. Then, $f-1$ will be non-zero in all but finitely many components. This shows that the NZDC of [4, Theorem 2.4] is satified at ∞; the condition at the other points of $\operatorname{Spec}\mathbf{B}(R)$ is clear since $n \in \mathbb{N}$ is a discrete point in $\operatorname{Spec}\mathbf{B}(R)$. Thus, R is almost clean and, as already mentioned, reduced.

Then, by [4, Proposition 2.9(1)], $\mathbf{B}(Q_{\mathrm{cl}}(R)) = \mathbf{B}(R)$. Moreover, $Q_{\mathrm{cl}}(R) \subseteq (\mathbb{Z}/3\mathbb{Z})(X) \times \mathbb{Q} \times (\mathbb{Z}/p_1\mathbb{Z})(X) \times \cdots$ and the only denominators are "eventually" those $f \in \mathbb{Z}[X]$ not in $(X) \cup (3)$, i.e., $Q_{\mathrm{cl}}(R)_\infty = \mathbb{Z}[X]_{(X)} \cap \mathbb{Z}[X]_{(3)}$, which is not a local ring. Hence, $Q_{\mathrm{cl}}(R)$ is not clean because all the stalks of a clean ring must be local (e.g., [5, Proposition 1.2]). □

A variant of Example 2.3 gives an example of a reduced clean ring R such that $Q_{\mathrm{cl}}(R)$ is not clean. In the example, the rings R_n when 3 divides n are localized at (X), when $n \in \mathbb{N}+1$, R_n is $\mathbb{Z}$ localized at (3), and, when $n \in \mathbb{N}+2$, R_n is localized at (X). The sequences now must eventually be in $\mathbb{Z}[X]_{(3,X)}$. The conclusion will again be that $Q_{\mathrm{cl}}(R)_\infty = \mathbb{Z}[X]_{(3)} \cap \mathbb{Z}[X]_{(X)} \supseteq \mathbb{Z}[X]_{(3,X)}$.

Taking the classical ring of quotients of an indecomposable almost clean ring can add infinitely many idempotents. Consider the following examples. Let K be a field and $\{X_1, X_2, \ldots\}$ a set of variables. Put $R = K[X_1, X_2, \ldots]/I$, where I is generated by all monomials

$$X_1X_2, X_3X_4, \ldots, X_{2n-1}X_{2n}, \ldots$$

(see [15, Example 167]). A local version of this is $S = R_{\mathfrak{m}}$, where $\mathfrak{m}$ is the maximal ideal generated by the images of the X_i, $i \in \mathbb{N}$. Now R is indecomposable and almost clean because elements not in $\mathfrak{m}$ are non zero-divisors while $1+\mathfrak{m} \subseteq \mathfrak{R}(R)$. Moreover, S is local and hence clean. It is easy to verify the criteria of, for example, [14, Theorem 4.7] to see that our rings are coherent and satisfy the annihilator condition (i.e., for $a, b \in R$ there is $c \in R$ such that $\operatorname{ann} a \cap \operatorname{ann} b = \operatorname{ann} c$). Thus $Q_{\mathrm{cl}}(R)$ and $Q_{\mathrm{cl}}(S)$ are von Neumann regular. They have infinitely many idempotents. If we write x_i for the image of X_i, $x_{2n-1}(x_{2n-1}+x_{2n})^{-1}$ is, for any $n \in \mathbb{N}$, an idempotent in the classical ring of quotients.

Some results on $Q_{\mathrm{cl}}(R)$ which avoid finiteness conditions are found in [4, Corollary 2.7 and Proposition 2.9].

2.2. *Directed limits of clean and almost clean rings.*

Various kinds of extensions of clean and almost clean rings are studied in [1], [2], [12] and [22]; we have begun our study of the classical ring of quotients above. In this part, we look at directed limits.

Proposition 2.4. *Let $\{R_\alpha, \phi_{\alpha,\beta}\}$ be a directed family of clean rings. Then, the direct limit (R, ϕ_α) is a clean ring.*

Proof. This is immediate since both units and idempotents are preserved by homomorphisms. $\square$

A corollary of this proposition about rings of continuous functions will be left to Section 2.

We now look at directed limits of almost clean rings.

Lemma 2.5. *Let R be a commutative ring and T an essential ring extension of R. If $r \in \mathfrak{R}(R)$ then $r \in \mathfrak{R}(T)$.*

Proof. If $r \in \mathfrak{R}(R)$ and $0 \neq t \in T$, then there is $t' \in T$ with $0 \neq t't \in R$. Then, $rt't \neq 0$ and, hence, $rt \neq 0$. $\square$

Proposition 2.6. *Let $\{R_\alpha; \phi_{\alpha,\beta}\}$ a directed family of almost clean rings where each $\phi_{\alpha,\beta}$ is an essential ring monomorphism. Then, the direct limit, S of the system is an almost clean ring.*

Proof. Indeed, for $r \in \mathfrak{R}(R_\alpha)$, some α, if $0 \neq s \in S$ is such that $rs = 0$ (identifying r with its image in S) then for for $\beta \geq \alpha$ there is $0 \neq t \in R_\beta$ with image s and $\phi_{\alpha,\beta}(r)t = 0$. We get a contradiction using Lemma 2.5. Hence, images of non zero-divisors in the system are non zero-divisors in S. Hence, S is almost clean. $\square$

However, some restrictions on the directed system of almost clean rings is needed in order that the limit be almost clean, as the following example illustrates.

Example 2.7. There is a directed family of almost clean rings with monomorphic maps whose limit is not almost clean.

Proof. It is shown in [4, Example 2.8(i)] that the following ring is not almost clean. It will be expressed here as a limit of a directed system of almost clean rings. The set $\mathbb{N}$ is partitioned into infinitely many infinite subsets $\{N_k\}_{k\in\mathbb{N}}$ and each N_k is well ordered as $\{m_{k,1}, m_{k,2}, \ldots\}$. If $n \in \mathbb{N}$ is $m_{k,j}$, then R_n is defined to be $\mathbb{Z}/p_j\mathbb{Z}$, where p_j is the jth prime. The ring R is defined to be the ring of sequences of the form $(\bar{a}_1, \bar{a}_2, \ldots)$ where $\bar{a_n} \in R_n$ and, for some $m \in \mathbb{N}$, $\bar{a}_n = \bar{z}$, for all $n \geq m$ and some $z \in \mathbb{Z}$. Then $\operatorname{Spec}\mathbf{B}(R) = \mathbb{N} \cup \{\infty\}$ with R_n the Pierce stalk over $n \in \mathbb{N}$ and $R_\infty = \mathbb{Z}$.

We now define, for $n \in \mathbb{N}$, $S_n = R_1 \times \cdots \times R_n \times \mathbb{Z}$ and $\phi_{n,n+1} \colon S_n \to S_{n+1}$ by $\phi_{n,n+1}((\bar{a}_1, \ldots, \bar{a}_n, z)) = (\bar{a}_1, \ldots, \bar{a}_n, \bar{z}, z)$. It needs to be verified that R is the limit of the system $\{S_n, \phi_{n,n+1}\}$. We define, for $n \in \mathbb{N}$, $\phi_n \colon S_n \to R$ by $\phi_n((\bar{a}_1, \ldots, \bar{a}_n, z)) = (\bar{a}_1, \ldots, \bar{a}_n, \bar{z}, \bar{z}, \ldots) \in R$. It is now easy to verify that R satisfies the universal property of a direct limit because R can be viewed as a union of subrings. □

2.3. *Extensions by idempotents.*

In this paragraph we look at extensions of (almost) clean rings by idempotents, that is, a ring extension $R \subseteq S$ where S is generated, as a ring, by R and $\mathbf{B}(S)$.

Proposition 2.8. *Let R be an almost clean ring, T an essential ring extension of R and $E \subseteq \mathbf{B}(T)$. If S is the subring of T generated by R and E then S is almost clean. This applies, in particular, when $T = Q(R)$, the complete ring of quotients of R.*

Proof. A typical element of S may be written $s = r_1e_1 + \cdots + r_ne_n$, for $r_1, \ldots, r_n \in R$ and $\{e_1, \ldots, e_n\}$ a complete orthogonal set of idempotents from $\mathbf{B}(S)$. For each $i = 1, \ldots, n$, we can write $r_i = r'_i + f_i$, where $r'_i \in \mathfrak{R}(R)$ and $f_i \in \mathbf{B}(R)$. Then, $s = \sum_{i=1}^n r'_ie_i + \sum_{i=1}^n f_ie_i$. The second term is an idempotent so we need to show that $u = \sum_{i=1}^n r'_ie_i \in \mathfrak{R}(S)$.

To verify this, let $v = \sum_{j=1}^m a_jg_j \in S$, where, for $j = 1, \ldots, m$, $a_j \in R$ and $\{g_1, \ldots, g_m\}$ is a complete orthogonal set of idempotents from $\mathbf{B}(S)$. Suppose $uv = 0$ and, hence, that for each pair i, j, $r'_ia_je_ig_j = 0$. Since $r'_i \in \mathfrak{R}(S)$, by Lemma 2.5, each $a_je_ig_j = 0$. Summing over i yields that $a_jg_j = 0$ and, hence, that $v = 0$, as required. □

An extension of a clean ring by idempotents is clean without any restrictions.

Proposition 2.9. *Suppose R is a clean ring which is a subring of a ring T. Let $E \subseteq \mathbf{B}(T)$ and let S be the subring of T generated by R and E. Then, S is clean.*

Proof. The argument is like that of Proposition 2.8. As before, a typical element $s = \sum_{i=1}^{n} r_i e_i$, with each $r_i \in R$ and $\{e_1, \ldots, e_n\}$ a complete orthogonal set of idempotents from $\mathbf{B}(S)$. Each $r_i = u_i + f_i$, $u_i \in \mathfrak{U}(R)$ and $f_i \in \mathbf{B}(R)$, with, say, $u_i^{-1} = v_i$. Then, $s = (\sum_{i=1}^{n} u_i e_i) + (\sum_{i=1}^{n} f_i e_i)$ where the first term is a unit in S, with inverse $\sum_{i=1}^{n} v_i e_i$ and the second is an idempotent. $\square$

The extra hypothesis in Proposition 2.8 is needed, as witnessed by the following example.

Example 2.10. There is an almost clean ring T and an idempotent e from an extension ring W such that T', the subring of W generated by T and e, is not almost clean

Proof. We can use the ring of [1, Example 2.9] or a similar one given here. Let $R = \mathbb{Z}[X]$ and $S = \mathbb{Z}[X]/(6X)$. Set T to be a subring of $R \times R \times S$ as follows:

$$T = \{(r_1, r_2, \bar{s}) \mid r_1 \equiv s \pmod{2X}, r_2 \equiv s \pmod{3X}\}\ .$$

Just as in [1, Example 2.9], S is indecomposable and not almost clean ($\bar{3}$ is a zero-divisor as is $\overline{3-1}$)). Next, T is indecomposable. Note that $(0, 0, \bar{s}) \notin T$ unless $\bar{s} = \bar{0}$. Hence, a non-trivial idempotent would look like $(1, 0, \bar{1})$ or $(0, 1, \bar{1})$; however, these are not possible because s must be in $(2X)$ or $(3X)$ and have constant term 1. The element $(1, 0, \bar{0})$ (or $(0, 1, \bar{0})$) is also excluded since 0 would have to be in $(3X)$ and have constant term 1.

Next, T is almost clean. The non zero-divisors are the elements of the form $(r_1, r_2, \bar{s})$ with $r_1, r_2 \neq 0$. Indeed, $(0, r_2, \bar{s})$ is annihilated by $(2X, 0, \bar{0})$ and $(r_1, 0, \bar{s})$ is annihilated by $(0, 3X, \bar{0})$. If $(0, r_2, \bar{s}) \in T$ then $(0, r_2, \bar{s}) - (1, 1, \bar{1})$ is a non-zero divisor (similarly for $(r_1, 0, \bar{s})$). This is because $s \in (2X)$ and then $r_2 = 1$ would imply $s - 1 \in (3X)$ showing that the constant term of s is 1, however, we know it to be 0. Hence, $(0, r_2, \bar{s}) - (1, 1, \bar{1}) = (-1, r_2 - 1, \bar{s} - \bar{1})$ is a non zero-divisor because the middle component is non-zero.

Now, $T \subset R \times R \times S$ and if we adjoin $(1, 1, \bar{0})$ to T to get T', the ring S splits off showing that T' is not almost clean, since it is clear that a direct factor of an almost clean ring is almost clean. (Note, however, that $Q_{\mathrm{cl}}(T) \cong \mathbb{Q} \times \mathbb{Q}$ and the idempotent we have added is not from $Q_{\mathrm{cl}}(T)$. This can be seen when T is presented as a subring of $\mathbb{Z} \times \mathbb{Z}$, as is possible.) $\square$

There is also a "generic" extension of a ring by idempotents which is best described in terms of the Pierce sheaf and thus is more properly the subject of [4]. It is shown there that a generic extension of an almost clean ring by idempotents does yield an almost clean ring.

3. On $C(X)$ and $Q_{\mathrm{cl}}(X)$.

The principal aim of this section is to answer the question: When is $Q_{\mathrm{cl}}(X)$ clean? The characterization of when $C(X)$ is clean derives readily from one of the characterizations of clean ring in [20, Theorem 1.7], due to Johnstone ([16, Theorem V.3.9]), which says that a ring R is clean if and only if it is pm and $\operatorname{maxSpec} R$ is 0-dimensional. However, $C(X)$ is always a pm ring ([10, Theorem 2.11]) and $\operatorname{maxSpec} C(X)$ is βX, the Čech-Stone compactification of X ([10, 7.11]). Hence, $C(X)$ is clean if and only if βX is 0-dimensional, i.e., if and only if X is *strongly 0-dimensional*. For several other characterizations of when $C(X)$ is clean see McGovern [21, Theorem 13]. Note that $C(X)$ is clean if and only if it is almost clean, since this is a property of pm rings ([4, Theorem 3.3]).

It is clear that if R is a pm ring then so is $Q_{\mathrm{cl}}(R)$. In particular, $C(X)$ is always a pm ring and, hence, $Q_{\mathrm{cl}}(X)$ is almost clean if and only if it is clean by [4, Theorem 3.3].

We have seen (Proposition 2.4) that a directed limit of clean rings is clean. A special case of this follows.

Proposition 3.1. *For a completely regular topological space X, if $C(X)$ is clean (i.e., X is strongly 0-dimensional) then $Q_{cl}(X)$ is clean.*

Proof. We have by, e.g., [21, Theorem 13] that X is strongly zero-dimensional and, by definition, so is βX. By [9, 3.1], $Q_{\mathrm{cl}}(X) = Q_{\mathrm{cl}}(\beta X)$. By [9, Corollary 1.10], $Q_{\mathrm{cl}}(X)$ is a direct limit of the rings $C(V)$, where V is a dense cozero-set of βX. Hence, by Proposition 2.4, it suffices to show that V is strongly zero-dimensional. Now V has a strongly zero-dimensional compactification, namely βX, and so is zero-dimensional by [10, 16D(2)] and, hence, strongly zero-dimensional by [10, 16.11] because V is Lindelöf. $\square$

We have just seen that $C(X)$ clean implies that $Q_{\mathrm{cl}}(X)$ is clean. Recall that a space X is *cozero complemented* if, for every cozero set V, there is a cozero set U such that $V \cap U = \emptyset$ and $V \cup U$ is dense in X. It is easy to find spaces where $C(X)$ is not clean but $Q_{\mathrm{cl}}(X)$ is. It is shown in [13, Theorem 1.3] that X is cozero complemented if and only if $Q_{\mathrm{cl}}(X)$ is von Neumann regular, and, hence, clean. A long list of cozero complemented spaces is found in [13] and these include all metric spaces. If we take a space like $\mathbb{R}$, then the connectedness of $\mathbb{R}$ shows that $C(\mathbb{R})$ is not clean but $Q_{\mathrm{cl}}(\mathbb{R})$ is clean.

The proof of Proposition 3.1 also applies to any ring of fractions A obtained from a filter of dense cozero sets in X; this shows that A is a clean ring if $C(X)$ is. As just mentioned, the converse of Proposition 3.1 is false. However, the converse holds for F-spaces. Recall ([10, 14.25]) two of the characterizations of these spaces: X is an F-space if and only if for each $p \in \beta X$, the ideal O^p of elements zero on a neighbourhood of p is prime; equivalently, if and only if the primes contained in a given maximal ideal form a chain.

In what follows, homeomorphism is denoted by $\sim$.

Proposition 3.2. *Let X be a topological space and $S \subseteq \mathfrak{R}(C(X))$ be multiplicatively closed.*
(1) There is a continuous surjection $\Theta\colon \operatorname{maxSpec} S^{-1}C(X) \to \beta X$.
(2) If, in addition, X is an F-space, Θ is a homeomorphism.

Proof. (1) Since $C(X)$ is a pm ring ([7, Theorem 1.2]) $\Psi\colon \operatorname{Spec} C(X) \to \operatorname{maxSpec} C(X)$, sending a prime ideal to the unique maximal ideal containing it, is a continuous surjection. For any $\mathfrak{p} \in \operatorname{minSpec} C(X)$ there is a unique largest element $\mathfrak{q}$ in the chain of prime ideals from $\mathfrak{p}$ to the maximal ideal containing it which does not meet S. Call the set of such prime ideals Max-S. Since $C(X) \to S^{-1}C(X)$ is a ring epimorphism, the natural continuous map $\phi\colon \operatorname{Spec} S^{-1}C(X) \to \operatorname{Spec} C(X)$ is one-to-one. Moreover, ϕ, restricted to $\operatorname{maxSpec} S^{-1}C(X)$, call it ϕ_S, is a continuous bijection between the subspaces $\operatorname{maxSpec} S^{-1}C(X)$ and Max-S. Keeping in mind that $\operatorname{maxSpec} C(X) \sim \beta X$, we put $\Theta = \Psi \circ \phi_S\colon \operatorname{maxSpec} S^{-1}C(X) \to \beta X$. Since every chain of prime ideals of $C(X)$ from a minimal prime to a maximal ideal passes through Max-S, Θ is surjective.

(2) The extra hypothesis ensures that Θ is one-to-one. We now have a continuous bijection $\Theta\colon \operatorname{maxSpec} S^{-1}C(X) \to \beta X$. However, βX is compact and Hausdorff, showing that Θ is a homeomorphism. □

Proposition 3.3. *Let X be an F-space. The following statements are equivalent.*
(1) $C(X)$ is clean.
(2) $S^{-1}C(X)$ is clean for each multiplicatively closed set $S \subseteq \mathfrak{R}(C(X))$.
(3) $S^{-1}C(X)$ is clean for some multiplicatively closed set $S \subseteq \mathfrak{R}(C(X))$.

Proof. (1) $\Rightarrow$ (3) by Theorem 3.1 using $S = \mathfrak{R}(C(X))$ or $S = \emptyset$. Clearly (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) using $S = \emptyset$.

(3) $\Rightarrow$ (2): We fix S with $S^{-1}C(X)$ clean. Proposition 3.2 (2) shows that $\operatorname{maxSpec} S^{-1}C(X) \sim \beta X$. By hypothesis and using [20, Theorem 1.7], $\operatorname{maxSpec} S^{-1}C(X)$ is strongly 0-dimensional. Hence, βX and X are strongly 0-dimensional. Now we can apply Proposition 3.2 (2) to any multiplicatively closed set $T \subseteq \mathfrak{R}(C(X))$ to get that $\operatorname{maxSpec} T^{-1}C(X)$ is strongly 0-dimensional. Since $T^{-1}C(X)$ is a pm ring, [20, Theorem 1.7], again, says that $T^{-1}C(X)$ is clean. This proves (2). □

There are F-spaces where $C(X)$ is clean and F-spaces where it is not. Any extremally disconnected space will give a strongly 0-dimensional F-space while the connected F-space $\beta\mathbb{R}^+ \setminus \mathbb{R}^+$ does not (see [10, 14.27]). As an illustration of a ring strictly between $C(X)$ and $Q_{\mathrm{cl}}(X)$ in Proposition 3.3, we take the strongly 0-dimensional F-space $\beta\mathbb{N}$; let $f \in \mathfrak{R}(C(\beta\mathbb{N}))$ be given by $f(n) = 1/n$, $n \in \mathbb{N}$ and define $S = \{f^m \mid m \in \mathbb{N}\}$. Then $S^{-1}C(\beta\mathbb{N})$ is such a ring.

The conclusion of Proposition 3.3 does not imply that X is an F-space as the following shows. Recall ([19]) that a space X is an *almost P-space* if each non-empty zero-set has non-empty interior. If X is an almost P-space, X is the only dense cozero set and hence ([24, page 72]) $Q_{\mathrm{cl}}(X) = C(X)$.

Proposition 3.4. *Suppose K is a compact strongly 0-dimensional almost P-space, X is the free sum $\bigcup_{n\in\mathbb{N}} K_n$, where each $K_n = K$, and ωX is the one-point compactification of X. Then, for each multiplicatively closed $S \subseteq \mathfrak{R}(C(\omega X))$, $S^{-1}C(\omega X)$ is clean. There are choices of K for which ωX is not an F-space.*

Proof. Note that $Q_{\mathrm{cl}}(\omega X) = \prod_{n\in\mathbb{N}} C(K_n)$. It suffices to pick $S = \{s, s^2, s^3, \ldots\}$, for some non zero-divisor $s \in C(\alpha X)$ which is not a unit. A separate proof for $S = \emptyset$ will follow. If we write $\omega X = X \cup \{\infty\}$, then, by the hypotheses on K, s is non-zero everywhere on X and $s(\infty) = 0$. For $f \in C(\omega X)$ and $m \in \mathbb{N}$, we need to look at $fs^{-m} \in S^{-1}C(\omega X)$. We have that $s_n = s|_{K_n}$ is a unit in $C(K_n)$ and if $f|_{K_n} = f_n$, $f_n s_n^{-m} = u_n + e_n$, u_n a unit in $C(K_n)$ and $e_n \in \mathbf{B}(C(K_n))$. We next define $u, e \in \prod_{n\in\mathbb{N}} C(K_n)$ to have components u_n and e_n, respectively.

It will be seen that $es \in C(X)$ can be extended to an element of $C(\omega X)$. For each $m \in \mathbb{N}$, $\{x \in X \mid |es(x)| < 1/m\} = (s^{-1}((-1/m, 1/m)) \cap X) \cup \mathrm{Z}(e)$, whose complement in X is an intersection of a compact with a closed set; it is, hence, compact. This means that es can be extended to ∞ by making the function 0 there. Hence, $es \cdot s^{-1} = e \in S^{-1}C(\omega X)$. The equation $fs^{-m} = u + e$ now has $e \in S^{-1}C(\omega X)$, showing that $u \in S^{-1}C(\omega X)$, as well. Since u is a unit in $Q_{\mathrm{cl}}(X)$, it is in $\mathfrak{R}(S^{-1}C(X))$. Hence, $S^{-1}C(\omega X)$ is almost clean. By [4, Theorem 3.3], an almost clean pm ring is clean, hence, $S^{-1}C(X)$ is clean.

Next suppose $S = \emptyset$. We need to show that ωX is strongly 0-dimensional. Since it is compact, it suffices to show it has a basis of clopen neighbourhoods. If $x \in X$ and U is an open set with $x \in U$, we have that $x \in K_n$ for some n. However, K_n is strongly 0-dimensional and we can pick a clopen neighbourhood V of x in K_n which is inside U. Then, V is clopen in ωX. Next, let U be a neighbourhood of ∞. Since $X \setminus U$ is compact, $X \setminus U \subseteq \bigcup_{n=1}^{m} K_n$, for some $m \in \mathbb{N}$. Then, $\omega X \setminus \bigcup_{n=1}^{m} K_n$ is a clopen neighbourhood of ∞ inside U.

The particular case of $\omega\mathbb{N}$, with $K = \{x\}$, which is not an F-space, finishes the proof. □

We now characterize completely regular spaces X such that $Q_{\mathrm{cl}}(X)$ is clean. The condition, called "WZD" in the next result, is a strict generalization both of "cozero complemented" and of "strongly 0-dimensional" as examples will show. That is, there are spaces X such that $Q_{\mathrm{cl}}(X)$ is clean but not regular and, moreover, $C(X)$ is not clean. Recall from [9] that $\mathcal{V}_0(X)$, or simply, $\mathcal{V}_0$, is the set of dense cozero sets of a space X. The condition is labeled WZD (for *weak zero dimensional*) because it is a weakening of the strong 0-dimensional condition. Since the characterization of the WZD property is algebraic ($Q_{\mathrm{cl}}(X)$ is clean) a space X satisfies the property if and only if its realcompactification υX does.

Theorem 3.5. *Let X be a completely regular space. Then, $Q_{cl}(X)$ is clean if and only if the following condition holds*

> *WZD: Given $V \in \mathcal{V}_0$ and disjoint zero sets K_1 and K_2 of V, there exist $W \in \mathcal{V}_0$, $W \subseteq V$, and a clopen decomposition $W = W_1 \dot\cup W_2$ such that $K_1 \cap W \subseteq W_1$ and $K_2 \cap W \subseteq W_2$.*

Proof. We first assume that $Q_{\mathrm{cl}}(X)$ is clean. Suppose that $V \in \mathcal{V}_0$ and let K_1, K_2 be disjoint zero-sets in V. There is an $f \in C(V)$ such that f is 0 on K_1 and 1 on K_2. Then, f gives rise to an element $q \in Q_{\mathrm{cl}}(X)$ which can be written $q = u+e$, u a unit and e an idempotent. For some $W \in \mathcal{V}_0$, for which we may assume $W \subseteq V$, $f|_W = u + e$, where now, $u, e \in C(W)$. The idempotent e yields a clopen decomposition $W = W_1 \dot\cup W_2$ where $\mathrm{Z}(e) = W_2$. It follows that $K_1 \cap W \subseteq \mathrm{Z}(f) \cap W \subseteq W_1$ and $K_2 \cap W \subseteq W_2$, as required.

Now assume that X is a WZD-space. Let $q \in Q_{\mathrm{cl}}(X)$ coming from some $f \in C(V)$, $V \in \mathcal{V}_0$. Let $K_1 = \mathrm{Z}(f)$ and $K_2 = \{v \in V \mid f(v) = 1\}$. We invoke the condition to get $W \in \mathcal{V}_0$, $W \subseteq V$, and a clopen decomposition $W = W_1 \dot\cup W_2$ with $K_1 \cap W \subseteq W_1$ and $K_2 \cap W \subseteq W_2$. Let $e = e^2 \in C(W)$ with $\mathrm{Z}(e) = W_2$. Then, $f|_W - e = u$ is a unit in $C(W)$, as needed. This expression lifts to an expression of q as a unit plus an idempotent in $Q_{\mathrm{cl}}(X)$. □

Recall ([6, Definition 7.4 and Theorem 7.8] that if Y is a locally compact, non pseudocompact space and Z is a weak Peano space then there is a compactification X of Y such that $Z = X \backslash Y$; a *weak Peano space* is a compact Hausdorff space which contains a dense continuous image of $\mathbb{R}$. Thus if D is an infinite discrete space then βD and the one-point compactification D are both strongly 0-dimensional but there are compactifications αD which are not. More details are found in Examples 3.7, below.

Lemma 3.6. *Suppose that Y is a dense C^*-embedded subset or a dense cozero set of a space X. Then $Q_{cl}(Y) = Q_{cl}(X)$. In particular, if Y is a locally compact Lindelöf space and X a compactification of Y, then $Q_{cl}(X) \cong Q_{cl}(Y)$.*

Proof. We use a standard argument for the first case. Let $\rho\colon C(X) \to C(Y)$ be the restriction map. By hypothesis, ρ is one-to-one and $C^*(Y) \subseteq \operatorname{Im}\rho$. If $f \in \mathfrak{R}(C(X))$ then $\operatorname{coz}\rho(f)$ is dense in Y, i.e., $\rho(f) \in \mathfrak{R}(C(Y))$. This implies that ρ extends to $\sigma\colon Q_{\mathrm{cl}}(X) \to Q_{\mathrm{cl}}(Y)$, which is also one-to-one. For $g \in C(Y)$, $g/(1+g^2), 1/(1+g^2) \in \operatorname{Im}\rho$ and the extension of $1/(1+g^2)$ is in $\mathfrak{R}(C(X))$. Hence, $g \in \operatorname{Im}\sigma$. Moreover, if $g \in \mathfrak{R}(C(Y))$ then the extension of $g/(1+g^2)$ is also in $\mathfrak{R}(C(X))$. This shows that σ is onto. In the case when Y is a cozero set of X, $C(Y)$ lies between $C(X)$ and $Q_{\mathrm{cl}}(X)$ so $Q_{\mathrm{cl}}(X) \cong Q_{\mathrm{cl}}(Y)$.

In the particular case where Y is locally compact Lindelöf, Y is a dense cozero set in X and the second statement applies. □

It follows from Lemma 3.6 that if a space X has a dense C*-embedded subset or a dense cozero set satisfying WZD then X is a WZD-space.

The next aim is to specialize the spaces X and Y in the lemma. The two families of examples which follow give WZD-spaces but which are neither cozero complemented nor strongly 0-dimensional.

Examples 3.7. (1) There is an example of a locally compact Lindelöf space Y and a compactification X such that $Q_{\mathrm{cl}}(X) = C(Y)$ is clean but $C(X)$ is not clean.

(2) There is an example of a locally compact Lindelöf space Y and a compactification X such that $Q_{\mathrm{cl}}(X) = Q_{\mathrm{cl}}(Y)$ is clean but $C(Y)$ and $C(X)$ are not clean.

Proof. (1) Let Y be an almost P strongly 0-dimensional space which is locally compact but not compact. (For example, a free union of infinitely many copies of $\beta\mathbb{N} \setminus \mathbb{N}$.) Then Y has a compactification X that is not 0-dimensional. For example we may take a compactification X of Y, where $X \setminus Y$ is a connected Peano space, say, $[0, 1]$. If X were strongly 0-dimensional then $X \setminus Y$ would not be connected. When Y is Lindelöf as well then it is a cozero set of X showing that $C(Y)$ is a ring of fractions of $C(X)$ and, hence, $Q_{\mathrm{cl}}(Y) = Q_{\mathrm{cl}}(X)$. However, here, $Q_{\mathrm{cl}}(Y) = C(Y)$.

As a specific example we can take V_n, $n \in \mathbb{N}$, each a copy of the one-point compactification of an uncountable discrete space and $Y = \bigcup_{n\in\mathbb{N}} V_n$, a free union. The space Y is locally compact and Lindelöf and is neither compact nor pseudocompact. In this particular example, $C(Y)$ is not von Neumann regular and thus X is not cozero complemented.

(2) An example of this type may be constructed using (1). Suppose, for each $n \in \mathbb{N}$, X_n is the compact space constructed in (1). Set $Y = \bigcup_{n\in\mathbb{N}} X_n$, a free union, and let X be a compactification of Y such that $X \setminus Y$ is a connected Peano space. Once again Y is a dense cozero set of X and $C(Y)$ is a ring of fractions of $C(X)$. It follows that $Q_{\mathrm{cl}}(X) = Q_{\mathrm{cl}}(Y)$. However, Y is not strongly 0-dimensional showing that $Q_{\mathrm{cl}}(X) \neq C(Y)$. We have that $C(Y) = \prod_{n\in\mathbb{N}} C(X_n)$ and, hence, $Q_{\mathrm{cl}}(Y) = \prod_{n\in\mathbb{N}} Q_{\mathrm{cl}}(X_n)$, a product of clean rings and, thus, a clean ring ([12, Proposition 7]). □

If X is a WZD-space and V is a dense cozero set of X or a dense C*-embedded subspace of X then, clearly, V is a WZD-space. A compact subspace of a WZD-space need not be WZD; for example, take $X = \beta\mathbb{R}^+$ and $Y = \beta\mathbb{R}^+ \setminus \mathbb{R}^+$. Moreover, Gruenhage, [11, Theorem 1.5], has shown that *any* space X can be embedded as a closed subspace of a cozero complemented and, hence, a WZD-space; in fact there is a metrizable space M such that $X \times M$ is cozero complemented. We do have the following using the property z-embedded, a generalization of "C*-embedded".

Proposition 3.8. *Let X be a WZD-space and suppose X is a dense z-embedded subspace of a space T. Then, T is a WZD-space.*

Proof. Consider $V \in \mathcal{V}_0(T)$ and Z_1, Z_2 disjoint zero-sets of V. The WZD condition needs to be verified. Put $V' = V \cap X \in \mathcal{V}_0(X)$ and $Y_i = Z_i \cap X \subseteq V'$, $i = 1, 2$, which are zero-sets of V'. There exists $U \in \mathcal{V}_0(X)$, $U \subseteq V'$ and a clopen decomposition $U_1 \dot\cup U_2$ of U such that $Y_i \cap U \subseteq U_i$, $i = 1, 2$. Since each U_i is a cozero set of the cozero set U of X, there is a cozero set $\tilde{U}_i$ of T, $i = 1, 2$ such that $\tilde{U}_i \cap X = U_i$, because X is z-embedded in T.

Let $Z_i' = Z_i \cup (T \setminus V)$, $i = 1, 2$, a zero-set of T and set $S_1 = \tilde{U}_1 \cap (T \setminus Z_2')$ and $S_2 = \tilde{U}_2 \cap (T \setminus Z_1')$ both cozero sets of T. Now set $S = (S_1 \cup S_2) \cap V$, a cozero set of T.

$$\begin{aligned} (1) \qquad \mathrm{S}_1 \cap \mathrm{X} &= U_1 \cap (T \setminus Z_2') = U_1 \cap (T \setminus (Z_2 \cup (T \setminus V))) \\ &= U_1 \cap X \cap (T \setminus Z_2 \cap V) = U_1 \cap X \cap (V \setminus Z_2) \\ &= U_1 \cap (V \setminus Z_2) = U_1 \end{aligned}$$

and, similarly, $S_2 \cap X = U_2$.

(2) $S_1 \cap S_2 \cap X = U_1 \cap U_2 = \emptyset$ and, hence, $S_1 \cap S_2 = \emptyset$.

(3) $S_1 \cup S_2$ is dense since it contains U. Hence, S is dense.

The required splitting of S is using $S_1 \cap V$ and $S_2 \cap V$. □

By [3, Theorem 4.1], if X is Lindelöf or almost compact, it is z-embedded in any space T of which it is a subspace. Hence, Proposition 3.8 will apply to any Lindelöf or almost compact space X and space T in which X is a dense subspace.

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