October 26, 2010

On primitively divisible modules and related rings

W.D. Burgess
Department of Mathematics and Statistics,
University of Ottawa, Ottawa, ON K1N 6N5
wburgess@uottawa.ca

A. Mojiri
Department of Mathematics,
Saint Xavier University, Chicago, IL 60655
mojiri@sxu.edu

Key words: primitively divisible, π-regular, max-ring.

Mathematics Subject Classification: 16D60, 16D50, 16E50.

Abstract. For a ring R, $\mathbf{0} \neq M \in R$-Mod is *primitively divisible* if for each left primitive ideal P, $PM = M$; R is a *left noprimdiv ring* if it has no primitively divisible left modules. The class of left noprimdiv rings strictly includes that of left max-rings. Rings which are left primitive or regular or biregular are left noprimdiv. Properties of such rings are given and closure properties of the class of left noprimdiv ring is studied. A theorem of Snider on π-regular max-rings is extended by including some rings not of finite index. At the other extreme are rings R where ${}_RR$ can be embedded in a primitively divisible module; these are known to be the rings over which every injective left module is primitively divisible. Analogies with classical divisibility are studied and examples found. Commutative rings with this property are characterized.

1. Introduction and basic examples. The class of rings R over which every non-zero left module has a maximal (proper) submodule, called *left max-rings*, has been extensively studied (see Tuganbaev, [17, §26] and the many references of that section). If R is a left max-ring and $\mathbf{0} \neq M$ is a left R-module then there is a left primitive ideal P such that $PM \neq M$. This property will be seen to be weaker than "left max-ring" but is still quite restrictive. The two concepts coincide for PI-rings ([16]). When this property fails there is a module $\mathbf{0} \neq M \in R$-Mod such that $PM = M$ for all left primitive ideals P.

Definition 1.1 (See [16])**.** *For a ring R, if for some $\mathbf{0} \neq M \in R$-Mod, for each left primitive ideal P, $PM = M$ then M is called* primitively divisible.

Definition 1.2. *If R is a ring with no primitively divisible left modules it is called a* left noprimdiv ring*. Similarly for* right noprimdiv rings*.*

Notation and terminology. In a ring R, if A is a subset then $\langle A \rangle$ is the ideal of R generated by A. The class of left modules over a ring R is denoted R-Mod and that of right modules is Mod-R. The set of maximal ideals of a ring R is $\operatorname{Max} R$, the set of minimal prime ideals is $\operatorname{Min} R$, and the set (boolean algebra) of *central* idempotents of R is denoted $\mathbf{B}(R)$. For an R-module M, $E(M)$ denotes any injective hull. The Jacobson radical of R is written $\mathbf{J}(R)$ and its prime radical is $\mathbf{P}(R)$. The left (right) annihilator in X of Y is denoted $\operatorname{lann}_X Y$ ($\operatorname{rann}_X Y$). The set of left primitive ideals of a ring R is denoted $\mathfrak{Pl}(R)$, or just $\mathfrak{Pl}$. The term "regular ring" will always mean "von Neumann regular ring".

A. Introduction. It will be seen (Examples 1.7) that there are left noprimdiv rings which are not left max-rings. The former are known to include, for example, all regular and all biregular rings. A characterization of PI-rings which are left noprimdiv rings is known and is quoted below ([16, Theorem 2]). Many of the known examples are found in [16], due to Tuganbaev. The purpose here, in the first two sections, is to give constructions of families of examples of left noprimdiv rings, to examine the class of left noprimdiv rings and to look at rings which are not left noprimdiv rings.

The paper begins with a list of known classes of left noprimdiv rings and then (Proposition 1.4) some examples where the condition fails. Proposition 1.10 shows that it suffices to test the submodules of any cogenerator module. Then it is shown that a ring R is a left noprimdiv ring if and only if $R/\mathbf{J}(R)$ is and $\mathbf{J}(R)$ is left T-nilpotent (Proposition 1.11). A basic example due to Snider (Example 1.13) shows that even when "noprimdiv" and "max" coincide, the condition is not right-left symmetric. By Snider's [15, Theorem 1], it is known that a semiprimitive π-regular ring of finite index is a left (and right) max-ring. Theorem 1.15 extends this to some π-regular rings not of finite index using a condition on the structure space. Examples 1.16 illustrate this.

The next topic (Section 2) is to examine closure properties of the class of left noprimdiv rings. The class is not closed under homomorphic images, subrings, direct or inverse limits or infinite products (Propositions 2.1 and 2.7). On the other hand, "left noprimdiv" is a Morita invariant (Theorem 2.5) and a ring is a left noprimdiv ring if and only if each of its Pierce stalks is (Theorem 2.4). The property is preserved under some ring extensions such as polynomial rings as well as finite products (Proposition 2.6).

The third section looks at rings, called left IPD rings, where each injective left module is primitively divisible. This can be expressed in terms of a form of divisibility by finite sets of elements (Proposition 3.5 and Theorem 3.4) and is classical divisibility in PIDs. Every domain which is not left primitive is a left IPD ring (Proposition 1.4). Most of the discussion in Section 3 is of commutative rings. A commutative ring R is an IPD ring if and only if each $P \in \operatorname{Max} R$ contains a finite subset with zero annihilator (Theorem 3.4). If R is a commutative IPD ring then $\operatorname{Min} R \cap \operatorname{Max} R = \emptyset$ (Theorem 3.6) and the condition is sufficient when R is reduced and $\operatorname{Min} R$ is compact (Proposition 3.7), but not in general (Example 3.10). Examples, including of rings of continuous functions, are presented as illustrations.

B. Classes of rings which are left noprimdiv rings and some which are not. Some classes of left noprimdiv rings (although the name was not used) are found in Tuganbaev, [16].

Definition 1.3 ([16])**.** *A left R-module M is called* primitively pure *if for each $N \subseteq M$ and each left primitive ideal P, $PM \cap N = PN$. The ring R is* left primitively pure *if ${}_RR$ is.*

It is shown in [16, Lemma 1] that to show a module is primitively pure, it suffices to look at cyclic submodules. If ${}_RR$ is primitively pure then R is a left noprimdiv ring ([16, Lemma 4]). This result gives some of the following classes of examples of left noprimdiv rings. Others will be added later.

Class 1. Every left primitive ring is a left noprimdiv ring.

Class 2. Every left max-ring is a left noprimdiv ring.

Class 3. ([16, Theorem 2]) If R is a PI-ring then R is a left noprimdiv ring if and only if it is a left max-ring and if and only if R is a right noprimdiv ring if and only if R is a right max-ring.

Class 4. ([16, Lemma 4]) If R is *left weakly regular* ($I^2 = I$ for all left ideals I) then ([17, Lemma 20.3(7)]) R is left primitively pure and, hence, a left noprimdiv ring. Regular rings and biregular ring are both left and right weakly regular; hence, they are left and right noprimdiv rings.

Class 5. ([4, Lemma 1]) All *right* semi-artinian rings are *left* noprimdiv rings.

The rings of Class 3 are characterized in [16, Theorem 2] as those PI-rings R where $\mathbf{J}(R)$ is left (and right) T-nilpotent and $R/\mathbf{J}(R)$ is π-regular.

The paper cited in Class 5, above, has as its main purpose the construction of a right semi-artinian regular ring R which is a right max-ring but is not a left max-ring. However, R is (by Class 4) a left noprimdiv ring. Hence, the class of left noprimdiv rings is strictly larger than that of left max-rings. (See also Examples 1.7.)

In the other direction, a ring like $\mathbf{Z}$ is not a noprimdiv ring because $\mathbf{Q}$, as a $\mathbf{Z}$-module is primitively divisible. This observation can be generalized to include, as examples, all domains which are not left primitive and all commutative rings in which each maximal ideal contains a non zero-divisor. (This topic is expanded upon in Section 3.)

Proposition 1.4. *(i) Let R be a ring such that it has a left or right Ore set T of non zero-divisors such that for each left primitive ideal P, $P \cap T \neq \emptyset$. Then, R is not a left noprimdiv ring.*

(ii) Let R be a left non-singular ring such that every left primitive ideal P contains $a \in P$ such that $ra = 0$ implies $r = 0$. Then, R is not a left noprimdiv ring.

Proof. (i) Let S be the ring of left or right fractions corresponding to T. Then ${}_RS$ is primitively divisible since for each left primitive ideal P, there is $t \in P \cap T$ and $PS \supseteq tS = S$.

(ii) Let $S = Q^{\mathrm{l}}_{\max}(R)$ be the left maximal ring of quotients of R, a regular ring. If $a \in R$ is such that $ra = 0 \Rightarrow r = 0$, let $a' \in S$ be such that $aa'a = a$. Then, $aa' = e$ is an idempotent and, if $e \neq 1$, $S(1-e) \cap R \neq \mathbf{0}$. For $0 \neq r \in S(1-e) \cap R$, $ra = rea = 0$, showing that $e = 1$ and a is left invertible in S. If $a \in P$, some ideal P, $aa' = 1 \in PS = S$. $\square$

Any domain, whether Ore or not, which is not left primitive satisfies the conditions of Proposition 1.4 (ii). In [16] there is also a discussion of R-modules which are *maximally divisible*, i.e., $\mathbf{0} \neq M \in R$-Mod such that for each *maximal* ideal P, $PM = M$. The reasoning above will show that there can be a left noprimdiv ring which has a maximally divisible module.

Remark 1.5. *There is a left noprimdiv ring R and $\mathbf{0} \neq M \in R$-Mod such that for any maximal ideal P, $PM = M$.*

Proof. By [11, Theorem 11.27] (due to Formanek) the free ring over $\mathbf{Z}$ in more than one variable is left primitive. Take R to be a primitive domain which is not a division ring. Let $S = Q^{\mathrm{l}}_{\max}(R)$. Then, as in the proof of Proposition 1.4 (ii), $IS = S$ for any *non-zero* ideal I. $\square$

The example just recalled is one of a class of left noprimdiv rings which are not left max-rings. Since, by [17, Lemma 26.1 (6)] the centre of a prime left max-ring is a field, the next remark follows.

Remark 1.6. *If R is a left primitive ring whose centre is not a field then R is a left noprimdiv ring but not a left max-ring.*

Examples 1.7. *A free $\mathbf{Z}$-ring in more than one variable and, for any field K, a free K-algebra in more than one variable are left (and right) noprimdiv rings which are not left max-rings. If K is a division ring not algebraic over its centre C then $R = K[X]$ is left and right primitive but not a left or right max-ring.*

Proof. The free rings and algebras are left primitive but are not left max-rings because they each have a homomorphic image of the form $K[X]$, K a field, which is not a left max-ring.

The second statement follows from [11, Proposition 11.14] which says that R is left and right primitive. The centre of R is $C[X]$ and so R is not a left (or right) max-ring by Remark 1.6. □

As a special case of Class 3, above, if R is a commutative ring it is a noprimidiv ring if and only if $\mathbf{J}(R)$ is T-nilpotent and all prime ideals are maximal. The following remark shows how to construct a primitively divisible module when there is a prime ideal which is not maximal.

Remark 1.8. *Let R be a commutative ring which has a prime ideal P which is not maximal. Then the localization R_P, as an R-module, is primitively divisible.*

Proof. Let Q be any maximal ideal of R. Then, there is $r \in Q \setminus P$ whose image in R_P is invertible. It follows that $QR_P = R_P$. □

The following remark is easily shown.

Remark 1.9. *A ring R is a left noprimdiv ring if and only if no subdirectly irreducible left module is primitively divisible.*

The following criterion shows an analogy with one given by Faith for max-rings ([5, One-Module Theorem]); the proofs are similar.

Proposition 1.10. *A ring R is a left noprimdiv ring if and only if there is a cogenerator C in R-Mod such that no non-zero submodule of C is primitively divisible. This occurs when for each simple module S, no non-zero submodule of $E(S)$ is primitively divisible.*

Proof. One direction is clear. Suppose C has the stated properties. If $\mathbf{0} \neq M$ were primitively divisible then, for some index set K, there is an embedding $M \hookrightarrow C^K$ and, for some $k \in K$, the k^{th} projection $\pi_k \colon C^K \to C$ is non-zero on M. Then, $\pi_k(M)$ would be a non-zero primitively divisible submodule of C. This is not possible.

The second statement follows similarly since $C = \bigoplus_{S\,\text{simple}} E(S)$ is a cogenerator and the projection of the first part of the proof can be followed by a non-zero projection onto one of the components $E(S)$. □

An idempotent radical on R-Mod suited to left max-rings is developed in [5]. The appropriate one for a left noprimdiv ring R is found for $M \in R$-Mod by transfinitely iterating $\mathfrak{r}(M) = \bigcap_{P\in\mathfrak{Pl}} PM$. It is zero when R is a left noprimdiv ring. This subject will not be pursued here.

The next step is to be able to reduce the study of left noprimdiv rings to the semiprimitive case (Cf. [17, Lemma 26.2] for left max-rings).

Proposition 1.11. *For any ring R, R is a left noprimdiv ring if and only if $R/\mathbf{J}(R)$ is a left noprimdiv ring and $\mathbf{J}(R)$ is left T-nilpotent.*

Proof. If R is a left noprimdiv ring then for any $\bar{R} = R/\mathbf{J}(R)$-module $M \neq \mathbf{0}$, there is a left primitive ideal P of R with $PM \neq M$. Since $P \supseteq \mathbf{J}(R)$, $\bar{P}M \neq M$. By, e.g., [11, Theorem 23.16], if $\mathbf{J}(R)$ is not left T-nilpotent there is $\mathbf{0} \neq M \in R$-Mod with $\mathbf{J}(R)M = M$, which is not possible.

In the other direction, if $\bar{R}$ is a left noprimdiv ring and $\mathbf{J}(R)$ is left T-nilpotent then for any $\mathbf{0} \neq M \in R$-Mod, $\mathbf{J}(R)M \neq M$ and $M/\mathbf{J}(R)M$ is not primitively divisible. Hence, M is not primitively divisible. □

The proof of Proposition 1.11 also shows that if N is any left T-nilpotent ideal of a ring R then R is a noprimdiv ring if and only if R/N is.

Corollary 1.12. *A left self injective ring is a left noprimdiv ring if and only if $\mathbf{J}(R)$ is left T-nilpotent.*

Proof. A left self-injective ring is regular modulo its Jacobson radical. The result follows from Proposition 1.11 and the fact that regular rings are left (and right) noprimdiv rings. □

C. Examples and counterexamples. The following example due to Snider ([15]) will be very useful in what follows. It shows, among other things that, even when the left and right primitive ideals coincide with the maximal ideals, the noprimdiv condition (which, here, coincides with the max-ring condition) is not left-right symmetric. By [16, Lemma 10], if the left primitive images of a ring R are artinian then R is a left noprimdiv ring if and only if it is a left max-ring. (In fact, if the left primitive images of R are any left max-rings then the same equivalence holds.) Details of the example are quoted because they will be used later.

Example 1.13 ([15]). *There is a semiprimitive π-regular ring R whose left and right primitive ideals coincide, whose primitive images are artinian and which is a right max-ring but not a left noprimdiv (max-)ring.*

Proof. Fix a field K; all the matrices will be over K. For $m > 1$, let B be an $m \times m$-matrix; then for $k \geq 0$ set $B(k)$ to be the $(m+k) \times (m+k)$-matrix formed from B by adding k columns of zeros on the right and k rows of zeros on the bottom. Let R be the ring of sequences (A_n) of $n \times n$-matrices, $n \in \mathbf{N}$, such that for some $m > 1$ there is a strictly upper triangular $m \times m$-matrix B and $f \in K$ such that for all $n = m + k$, $k \geq 0$, $A_n = B(k) + fI_n$. The left and right primitive ideals are maximal and they are all the prime ideals. They are of two sorts: P_m, $m \in N$, where $P_m = \{(A_n) \in R \mid A_m = 0\}$, and $P = \{(A_n) \in R \mid \text{such that } f = 0\}$. For each $m \in \mathbf{N}$, $R/P_m \cong M_m(K)$ while $R/P \cong K$. Now let $I = \bigoplus_{n \in \mathbf{N}} P_n$ and set $S = R/I$; $\mathbf{J}(S)$ is right but not left T-nilpotent. Proposition 1.11 then says that S it is not a left noprimdiv ring; however, it will be seen below that if the left primitive ideals of a ring are maximal then a homomorphic image of a left noprimdiv ring is a left noprimdiv ring (Proposition 2.2 (i)).

That R is a right noprimdiv ring follows from Proposition 1.14, below. □

Snider's [15, Theorem 1], which says that a semiprimitive π-regular ring of bounded index is a left max-ring, can be generalized to some π-regular rings which are not of finite index; however, the condition of bounded index in that result cannot simply be dropped because of Example 1.13 ([15, Example 1]). The following proposition is suggested by the manner in which many examples are constructed.

Proposition 1.14. *Suppose that in a ring R there is a set of central idempotents, $\{e_\alpha \mid \alpha \in A\}$ such that (i) for each $\alpha \in A$, $e_\alpha R$, as a ring, is a left noprimdiv (left max-)ring, and (ii) setting $I = \sum_{\alpha \in A} e_\alpha R$, R/I is a left noprimdiv (left max-)ring. Then, R is a left noprimidiv (left max-)ring.*

Proof. The proof is given for left noprimdiv rings; that for left max-rings is similar. For each $\alpha \in A$ and $P \in \mathfrak{Pl}(R)$, $e_\alpha \in P$ or $1 - e_\alpha \in P$. Then, $\mathfrak{Pl}(e_\alpha R) = \{e_\alpha P \mid P \in \mathfrak{Pl}(R), e_\alpha \notin P\}$.

Let $\mathbf{0} \neq M \in R$-Mod. If, for some $\alpha \in A$, $e_\alpha M \neq \mathbf{0}$, then, for some $P \in \mathfrak{Pl}(R)$ with $e_\alpha \notin P$, $e_\alpha PM \neq e_\alpha M$. It follows that $PM \neq M$. If $e_\alpha M = \mathbf{0}$ for all $\alpha \in A$ then M is an R/I-module and there is some $P \in \mathfrak{Pl}(R)$, $P \supseteq I$ with $PM \neq M$. $\square$

The next step is to find a family of rings satisfying the conditions of Proposition 1.14. In a ring whose left primitive images are artinian, the left primitive ideals coincide with the maximal ideals . In what follows the left primitive ideals of R will be maximal and so the (left) Jacobson structure space of R will be $\operatorname{Max} R$. A closed set in $\operatorname{Max} R$ will, as usual, be denoted $V(I) = \{P \in \operatorname{Max} R \mid P \supseteq I\}$, for some ideal I.

Theorem 1.15. *Let R be a π-regular ring whose primitive images are artinian and such that for any ideal I, $\mathbf{J}(R/I)$ is left T-nilpotent. Write $\mathcal{U}_n = \{P \in \operatorname{Max} R \mid \operatorname{index} R/P = n\}$ and suppose that for all $n \in \mathbf{N}$, except for a finite set $F \subset \mathbf{N}$, $\mathcal{U}_n$ is open in* $\operatorname{Max} R$*. Then, R is a left max-ring.*

Proof. Since, by hypothesis, $\mathbf{J}(R)$ is left T-nilpotent, it can immediately be assumed that R is semiprimitive (Proposition 1.11). In what follows, arguments from [15] will come into play where finite index can be used.

Put $Y = \{n \in \mathbf{N} \mid n \notin F \text{ and } \mathcal{U}_n \neq \emptyset\}$; by Snider's result it may be assumed that Y is infinite. For each $n \in Y$ let $L_n = \bigcap_{P \in \operatorname{Max} R \setminus \mathcal{U}_n} P$. Since $\operatorname{Max} R \setminus \mathcal{U}_n$ is closed, there is an ideal $K_n \neq \mathbf{0}$ so that $\operatorname{Max} R \setminus \mathcal{U}_n = V(K_n)$. It follows that $\operatorname{Max} R \setminus \mathcal{U}_n = V(L_n)$ and $L_n \neq \mathbf{0}$. If L_n is considered as a ring (possibly without 1) then its primitive ideals are those of $\operatorname{Max} R$ not containing L_n, i.e., those $P \in \operatorname{Max} R$ with $\operatorname{index} R/P = n$, and, since R/P is simple, $L_n/(P \cap L_n) \cong R/P$ is of index n. Hence, L_n, as a ring, is a homogeneous π-regular ring of index n. If Z_n is the centre of L_n then, [10, Theorem 4.3], $L_n Z_n$ is a regular and biregular ring of index n.

Now suppose that $e \neq 0$ is a central idempotent of $L_n Z_n$. If $P \in \operatorname{Max} R$ then, as in [15], if $e \notin P$, e has image $1 \in R/P$. Thus, e has image 0 or 1 in each R/P showing $e \in \mathbf{B}(R)$.

Now fix $\mathbf{0} \neq M \in R$-Mod; it will be shown not to be primitively divisible. If $L_n Z_n M \neq \mathbf{0}$, exactly as in [15], there is $e \in \mathbf{B}(L_n Z_n) \subseteq \mathbf{B}(R)$ with $eM \neq \mathbf{0}$. Then, $M = eM \oplus (1-e)M$ has a maximal submodule since eM, a module over the V-ring eR, does. (More precisely, $eR = eL_n Z_n R = eL_n Z_n$ is a regular, biregular ring with identity and is homogeneous of index n.) Thus, it may be supposed that for all $n \in Y$, $L_n Z_n M = \mathbf{0}$.

Set $I = \sum_{n \in Y} L_n Z_n$. By assumption, $IM = \mathbf{0}$. Since each $L_n/L_n Z_n$, $n \notin F$, is nil ([10, Theorem 4.3]), if $P \in \operatorname{Max} R$ contains $L_n Z_n$ it also contains L_n. In other words, the maximal ideals containing I are those containing each L_n, $n \in Y$. Thus, if $P \in \operatorname{Max} R$ is such that $P \supseteq I$, $P \notin \mathcal{U}_n$ for all $n \in Y$. Because F is finite, the ring R/I is π-regular whose left primitive images are of bounded index. It need not be semiprimitive (Example 1.13). However, by hypothesis $\mathbf{J}(R/I)$ is left T-nilpotent and, thus, $\mathbf{J}(R/L)M \neq M$ (e.g., [11, Theorem 23.16]). Put $\bar{R} = (R/I)/\mathbf{J}(R/I)$. By [15, Theorem 1], the $\bar{R}$-module $M/\mathbf{J}(R/I)M$ has a maximal submodule since $\bar{R}$ is is a semiprimitive π-regular of finite index. Hence, M has a maximal submodule and R is a left max-ring. □

Theorem 1.15 is illustrated by Example 1.13 ([15, Example 1]) where the right hand version of the hypotheses are satisfied but not the left. The space $\operatorname{Max} R$ can be identified with the one-point compactification $\mathbf{N} \cup \{\infty\}$ of $\mathbf{N}$ and, for each $n \geq 2, \mathcal{U}_n = \{n\}$ is both open and closed. The condition on the radicals of images in Theorem 1.15 is necessary by Corollary 2.3 although in the proof it is only used for one ideal. The following contains a more elaborate illustration of Theorem 1.15 using Example 1.13 as a model; in this ring, the subrings $L_n Z_n$ are no longer rings with 1.

Examples 1.16. *(i) Let K be a field and, for each $m \in \mathbf{N}$, S_m is a copy of the ring of Example 1.13. Put R to be $\bigoplus_{m \in \mathbf{N}} S_m$ with a copy of K adjoined to make a ring with 1. Then, R satisfies the conditions of Theorem 1.15 on the right.*

(ii) Let K be a field and for each $n > 1$, let S_n be the ring of sequences from $M_{2^n}(K)$ which are eventually constant of the form $\left(\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & A \end{smallmatrix}\right)$, where $A \in M_{2^{n-1}}(K)$. Let R be the direct sum of S_n, $n > 1$, with a copy of K adjoined. Then, R satisfies the conditions of Proposition 1.14 but not those of Theorem 1.15.

Proof. (i) $\operatorname{Max} R$ can be identified with the topological space which is the one-point compactification of a disjoint union of the spaces $\operatorname{Max} S_m, m \in \mathbf{N}$. For each $n \geq 2$, $\mathcal{U}_n$ is $\mathbf{N}$ as a discrete space and is open in $\operatorname{Max} R$. In the language of the proof of Theorem 1.15, $L_n Z_n = L_n$ is regular and biregular; the ideal I is $\sum_{n \geq 2} L_n$. If, in S_m, the ideal which is the sum of the matrix rings is I_m then R/I is $\bigoplus_{m \in \mathbf{N}} S_m/I_m$ with a copy of K adjoined to make a ring with 1. The radical of R/I is right T-nilpotent and $(R/I)/\mathbf{J}(R/I)$ is a commutative regular ring.

(ii) In this case, $\mathcal{U}_{2^n}$ consists of a copy of the discrete space $\mathbf{N}$ with one point added from the constant part of $S_{2^{n+1}}$. No neighbourhood of this point is contained in $\mathcal{U}_{2^n}$. However, each copy of $M_{2^n}(K)$ can be identified with a central idempotent. The sum of these, over $n > 1$, gives an ideal

I and R/I can be identified with the ring of sequences with components $\left(\begin{smallmatrix} A_n & \mathbf{0} \\ \mathbf{0} & A_n \end{smallmatrix}\right) \in M_{2^n}(K)$, $n > 1$, which are eventually constant and scalar. This ring satisfies the conditions of Proposition 1.14 and, hence, so does R. $\square$

2. The class of left noprimdiv rings. In this section the class of left noprimdiv rings is examined. The first thing to note is that it is not closed under homomorphic images even though the classes 2 through 5 listed in Section 1 all are closed under homomorphic images. Moreover, the class of rings which are left primitively pure is closed under homomorphic images ([16, Lemma 4(4)]).

Proposition 2.1. *The class of noprimdiv rings is not closed under homomorphic images; it in not closed under subrings.*

Proof. As already mentioned, a free ring over $\mathbf{Z}$ in more than one variable is left primitive. It follows that every ring is the homomorphic image of a left primitive, and, hence, left noprimdiv ring. For the second statement it suffices to look at $\mathbf{Z}$ and $\mathbf{Q}$. $\square$

Sometimes homomorphic images behave well.

Proposition 2.2. *Let R be a left noprimdiv ring.*

(i) If the left primitive ideals of R are maximal then any homomorphic image of R is a left noprimdiv ring.

(ii) If I is an ideal of R generated by central idempotents then R/I is a left noprimdiv ring.

Proof. (i) If I is an ideal of R and $\mathbf{0} \neq M \in R/I$-Mod then there is a left primitive ideal P of R with $PM \neq M$. If $P \supseteq I$ then P/I is left primitive in R/I giving what is required. Otherwise, $P + I = R$ and, since $IM = \mathbf{0}$, $(P + I)M = M = PM$, which is impossible.

(ii) This is done similarly. If $\mathbf{0} \neq M \in R/I$-Mod there is a left primitive ideal P of R with $PM \neq M$. Suppose, as above, that $P \not\supseteq I$. There is a central idempotent $e \in I$, $e \notin P$. Then, $1 - e \in P$ giving $P + I = R$. $\square$

An idempotent $e \in R$ is called *semi-central* is $eR(1-e) = \mathbf{0}$ or $(1-e)Re = \mathbf{0}$. In the proof of Proposition 2.2(ii) it suffices to assume that I is generated by semi-central idempotents. The left primitive skew polynomial rings of [11, Proposition 11.12] show that the converse to Proposition 2.2 (i) fails.

If the left primitive ideals of R are maximal then Proposition 2.2 (i) yields a necessary condition for R to be left noprimdiv.

Corollary 2.3. *Let R be a ring whose left primitive ideals are maximal. If R is a left noprimdiv ring then, for any ideal I, $\mathbf{J}(R/I)$ is left T-nilpotent.*

Proposition 2.2 (ii) has a further application. Recall the nature of the *Pierce sheaf* of a ring R. Every ring R is the ring of sections of a sheaf whose base space is $\operatorname{Spec}\mathbf{B}(R)$ and whose stalks are of the form R/Rx, where $x \in \operatorname{Spec}\mathbf{B}(R)$. For details, see [14] and [9, V §2].

Theorem 2.4. *A ring R is a left noprimdiv ring if and only each of the stalks of its Pierce sheaf is a left noprimdiv ring.*

Proof. First assume that R is a left noprimdiv ring. Then, since the Pierce stalks have the form $R_x = R/Rx$, for a maximal ideal x of the boolean algebra of central idempotents $\mathbf{B}(R)$, R_x is a left noprimdiv ring (Proposition 2.2.).

In the other direction assume that each Pierce stalk R_x is a left noprimdiv ring. For any $\mathbf{0} \neq M \in R$-Mod, there is some $x \in \operatorname{Spec}\mathbf{B}(R)$, $M_x = M/xM \neq \mathbf{0}$ ([14, Proposition 1.7], M is a subdirect product of the M_x). Then, there is a left primitive ideal P of R such that $PM_x = P_xM_x \neq M_x$. However, $(PM)_x = P_xM_x \neq M_x$, showing that $PM \neq M$. □

This result allows one to construct many examples. It was already mentioned that biregular rings are left and right noprimdiv rings. They are the rings whose Pierce stalks are simple rings.

The Pierce sheaf construction can be iterated, perhaps transfinitely, to get a presentation of a ring R as a subdirect product of indecomposable rings (with only 0 and 1 as central idempotents), called the *maximal indecomposable factors* (see [3]). Details will not be given here but as in [17, Lemma 26.2], for left max-rings, the methods of Theorem 2.4 and Remark 1.9 will show that *a ring R is a left noprimdiv ring if and only if each of its maximal indecomposable factors is a left noprimdiv ring.*

It is now shown that "left noprimdiv" is a Morita invariant.

Theorem 2.5. *Let R and S be Morita equivalent rings. Then, R is a left noprimdiv ring if and only if S is.*

Proof. Let $R \sim_M S$ be Morita equivalent rings. By [1, Theorem 22.2], there is a balanced module ${}_SP_R$ with ${}_SP$ and P_R progenerators and with

$$\operatorname{Hom}_R(P,-)\colon R\text{-Mod} \to S\text{-Mod} \text{ and } P \otimes_S -\colon S\text{-Mod} \to R\text{-Mod}$$

inverse equivalences. Moreover, $Q = \operatorname{Hom}_R(P, R)$ is a balanced bi-module ${}_RQ_S$ with ${}_RQ$ and Q_S progenerators. By [1, Proposition 21.11] there is a lattice isomorphism Φ from the lattice of two-sided ideals of R to that of S so that $R/I \sim_M S/\Phi(I)$. In fact, $\Phi(I) = \mathrm{Hom}_R(P, IP)$. Since primitiveness of a ring is a Morita invariant property, I is a left primitive ideal if and only if $\Phi(I)$ is a left primitive ideal. Suppose R is a left noprimdiv ring and $\mathbf{0} \neq M \in S$-Mod. Then, there is a left primitive ideal I in R such that $I(P \otimes_S M) \neq P \otimes_S M$. (Notice that $IP = P\Phi(I)$ [12, proof of 18.44].) Then, $I(P \otimes_S M) = IP \otimes_S M = P\Phi(I) \otimes_S M = P \otimes_S \Phi(I)M$. Therefore $\Phi(I)M \neq M$. □

Proposition 2.6. *(i) Let R be a left noprimdiv ring which is not left primitive and $S = R[T]$ an extension of R such that $R \cap \langle T \rangle = \mathbf{0}$. Then, S is a left noprimdiv ring. (ii) A finite direct product of left noprimdiv rings is a left noprimdiv ring. (iii) A trivial extension of a left noprimdiv is again one. (iv) The ring of upper triangular $n \times n$ matrices over a left noprimdiv ring is again one.*

Proof. (i) If $S = R[T]$ is as described, then for any ideal I of R, $I + \langle T \rangle$ is an ideal of S and $S/(I + \langle T \rangle) \cong R/I$. Suppose $\mathbf{0} \neq M \in S$-Mod is primitively divisible. For any left primitive ideal P of R, $P + \langle T \rangle$ is a left primitive ideal of S and $M = (P + \langle T \rangle)M \subseteq PM$. Hence, M would be a primitively divisible R-module, which is not possible. (ii) is clear (but see Proposition 2.7 (ii)); (iii) and (iv) follow from the remark after Proposition 1.11 and (ii). □

Polynomial rings and free R-rings are among the rings covered by Proposition 2.6 (i). The proviso that R not be left primitive in part (i) is essential. If K is a field, K is a noprimdiv ring but the polynomial ring $K[X]$ is not. There is no similar result for group rings. Let G be an infinite cyclic group and $R = (\mathbf{Z}/p^n\mathbf{Z})[G]$, where p is a prime and $n > 1$. Then, $R/pR \cong \mathbb{F}_p[G]$, a domain. By the remark after Proposition 1.11, since R/pR is not a noprimdiv ring, neither is R; however, $\mathbf{Z}/p^n\mathbf{Z}$ is a noprimdiv ring.

The class of left noprimdiv rings is not closed under direct limits or products, as will now be shown.

Proposition 2.7. *(i) There is a directed family of left noprimdiv rings whose direct limit is not a left noprimdiv ring. (ii) There is a direct product of left noprimdiv rings which is not a left noprimdiv ring. (iii) There is an inverse limit of left noprimdiv rings which is not a left noprimdiv ring.*

Proof. (i) Example 1.13 will provide an example. By Theorem 2.4, the stalks of the Pierce sheaf of a left noprimdiv ring are left noprimdiv rings.

The Pierce stalks of the ring R of Example 1.13 are as follows: for each $n \in \mathbf{N}$, there is a stalk $M_n(K)$. The remaining stalk, R_∞, is the direct limit over $\mathbf{N}$ of rings T_n which are $n\times n$-matrices which are upper triangular with constant diagonal; the embeddings are $B+fI_n \mapsto B(1)+fI_{n+1}$, where B is sent to an $(n+1)\times(n+1)$ matrix as in Example 1.13. Each T_n has nilpotent radical and $T_n/\mathbf{J}(T_n) \cong K$; hence, each T_n is a left (and right) noprimdiv ring. However, we know that the limit is not a left noprimdiv ring.

(ii) The rings $T_n, n \in \mathbf{N}$, from the proof of part (i) can be used here. Put $\Pi = \prod_{n\in\mathbf{N}} T_n$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbf{N}$ and $I = \{(r_n) \in \Pi \mid \text{for some } F \in \mathcal{U}, r_m = 0, \forall m \in F\}$. Consider $\Pi/I = S$. Define $L = \{(r_n) \in I \mid r_m \text{ is strictly upper triangular } \forall m \in F, \text{some } F \in \mathcal{U}\}$. It follows that $L \supseteq I$ is an ideal of Π and $\bar{L} = L/I$ is an ideal of S. Moreover, $S/\bar{L}$ is a field which is a non-standard model of K and for $a \in \bar{L}$, $1+a$ is invertible. Thus, $\bar{L} = \mathbf{J}(S)$ and, just as in Example 1.13, $\bar{L}$ is not left T-nilpotent. Hence, S, a homomorphic image of Π, is not a left noprimdiv ring. However, the left primitive ideals of Π are maximal and, hence, by Proposition 2.2(1), Π is not a noprimdiv ring. (It can also be observed that S is a Pierce stalk of Π and Theorem 2.4 can be used.)

(iii) For a prime number p, the ring of p-adic integers $\mathbf{Z}_p$ is not a noprimdiv ring but it is an inverse limit of $\mathbf{Z}/\langle p^n\rangle$, $n \in \mathbf{N}$, each of which is a noprimdiv ring. □

3. Ring whose injective left modules are primitively divisible. It was shown in [16, Proposition 4] that ${}_RR$ can be embedded in a primitively divisible module if and only if every injective left module is primitively divisible. This property resembles classical divisibility and coincides with it for commutative PIDs. In this section this phenomenon is examined.

Definition 3.1. *A ring R such that every injective left module is primitively divisible is called a* left IPD ring *(for* injective left modules primitively divisible*). A commutative left IPD ring is called an* IPD ring.

The classes of left noprimdiv and left IPD rings are not exhaustive; $\mathbf{Z}\times\mathbf{Q}$ is neither in one nor the other.

Tuganbaev [16] supplies the basic tools.

Theorem 3.2 ([16] Propositions 4 and 5)**.** *The following statements are equivalent: (1) R is a left IPD ring, (2) ${}_RR$ can be embedded in a primitively divisible module, (3) $E({}_RR)$ is primitively divisible, and (4) given $P \in \mathfrak{Pl}$ there are $a_1, \ldots, a_k \in P$ and $x_1, \ldots, x_k \in E({}_RR)$ such that $1 = \sum_{i=1}^k a_i x_i$.*

With this in hand it is possible to give a characterization of left IPD rings.

Theorem 3.3. *A ring R is a left IPD ring if and only if each $P \in \mathfrak{Pl}(R)$ contains a finite subset with zero left annihilator.*

Proof. Theorem 3.2 (4) says, in particular, that the condition is necessary.

Suppose now that each $P \in \mathfrak{Pl}(R)$ contains a finite subset with zero left annihilator. Let $\{a_1, \ldots, a_k\}$ be a subset of R with zero left annihilator. Define $S = R\langle X_1, X_2, \ldots\rangle$, the free R-ring in countably many variables. A free R-basis for S is the set of monomials in the variables, including the empty monomial, 1. Define $L = E_S({}_S S)$, the S injective hull of ${}_S S$. Consider now $f = a_1X_1 + \cdots a_kX_k \in S$. Because of the statement about the R-basis, it follows that $\operatorname{lann}_S f = \mathbf{0}$.

It follows that for each $u \in L$, the S-homomorphism $\phi_u \colon Sf \to S$ defined by $\phi_u(f) = u$ is well-defined. By injectivity, ϕ_u lifts to some $\psi_u \colon S \to L$. Let $\psi_u(1) = v$; then $\phi_u(f) = \psi_u(f) = f\psi_u(1) = fv = u$. Hence, $u = \sum_{i=1}^k a_i\psi_u(X_i)$. Now suppose that $\{a_1, \ldots, a_k\} \subseteq P \in \mathfrak{Pl}(R)$. Then, each $u \in L$ can be expressed as an element of PL. Hence, $PL = L$ for all $P \in \mathfrak{Pl}(R)$. When viewed as a left R-module, L is a primitively divisible R-module containing ${}_R R$. By Theorem 3.2 (2), R is a left IPD ring. □

Corollary 3.4. *A commutative ring is an IPD ring if and only if each maximal ideal contains a finite subset with zero annihilator.*

Similar reasoning yields the following.

Proposition 3.5. *A ring R is a left IPD ring if and only if for each $P \in \mathfrak{Pl}$ there are $a_1, \ldots, a_k \in P$ such that for any injective $E \in R$-Mod and $y \in E$ the equation $\sum_{i=1}^k a_i z_i = y$ has a solution $y_i \ldots, y_k \in E$.*

Proof. Assume that R is a left IPD ring. For $P \in \mathfrak{Pl}$ find $a_1, \ldots, a_k \in P$ and $x_1, \ldots, x_k \in E({}_R R)$ such that $1 = \sum_{i=1}^k a_i x_i$. For $y \in E$, define $\phi \colon {}_R R \to E$ by $\phi(1) = y$. This lifts to $\psi \colon E({}_R R) \to E$ and $y = \psi(1) = \psi(\sum_{i=1}^k a_i x_i) = \sum_{i=1}^k a_i \psi(x_i)$. The converse is clear. □

All domains which are not left primitive are examples of left IPD rings.

The rest of this section concerns commutative rings. A commutative ring which is a noprimdiv ring is 0-dimensional. Something quite different occurs in IPD rings; no maximal ideal can be a minimal prime ideal.

Theorem 3.6. *Let R be a commutative IPD ring. Then,* $\operatorname{Min} R \cap \operatorname{Max} R = \emptyset$.

Proof. First suppose that R is local with maximal ideal P where $P \in \operatorname{Min} R$. Since, here, P is a nil ideal, any finite subset of P has non-zero annihilator, which is not possible in an IPD ring.

In any ring R as in the statement, let $P \in \operatorname{Max} R$; R_P, as an R-module, embeds in a primitively divisible module, say M. In particular, $PM = M$. Since $R_P \cong R_P \otimes_R R_P$, R_P embeds, using the isomorphism and flatness, into $R_P \otimes_R M = R_P \otimes_R PM = R_P P \otimes_R M = P_P R_P \otimes_R M = P_P(R_P \otimes_R M)$. Hence, $R_P \otimes_R M$ is a primitively divisible R_P-module.

If $P \in \operatorname{Min} R \cap \operatorname{Max} R$ then R_P would be an IPD ring and this would be a contradiction. □

It will be seen shortly (Example 3.10) that the condition of Theorem 3.6 is not sufficient even for reduced rings. However, the next proposition gives an instance where the converse holds. It is illustrated by Example 3.8.

Proposition 3.7. *Let R be a reduced commutative ring. If* $\operatorname{Min} R$ *is compact and* $\operatorname{Min} R \cap \operatorname{Max} R = \emptyset$ *then R is a IPD ring.*

Proof. Put $Q = Q_{\max}(R)$, a regular ring since R is reduced. By [8, Theorem 4.3], $\operatorname{Min} R = \{N \cap R \mid N \in \operatorname{Spec} Q\}$. If, for $P \in \operatorname{Max} R$, $PQ \neq Q$, then there is $N \in \operatorname{Spec} Q$ with $PQ \subseteq N$. Then, $N \cap R \supseteq P$ is in $\operatorname{Min} R$, which is not possible. □

Recall that a commutative p.p. ring is one where the annihilator of an element is generated by an idempotent ([2]).

Example 3.8. *If R is a commutative p.p. ring with* $\operatorname{Min} R \cap \operatorname{Max} R = \emptyset$ *then R is an IPD ring.*

Proof. It only needs to be shown that $\operatorname{Min} R$ is compact. For any $a \in R$, $\operatorname{ann} a = Re$, some $e^2 = e \in R$. Then, $\operatorname{ann}(\langle a \rangle + Re) = \mathbf{0}$. It follows from [8, Theorem 4.3] that $\operatorname{Min} R$ is compact. (It is also easy to show directly, using [2, Lemma 3.1], that $Q_{\mathrm{cl}}(R)$ is primitively divisible; here the Pierce stalks are domains which are not fields.) □

Rings of continuous real valued functions form a rich class of reduced commutative rings. All terminology is from [7] and [6]. Recall that if X is a compact Hausdorff space the maximal ideals of $C(X)$ have the form: for $x \in X$, $M_x = \{f \in C(X) \mid f(x) = 0\}$. A point $x \in X$ is called a *P-point* if

M_x is a minimal prime ideal, i.e., if $f \in C(X)$ is such that $f(x) = 0$ then f is zero on a neighbourhood of x. If every point is a P-point, X is called a *P-space*; these are exactly the spaces for which $C(X)$ is a regular ring ([7, 4J]). A point x is an *almost P-point* ([13]) if $f(x) = 0$ implies the zero-set of f has non-empty interior.

For these rings $Q_{\rm cl}(C(X))$ is written $Q_{\rm cl}(X)$ and $Q_{\max}(C(X))$ is $Q(X)$.

Proposition 3.9. *Let X be a compact Hausdorff space and $C(X)$ the corresponding ring of continuous real valued functions. (1) The ring $C(X)$ is a noprimdiv ring if and only if X is a P-space (i.e., $C(X)$ is regular). (2) If X has a P-point then $C(X)$ is not an IPD ring. (3) $C(X)$ is an IPD ring if and only if for each $x \in X$, there is a dense cozero set U with $x \notin U$. In this case, $Q_{cl}(X)$ is primitively divisible. (4) For all $M \in \operatorname{Max} C(X)$, $MQ(X) \neq Q(X)$ if and only if X is an almost P-space.*

Proof. (1) Since a reduced commutative noprimdiv ring is one with all prime ideals maximal, the only rings of the form $C(X)$ of this type are those where X is a P-space.

(2) Since a maximal ideal in an IPD ring cannot be a minimal prime ideal, X cannot have a P-point.

(3) If, for some $x \in X$, $M_x(X) = Q(X)$ then there are $f_1, \dots, f_k \in M_x$ such that $\{f_1, \dots, f_k\}$ has zero annihilator. It follows that $f = f_1^2 + \cdots + f_k^2 \in M_x$ is a non zero-divisor in $C(X)$. Hence, $\operatorname{coz} f$ is dense. Conversely, if, for each $x \in X$ there is $f \in M_x$ with $\operatorname{coz} f$ dense in X, then f is invertible in $Q_{\rm cl}(X)$ ([6, Theorem, p. 15]).

(4) Follows from the proof of (3) since an almost P-space ([13, Proposition 1.1]) is one where every non-empty zero-set has non-empty interior, which is saying that $f \in C(X)$ is either a zero-divisor or invertible. $\square$

Any compact metric space without isolated points, such as the unit interval, will satisfy the conditions of Proposition 3.9 (3).

Recall that in $C(X)$, X compact, a maximal ideal M_x is not minimal if and only if there is $f \in M_x$ such that the zero-set of f does not contain a neighbourhood of x ([7, §14.12]).

Example 3.10. *There is a reduced commutative ring R such that* $\operatorname{Min} R \cap \operatorname{Max} R = \emptyset$ *but R is not an IPD ring.*

Proof. A space X is constructed so that $C(X)$ is the required example. In order to have $\operatorname{Min} C(X) \cap \operatorname{Max} C(X) = \emptyset$, it suffices that X have no P-points; to prevent $C(X)$ from being an IPD ring X must have an almost P-point. Take Y be a topological sum (disjoint union) of uncountably many copies of the unit interval and $X = Y \cup \{\infty\}$, the one-point compactification. Every $y \in Y$ has the property that $M_y Q(X) = Q(X)$, however, when $0 \neq f \in C(X)$ has value $f(\infty) = 0$ then its zero-set need not be a neighbourhood of ∞ but is always contains a non-empty open set. Hence, ∞ is almost P-point which is not a P-point. Thus, $M_\infty Q(X) \neq Q(X)$. $\square$

The next example illustrates Theorem 3.4. The "$A + B$" construction of [8, §26] is used. The construction begins with a commutative reduced ring D and a subset $\mathcal{P}$ of $\operatorname{Spec} D$ indexed by $\mathcal{A}$. Then, $I = \mathcal{A} \times \mathbf{N}$ and, for each $i = (\alpha, n) \in I$, $P_i = P_\alpha$. Consider the ideal $B = \bigoplus_{i \in I} D_i$, where $D_i = D/P_i$, in $\Pi = \prod_{i \in I} D_i$; $\phi\colon D \to \Pi$ is defined by $\phi(d) = (d + P_i)$ and A is the image of ϕ. Then, the ring $R = A + B \subseteq \Pi$. Note that if $\bigcap_{P \in \mathcal{P}} P = \mathbf{0}$ then $R/B \cong D$.

Example 3.11. *There is a commutative ring R satisfying the requirements of Theorem 3.4 which has a maximal ideal consisting of zero-divisors.*

Proof. The "A+B" construction is used. Let K be an algebraically closed field and $D = K[X, Y]$. Let $\mathcal{P}$ be the set of all one-generator, nonzero prime ideals of A. The resulting ring R has two sorts of maximal ideals: (1) those of the form $M(i, Q_i) = \{\phi(a) + b \mid (\phi(a) + b)_i \in Q_i\}$, where $Q_i \in \operatorname{Max} D_i$; (2) an ideal of the form $\phi(Q) + B$, where $Q \in \operatorname{Max} D$. Each maximal ideal of the first sort contains a non zero-divisor. If $\phi(Q) + B$ is of the second kind then Q can be written $Q = \langle f, g \rangle$, f, g irreducible. Any single element of $h \in Q$ has an irreducible factor and so each $\phi(h) + b$, $b \in B$, is a zero-divisor. However, $\{(\phi(f), \phi(g)\}$ has zero annihilator. $\square$

Much of this section has been about reduced commutative rings. The next and final example shows that the IPD property does not lift modulo the prime radical, even when it it is nilpotent.

Example 3.12. *Let M be the $\mathbf{Z}$-module $\bigoplus_{p\ prime} \mathbf{Z}/\langle p \rangle$ and R the trivial extension of $\mathbf{Z}$ by M. Then, R is not an IPD ring while $R/\mathbf{P}(R) \cong \mathbf{Z}$ is.*

Proof. The maximal ideals of R are, for a prime p, $Q_p = \{(pz, x) \mid z \in \mathbf{Z}, x \in M\}$. If $Q_p E({}_R R) = E({}_R R)$, then (Proposition 3.5), Q_p has a finite subset with trivial annihilator. However, any finite subset of Q_p will be annihilated by $(0, m)$, where the only non-zero component of m is in $\mathbf{Z}/\langle p \rangle$. $\square$

References

[1] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules, Second Edition.* Springer-Verlag, 1992.

[2] G.M. Bergman, *Hereditary commutative rings and centres of hereditary rings*, Proc. London Math. Soc. **23** (1971), 214–236.

[3] W.D. Burgess and W. Stephenson, *An analogue of the Pierce sheaf for non-commutative rings*, Comm. Algebra **6** (1978). 863–886.

[4] V.P. Camillo and K.R. Fuller, *A note on Loewy rings and chain conditions on primitive ideals*, Lecture Notes in Mathematics **700** (1979), 75–86.

[5] C. Faith, *Rings whose modules have maximal submodules*, Pub. Mat. **39** (1995), 201–214.

[6] N.J. Fine, L. Gillman and J. Lambek, *Rings of Quotients of Rings of Functions.* Network RAAG, 2005.

[7] L. Gillman and M. Jerison, *Rings of Continuous Functions*. Graduate Texts in Mathematics **43**, Springer-Verlag, 1976.

[8] J.A. Huckaba, *Commutative Rings with Zero Divisors.* Marcel Dekker, Pure and Applied Math. **117**, 1988.

[9] P.T. Johnstone, *Stone Spaces*. Cambridge Studies in Advanced Mathematics, **3**, Cambridge University Press, 1982.

[10] I. Kaplansky, *Topological representation of algebras. II*, Trans. Amer. Math. Soc. **68** (1950), 62–75.

[11] T.Y. Lam, *A First Course in Noncommutative Ring Theory, Second Edition*. Graduate Texts in Mathematics, Springer-Verlag, 2001.

[12] T.Y. Lam, *Lectures in Modules and Rings*, Graduate Texts in Mathematics, Springer-Verlag, 1999.

[13] R. Levy, *Almost-P-spaces*, Canad. J. Math. **29** (1977), 284–288.

[14] R.S. Pierce, *Modules over Commutative Regular Rings*. Memoirs of the A.M.S. **70** (1967).

[15] R.L. Snider, *Semiprimitive π-regular rings of bounded index are left max rings*, Comm. Algebra **29** (2001), 5433–5437.

[16] A.A. Tuganbaev, *Primitively pure submodules and primitively divisible modules*, J. Math. Sci. **110** (2002), 2746–2754.

[17] A.A. Tuganbaev, *Rings Close to Regular*. Mathematics and its Applications, Kluwer, 2002.