# Elements of minimal prime ideals in general rings

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*Dedicated to S.K. Jain on his seventieth birthday*

**Abstract.** Let $R$ be any ring; $a \in R$ is called a *weak zero-divisor* if there are $r, s \in R$ with $ras = 0$ and $rs \neq 0$. It is shown that, in any ring $R$, the elements of a minimal prime ideal are weak zero-divisors. Examples show that a minimal prime ideal may have elements which are neither left nor right zero-divisors. However, every $R$ has a minimal prime ideal consisting of left zero-divisors and one of right zero-divisors. The union of the minimal prime ideals is studied in 2-primal rings and the union of the minimal strongly prime ideals (in the sense of Rowen) in NI-rings.

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**Introduction.** E. Armendariz asked, during a conference lecture, if, in any ring, the elements of a minimal prime ideal were zero-divisors of some sort. In what follows this question will be answered in the positive with an appropriate interpretation of "zero-divisor".

Two very basic statements about minimal prime ideals hold in a commutative ring $R$: (I) If $P$ is a minimal prime ideal then the elements of $P$ are zero-divisors, and (II) the union of the minimal prime ideals is $M = \{a \in R \mid \exists\, r \in R \text{ with } ar \in \mathbf{N}_*(R) \text{ but } r \notin \mathbf{N}_*(R)\}$, where $\mathbf{N}_*(R)$ is the prime radical. We will see that (I), suitably interpreted, is true for all rings. The statement (II) is false in general non-commutative rings but a version of it does hold in rings where the set of nilpotent elements forms an ideal.

In a commutative ring $R$ we always have that $R/\mathbf{N}_*(R)$ is reduced (i.e., has no non-zero nilpotent elements); this fails in the non-commutative case. Hence we can expect "commutative-like" behaviour when, for a non-commutative ring $R$, $R/\mathbf{N}_*(R)$ is reduced; these rings are called *2-primal* and have been extensively studied. Statement (II), above, holds for these rings. A larger class of rings is where the set of nilpotent elements, $\mathbf{N}(R)$, forms an ideal (called *NI-rings*). Once again statements (I) (Corollary 2.9) and (II) (Corollary 2.11) hold when "minimal prime ideals" are replaced by "minimal **r**-strongly prime ideals" whose definition is recalled below. (The two types of prime ideal coincide in commutative rings.)

Various weakened forms of commutativity yield results which show that minimal prime ideals consist of (left or right) zero-divisors. A thorough study of this is in [2, e.g., Corollary 2.7]. Our purpose here is to look at minimal prime ideals in general where elements need not be zero-divisors but always are what we call *weak zero-divisors* (Theorem 2.2); an element $a$ in a ring $R$ is a *weak zero-divisor* if there are $r, s \in R$ with $ras = 0$ and $rs \neq 0$. It will also be seen that, in special cases, other sorts of prime ideals consist of weak zero-divisors. Examples will show that "weak zero-divisor" cannot be replaced by "left (or right) zero-divisor" (Examples 3.2, 3.3 and the semiprime Example 3.4), however, in any ring $R$ there is a minimal prime ideal consisting of left zero-divisors and one consisting of right zero-divisors (Proposition 2.7).

**Terminology:** For a ring $R$ (always unital) the prime radical is denoted $\mathbf{N}_*(R)$, the upper nil radical $\mathbf{N}^*(R)$ and the set of nilpotent elements $\mathbf{N}(R)$. As usual, $R$ is called *semiprime* if $\mathbf{N}_*(R) = \mathbf{0}$, while $R$ is called an *NI-ring* if $\mathbf{N}^*(R) = \mathbf{N}(R)$. Recall that an ideal $P$ in a ring $R$ is called *completely prime* if $R/P$ is a domain.

There are several uses of the term "strongly prime". In the sequel we will use the definition chosen by Rowen (see [13] and [6]). In order to avoid confusion we will say that a prime ideal $P$ in a ring $R$ is an **r**-*strongly prime ideal* if $R/P$ has no non-zero nil ideals. (Since every maximal ideal of $R$ is an **r**-strongly prime ideal, there are **r**-strongly prime ideals which are not completely prime.) A ring in which every minimal prime ideal is completely prime is called *2-primal*. The 2-primal rings are special cases of NI-rings.

The (two-sided) ideal of a ring $R$ generated by a subset $X$ is written $\langle X\rangle$ or by an element $a \in R$ written $\langle a\rangle$.

Section 1 is devoted to a brief look at **r**-strongly prime ideals. Section 2 contains the main results and Section 3 is devoted to examples, counterexamples and special cases.

## 1. On r-strongly prime ideals.

The main topic will be deferred to the next section. Since **r**-strongly prime ideals will show up in several places we first briefly study these ideals. We get a description of **r**-strongly prime ideals in terms of special sorts of $m$-systems. Recall that an $m$-system $S$ in a ring $R$ is a subset of $R \setminus \{0\}$ such that $1 \in S$ and for $r, s \in S$ there is $t \in R$ such that $rts \in S$. The complement of a prime ideal is an $m$-system and an ideal maximal with respect to not meeting an $m$-system $S$ is a prime ideal (e.g., [11, §10]). A subset $S$ of $R \setminus \{0\}$ containing 1 and which is closed under multiplication is an example of an $m$-system.

A ring $R$, viewed as an algebra over $\mathbb{Z}$, has an *enveloping algebra* $R^e = R \otimes_{\mathbb{Z}} R^{op}$. The bimodule ${}_R R_R$ can be thought of as a left $R^e$-module. The ring $\mathbf{M}(R) = R^e / \operatorname{ann}_{R^e} R$ is called the *multiplication ring* of $R$. Then, $R$ is a faithful $\mathbf{M}(R)$-module. For $\lambda \in \mathbf{M}(R)$ we can lift $\lambda$ to some $\sum_{i=1}^n r_i \otimes s_i \in R^e$ and, for $a \in R$, think of $\lambda a$ as $\sum_{i=1}^n r_i a s_i$. We now formalize the definitions (cf. [13, Definition 2.6.5]). (In [9], the multiplication algebra was used in the definition of a different sort of "strongly prime" ideal.)

**Definition 1.1.** *Let $R$ be a ring. (1) A prime ideal $P$ of $R$ is called an* $\mathbf{r}$-strongly prime ideal *if $R/P$ has no non-zero nil ideals. (2) A subset $S$ of $R \setminus \{0\}$ is called an* $nm$-system *if (i) $S$ is an $m$-system and (ii) for $t \in S$ there is $\lambda \in \mathbf{M}(R)$, depending on $t$, such that $(\lambda t)^n \in S$ for all $n \geq 1$.*

It is readily seen that any $\mathbf{r}$-strongly prime ideal contains an $\mathbf{r}$-strongly prime ideal which is minimal among $\mathbf{r}$-strongly prime ideals. The intersection of the (minimal) $\mathbf{r}$-strongly prime ideals of a ring $R$ is $\mathbf{N}^*(R)$ (see [13, Proposition 2.6.7]). The connection between $\mathbf{r}$-strongly prime ideals and $nm$-systems is clear. The basic information is contained in the following.

**Proposition 1.2.** *Let $R$ be a ring. Then*

*(i) If $S \subseteq R\setminus\{0\}$ with $1 \in S$ is multiplicatively closed then $S$ is an $nm$-system.*

*(ii) If $P$ is an $\mathbf{r}$-strongly prime ideal then $R \setminus P$ is an $nm$-system.*

*(iii) If $S$ is an $nm$-system and $I$ is an ideal maximal with respect to not meeting $S$, then $I$ is an $\mathbf{r}$-strongly prime ideal.*

*(iv) Every $\mathbf{r}$-strongly prime ideal in $R$ contains a minimal $\mathbf{r}$-strongly prime ideal (i.e., minimal among the $\mathbf{r}$-strongly prime ideals).*

Proof. (i) This is clear since for $t \in S$ we can use $\lambda = 1 \in \mathbf{M}(R)$ and then $(\lambda t)^n = t^n \in S$ for all $n \geq 1$.

(ii) If $P$ is an $\mathbf{r}$-strongly prime ideal and $S = R \setminus P$, $S$ is an $m$-system and because $R/P$ has no non-zero nil ideals, for $t \in S$ there is $\lambda \in \mathbf{M}(R)$ such that $\lambda t$ is not nil modulo $P$, which is exactly the defining feature of an $mn$-system.

(iii) If $S$ is an $nm$-system and $I$ an ideal maximal with respect to not meeting $S$ then $I$ is prime since $S$ is an $m$-system. Suppose that $x \notin I$ generates an ideal which is nil modulo $I$. Consider the ideal $I + \langle x \rangle$. Using maximality we pick, $t \in (I + \langle x \rangle) \cap S$ and write $t = a + y$, $a \in I$, $y \in \langle x \rangle$. There is $\lambda \in \mathbf{M}(R)$ such that $(\lambda t)^n \in S$ for all $n \geq 1$. Now $(\lambda t)^n = (\lambda a + \lambda y)^n = b + (\lambda y)^n$, where $b \in I$ . However, for some $m \geq 1$, $(\lambda y)^m \in I$, which is impossible. Hence, $I$ is an $\mathbf{r}$-strongly prime ideal.

(iv) Clear. □

The result [6, Lemma 2.2], using a multiplicatively closed set for $S$, is a special case of Proposition 1.2(iii).

In a commutative ring $R$ a multiplicatively closed set $S \subseteq R \setminus \{0\}$, $1 \in S$, has a "saturation" $T = \{t \in R \mid \langle t \rangle \cap S \neq \emptyset\}$ which is a multiplicatively closed set and is the complement of the union of the prime ideals maximal with respect to not meeting $S$. There is a similar result, [9, Proposition 3.6], in connection with the "strongly prime ideals" of that paper. However, there is no "saturation" for $nm$-systems, in general. A given $nm$-system can in some cases be enlarged but Example 3.1 will show that there is not always a "saturation".

**Remark 1.3.** *Let $R$ be a ring and $S \subseteq R \setminus \{0\}$ an nm-system. Define $T = \{t \in R \mid \exists\, r, s \in R \text{ with } rts \in S\}$. Then, $T$ is an nm-system whose complement contains the same ideals as the complement of $S$.*

Proof. We first show that $T$ is an $m$-system. If $t, u \in T$ there are $r, s, r's' \in R$ with $rts, r'us' \in S$. Since $S$ is, in particular, an $m$-system there is $x \in R$ with $rtsxr'us' \in S$. It follows that $tsxr'u \in T$, showing that $T$ is an $m$-system. Moreover, if $rts \in S$ there is $\lambda \in M(R)$ with $(\lambda(rts))^n \in S$, for all $n \in \mathbb{N}$. However, $rts \in T$. □

Theorem 2.10, below, gives examples of multiplicatively closed sets which are saturated. As a final remark in this section we have the following companion to a result of Shin, [14, Proposition 1.11]: $R$ is 2-primal if and only if each minimal prime ideal is completely prime.

**Proposition 1.4.** *A ring $R$ is an NI-ring if and only if each minimal* **r***-strongly prime ideal is completely prime.*

Proof. If $R$ is an NI-ring then each minimal **r**-strongly prime ideal is completely prime by [6, Theorem 2.3(1)]. In the other direction, if each minimal **r**-strongly prime ideal is completely prime then $R/\mathbf{N}^*(R)$ is reduced. This means that $\mathbf{N}^*(R) = \mathbf{N}(R)$. □

## 2. Weak zero-divisors.

The following definition contains some terminology to be used throughout.

**Definition 2.1.** *Let $R$ be a ring. (i) An element $a \in R$ is called a* left zero-divisor *if there is $0 \neq r \in R$ with $ar = 0$. The set of elements which are not left zero-divisors is denoted $S_{\mathrm{nl}}$. (Similarly for* right zero-divisors *and $S_{\mathrm{nr}}$.) (ii) An element $a \in R$ is called a* weak zero-divisor *if there are $r, s \in R$ with $ras = 0$ and $rs \neq 0$. The set of elements of $R$ which are not weak zero-divisors is denoted by $S_{\mathrm{nw}}$.*

The notion of a weak zero-divisor is what is needed to answer the question about elements of minimal primes.

**Theorem 2.2.** *Let $R$ be a ring and $P$ a minimal prime ideal of $R$. Then, for each $a \in P$, $a$ is a weak zero-divisor.*

Proof. Let $P$ be a minimal prime ideal and put $S = R \setminus P$. Suppose, on the contrary, that $a \in P$ is not a weak zero-divisor. Consider the set

$$T = \{r_1 a^{i_1} r_2 \cdots r_k a^{i_k} r_{k+1} \mid k \in \mathbb{N}, i_j \geq 0, r_1 \cdots r_{k+1} \in S\} .$$

It is clear that $T \supseteq S$. The claim is that $T$ is an $m$-system. It first must be shown that $0 \notin T$. If $0 = r_1 a^{i_1} r_2 \cdots r_k a^{i_k} r_{k+1} \in T$ then the product remains 0 if any factors $a$ are removed since $a \in S_{\text{nw}}$; once all the factors are removed from the expression we get $r_1 \cdots r_{k+1} = 0$, which is not possible since that product is in $S$. It is next shown that $T$ is an $m$-system: given two elements of $T$, $r_1 a^{i_1} r_2 \cdots r_k a^{i_k} r_{k+1}$ and $s_1 a^{j_1} s_2 \cdots s_l a^{j_l} s_{l+1}$, we know that there is $t \in R$ such that $r_1 \cdots r_{k+1} t s_1 \cdots s_{l+1} \in S$. From that, $r_1 a^{i_1} r_2 \cdots r_k a^{i_k} r_{k+1} t s_1 a^{j_1} s_2 \cdots s_l a^{j_l} s_{l+1} \in T$, as required. □

Examples 3.2 and 3.3, below, show that left or right zero-divisors cannot replace weak zero-divisors in Theorem 2.2. However, in a reduced ring weak zero-divisors are both left and right zero-divisors.

**Corollary 2.3.** *In a ring $R$, if $a$ is an element of a minimal prime ideal then there are $r, s \in R$ such that $ras \in \mathbf{N}_*(R)$ and $rs \notin \mathbf{N}_*(R)$. If $R/\mathbf{N}_*(R)$ is reduced (i.e., $\mathbf{N}_*(R) = \mathbf{N}(R)$) then there is $r \notin \mathbf{N}_*(R)$ such that $ra \in \mathbf{N}_*(R)$ and $ar \in \mathbf{N}_*(R)$.*

Proof. The first part is Theorem 2.2 applied to $R/\mathbf{N}_*(R)$. The second follows since in a reduced ring $S$, $abc = 0$ implies $acb = bac = 0$. □

Corollary 2.3 can, of course, be restated for any ideal $I$ of $R$ in place of $\mathbf{N}_*(R)$ and using the prime ideals minimal over $I$.

The following simple lemma will be used here and again later.

**Lemma 2.4.** *Let $R$ be any ring and $X$ a subset of $R$. Set $M(X) = \{a \in R \mid \exists\, r, s \in R$ with $ras \in X$ but $rs \notin X\}$ and $M_r(X) = \{a \in R \mid \exists\, r \in R$ with $ar \in X$ but $r \notin X\}$. Then, $R \setminus M(X)$ and $R \setminus M_r(X)$ are multiplicatively closed and both contain 1.*

Proof. We write $M$ for $M(X)$ and $M_r$ for $M_r(X)$. Suppose $a, b \in R \setminus M$ and $ab \in M$. Then, there are $r, s \in R$ with $rabs \in X$ while $rs \notin X$. Since $a \notin M$, $rbs \in X$ and then $b \in M$. This contradiction shows $ab \notin M$. The statements about $M_r$ are proved similarly. □

In Lemma 2.4 there is an analogous statement for $M_l = M_l(X) = \{a \in R \mid \exists\, r \in R$ with $ra \in X$ but $r \notin X\}$. Results about $M_r$ for various sets $X$ can be restated for $M_l$.

**Corollary 2.5.** *Let $R$ be a ring. Then, $S_{\text{nw}}$ and $S_{\text{nl}}$ are closed under multiplication and contain 1; in particular, $S_{\text{nw}}$ and $S_{\text{nl}}$ are nm-systems with $S_{\text{nw}} \subseteq S_{\text{nl}}$.*

Proof. In Lemma 2.4 we take $X = \{0\}$. Moreover, if $a$ is a left zero-divisor then it is a weak zero-divisor. □

**Remark 2.6.** *Let $R$ be a ring. If the set of weak zero-divisors in $R$ forms an ideal $W$ then $W$ is a completely prime ideal. Moreover, if a minimal prime ideal $P$ contains all the weak zero-divisors, then $P$ is completely prime and $P = \mathbf{N}_*(R) = \mathbf{N}(R)$.*

Proof. By Corollary 2.5, $R \setminus W$ is a multiplicatively closed set. Hence, $W$ is prime and if $rs \in W$ then $r \in W$ or $s \in W$.

For the remaining part, the minimal prime ideal $P$ is the only minimal prime and is, hence, $\mathbf{N}_*(R)$. □

Remark 2.6 can be illustrated by a trivial extension of a domain. If $R$ is the ring of column finite $\aleph_0 \times \aleph_0$ upper triangular matrices with constant diagonal over a domain $D$, then $R$ is an example of the situation of Remark 2.6 with $P = \mathbf{N}_*(R)$ nil but not nilpotent. Rings of the type in Remark 2.6 are the subject of [7].

According to Proposition 1.2(iii) or [6, Lemma 2.2], if $S$ is a multiplicatively closed set in a ring $R$ with $0 \notin S$ and $1 \in S$, then an ideal maximal with respect to not meeting $S$ is an $\mathbf{r}$-strongly prime ideal.

**Proposition 2.7.** *Let $R$ be any ring. (i) There is an $\mathbf{r}$-strongly prime ideal consisting of weak zero-divisors. (ii) There is an $\mathbf{r}$-strongly prime ideal consisting of left zero-divisors. There is a minimal prime ideal consisting of left zero-divisors. Similarly for right zero-divisors.*

Proof. By Proposition 1.2(i) and (iii), an ideal maximal with respect to not meeting the multiplicatively closed set $S_{\text{nw}}$ is an $\mathbf{r}$-strongly prime ideal. Similarly for $S_{\text{nl}}$. Moreover, among the prime ideals not meeting $S_{\text{nl}}$ there are minimal prime ideals. □

The ring of Example 3.2 has two minimal prime ideals, one consists of elements which are both left and right zero-divisors while the other has weak zero-divisors which are not left or right zero-divisors. See also Example 3.6.

**Proposition 2.8.** *Let $P$ be a completely prime ideal in a ring $R$ which is minimal among $\mathbf{r}$-strongly prime ideals. Then, the elements of $P$ are weak zero-divisors.*

Proof. We use Proposition 1.2(iii) and put $S = R \setminus P$. The argument of Theorem 2.2 is modified. If $a \in P$ is not a weak zero-divisor then put $T = \{r_1 a^{i_1} r_2 \cdots r_k a^{i_k} r_{k+1} \mid i_j \geq 0, r_j \in S, j = 1, \ldots, k\}$. It follows that $T$ is a multiplicatively closed set strictly containing $S$ and with $0 \notin T$. An ideal maximal with respect to not meeting $T$ is an $\mathbf{r}$-strongly prime and is contained in $P$. This is not possible. □

There is an example, [6, Proposition 1.3], based on [6, Example 1.2], of a prime NI-ring $R$ in which $\mathbf{N}^*(R) \neq \mathbf{0}$. Hence, there are $\mathbf{r}$-strongly prime ideals minimal among $\mathbf{r}$-strongly prime ideals but which are not minimal prime ideals ($\mathbf{0}$ is the only minimal prime ideal). Moreover, by [6, Theorem 2.3(1)], these minimal $\mathbf{r}$-strongly prime ideals are completely prime; then Proposition 2.8 applies and the elements of such ideals are weak zero-divisors. We will use the construction of [6, Example 1.2] and our Example 3.2 to show that in Proposition 2.8 weak zero-divisors are required (see Example 3.3, below). We collect some of the remarks above as follows.

**Corollary 2.9.** *Let $R$ be an NI-ring and $P$ a minimal* $\mathbf{r}$*-strongly prime ideal. Then, $P$ consists of weak zero-divisors.*

Proof. As already mentioned, [6, Theorem 2.3(1)] says that Proposition 2.8 applies. $\square$

It also follows from [6, Example 1.2] that, unlike the commutative semiprime case, the union of the minimal primes is not the set of weak zero-divisors. However, in an NI-ring there is an analogous result. Recall (e.g., [12, §2.1, Exercise 11]), that, in a commutative ring $R$ we always have $\mathbf{N}_*(R) = \mathbf{N}(R)$ and, also, the union of the minimal prime ideals is $\{r \in R \mid \exists s \notin \mathbf{N}_*(R)$ such that $rs \in \mathbf{N}_*(R)\}$.

Recall that an ideal $I$ of $R$ is called a *completely semiprime ideal* if $R/I$ is a reduced ring; if $I$ is a completely semiprime ideal then the prime ideals minimal over $I$ are completely prime. The next result mimics the commutative case.

**Theorem 2.10.** *Let $R$ be a ring and $I$ a completely semiprime ideal. Then, $M_r(I) = \{a \in R \mid \exists\, r \in R$ with $ar \in I$ but $r \notin I\}$ is the union of the completely prime ideals minimal with respect to containing $I$. In addition, $R \setminus M_r(I)$ is multiplicatively closed and contains 1. The sets $M_r(I)$ and $M_l(I)$ coincide.*

Proof. Let $\mathcal{P}$ be the set of completely prime ideals minimal over $I$. Suppose $a \in P$ for some $P \in \mathcal{P}$ and we can suppose $a \notin I$. In the reduced ring $R/I$, $a + I$ is a left zero-divisor. I.e., there is $r \notin I$ such that $ar \in I$, showing that $a \in M_r(I)$.

In the other direction, if we have $a \in M_r(I)$ with $r \notin I$ and $ar \in I$ but $a \notin P$ for each $P \in \mathcal{P}$, then, since these primes are completely prime, $r$ would be in $I$, which is impossible. Hence, $M_r(I) = \bigcup_{P\in\mathcal{P}} P$.

The next part is an application of the second part of Lemma 2.4 applied to $X = I$. The last observation follows since left and right zero-divisors coincide in a reduced ring. $\square$

The set $R \setminus M_r(I)$ in Theorem 2.10 is a saturated $nm$-system as discussed at the end of Section 1.

When $R$ is an NI-ring Theorem 2.10 yields a result analogous with the commutative case.

**Corollary 2.11.** *Let $R$ be an NI-ring and $M_r = M_r(\mathbf{N}(R)) = \{a \in R \mid \exists\, r \in R$ with $ar \in \mathbf{N}(R)$ but $r \notin \mathbf{N}(R)\}$. Then, $M_r$ is the union of the minimal* $\mathbf{r}$*-strongly prime ideals of $R$. Moreover, $R \setminus M_r$ is closed under multiplication and contains 1.*

Proof. We need only invoke Theorem 2.10 with $I = \mathbf{N}(R)$ and the fact that in an NI-ring $\mathbf{N}(R)$ is completely semiprime (e.g., [6, Lemma 2.1]). $\square$

In the special case of a 2-primal ring, the minimal $\mathbf{r}$-strongly prime ideals of Corollary 2.11 are, in fact, the minimal prime ideals; in a 2-primal ring Corollary 2.11 is exactly as for commutative rings.

It is remarked in [5, page 4869] that if $R$ is a PI-ring or a ring of bounded index then $R$ is a NI-ring if and only if $R$ is 2-primal.

The conclusion of Corollary 2.11 need not hold when the ring is not an NI-ring: see Example 3.5, below.

**3. Examples and special rings.**

Our first example is to illustrate how an $nm$-system can fail to have a saturation.

**Example 3.1.** *There is a ring $R$ such that $T = \{t \in R \mid \langle t \rangle = R\} = \{t \in R \mid \langle t \rangle \cap S \neq \emptyset\}$ is not an $m$-system and is not a saturation for $S = \{1\}$.*

Proof. Let $K$ be a field and $F = K\langle Y, X_1, X_2 \rangle$ a free algebra in 3 variables. Set $I$ to be the ideal of $F$ generated by $\rho = X_1YX_2 - 1$ and $R = F/I$. We write the images $Y + I = y, X_1 + I = x_1$ and $X_2 + I = x_2$. By construction, $y \in T$ (as are $x_1$ and $x_2$). However, in order to have $v, u_j, w_j \in F$, $j = 1, \ldots, m$ with $\sum_j u_jYvYw_j - 1 \in I$ we would need an equation of the form $\sum_j u_jYvYw_j - 1 = \sum_i r_i\rho s_i$ for some $r_i, s_i \in F$, $i = 1, \ldots, n$. The equation shows that for some $k$, $1 \leq k \leq n$, $r_ks_k$ has a non-zero constant term. The corresponding $r_kX_1YX_2s_k$, when split into monomial terms, has a monomial term with only one copy of $Y$. No such term can exist in the other expression. Hence, no element of $yRy$ is in $T$. This shows that $T$ is not an $m$-system. $\square$

When a semiprime ring $R$ has only finitely many minimal prime ideals (see [10, Theorem 11.43] for characterizations of such rings) then each element of a minimal prime is a left and a right zero-divisor. The following example shows that even when there are only finitely many minimal prime ideals weak zero-divisors may be required when the ring is not semiprime.

**Example 3.2.** *Let $K$ be a field and $R = K\langle X, Y \rangle/I$ where $I$ is generated by the monomials $XY^iX, i \geq 1$. Write $X + I = x$ and $Y + I = y$. Then, $\langle y \rangle$ is a minimal prime of $R$, $R$ has only two minimal primes and $\mathbf{N}_*(R) \neq 0$. Moreover, $y$ is neither a left nor a right zero-divisor but $xyx = 0$ while $x^2 \neq 0$.*

Proof. Since $R/\langle y \rangle \cong K[X]$, $\langle y \rangle$ is a prime ideal, and, similarly, $\langle x \rangle$ is a prime ideal. Put $L = \langle x \rangle \cap \langle y \rangle$. Then, $L^3 = 0$. Moreover, $R/L$ is reduced since if $r^2 \in L$ and $r$ is written as a polynomial with no terms containing a factor $xy^ix, i \geq 1$, then $r \notin L$ would mean that $r$ has a term purely in $x$ or in $y$. Then, $r^2$ would also have such a term. It follows that any prime ideal $Q$ of $R$ contains $L$ and, hence, $\langle x \rangle\langle y \rangle \subseteq Q$. Hence, the minimal primes are $\langle x \rangle$ and $\langle y \rangle$. However, $y$ is not a left or a right zero-divisor while $xyx = 0$ and $x^2 \neq 0$. On the other hand the elements of $\langle x \rangle$ are all left and right zero-divisors. $\square$

The ring $R$ in Example 3.2 is an NI-ring (even 2-primal) because $\langle x \rangle \cap \langle y \rangle = \mathbf{N}(R) = \mathbf{N}_*(R)$. The set $M_r$ from Corollary 2.11 is $\langle x \rangle \cup \langle y \rangle$ and $R/\mathbf{N}(R)$ is the reduced ring $K\langle X, Y \rangle/K$, where $K$ is generated by $\{XY, YX\}$. Moreover (cf., Corollary 2.5), $S_{\text{nw}} = R \setminus (\langle x \rangle \cup \langle y \rangle)$ and $S_{\text{nl}} = R \setminus \langle x \rangle = S_{\text{nr}}$.

**Example 3.3.** *There is an example of an NI-ring $R$ such that $\mathbf{N}_*(R) \neq \mathbf{N}(R)$ in which there is a prime ideal minimal over $\mathbf{N}(R)$ whose elements are neither left nor right zero-divisors (they are weak zero-divisors).*

Proof. We rename the ring from Example 3.2 as $S$ and use it as the seed ring in the construction of [6, Example 1.2]. To recall the construction: for each $n \in \mathbb{N}$ let $S_n$ be the ring of $2^n \times 2^n$ upper triangular matrices over $S$, and $S_n$ is embedded in $S_{n+1}$ by sending $A \in S_n$ to $\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Then, $R$ is the direct limit of this system of rings. According to [6, Example 1.2], $R$ is an NI-ring but $\mathbf{N}_*(R) \neq \mathbf{N}(R)$.

Now let $P$ be the set of elements $r$ from $R$ which from some $n \in \mathbb{N}$, the matrices representing $r$ have an element of $\langle y \rangle$ in the $(1,1)$ position. The claim is that $P$ is a prime ideal minimal over $\mathbf{N}(R)$. It is clear that it is an ideal. Moreover, $R/P \cong S/\langle y \rangle \cong K[x]$, a prime ring. Just as in Example 3.2, a prime ideal contained in $P$ and containing $\mathbf{N}(R)$ would have to contain the elements with $(1,1)$ entry equal to $y$. Again as in Example 3.2, if we take for $a \in P$ an element represented by $\left(\begin{smallmatrix} y & 0 \\ 0 & y \end{smallmatrix}\right) \in S_1$, then the equation $ar = 0$ in $R$ with $r \neq 0$ would imply that there is a representative of $r$ in, say, $S_n$. The element corresponding to $a$ in $S_n$ is $yI_{2^n}$ where $I_{2^n}$ is the identity matrix. Then the product $ar = 0$ in $S_n$ multiplies each row of $r$ by $y$. Since $y$ is not a left zero-divisor, we have a contradiction. Similarly, $a$ is not a right zero-divisor. $\square$

In the ring $R$ of Example 3.3, $r \in \mathbf{N}(R)$ if and only if $r$ has a representative whose diagonal elements are in $\mathbf{N}(S)$. Examples 3.2 and 3.3 are not semiprime; the next example is of a semiprime ring which has a minimal prime whose elements are neither left nor right zero-divisors.

**Example 3.4.** *There is a semiprime ring $R$ and a minimal prime ideal $P$ along with $a \in P$ such that $a$ is neither a left nor a right zero-divisor.*

Proof. We again use the ring of Example 3.2 as a starting point. We will here call that ring $R_0$. Let $R_1$ be the ring $K\langle X, Y, Z\rangle/I$ where $I$ is generated by $\{XY^iX \mid i \geq 1\}$, the same defining relations as for $R_0$. There is a natural embedding of $R_0$ into $R_1$. However, $R_1$ is a prime ring.

The ring $R$ is defined as follows: $R$ is the ring of all sequences $r = (r_n)$ from $R_1$ such that for some $k \in \mathbb{N}$, depending on $r$, $r_j \in R_0$ is constant for all $j \geq k$. The ring $R$ is semiprime. To see this, if $r = (r_n) \in R$ and, for some $k \in \mathbb{N}$, $r_k \neq 0$ then $r_k R_1 r_k \neq \mathbf{0}$, showing that $rRr \neq \mathbf{0}$.

We define $P = \{r = (r_n) \in R \mid r_n \text{ is eventually constant and in } \langle y \rangle\}$. Since $R/P \cong R_0/\langle y \rangle$, $P$ is a prime ideal. It now needs to be shown that $P$ is a minimal prime ideal. Suppose that $Q \subseteq P$ is a prime ideal. For any idempotent $e \in R$ (all the idempotents in $R$ are central), $eR(1-e) = \mathbf{0}$ means that $e \in Q$ or $1 - e \in Q$. However, if $e$ is eventually 1, $e \notin P$ and, hence, $e \notin Q$. Thus $\bigoplus_{i \in \mathbb{N}} R_1 \subseteq Q$. For $u \in R_0$, let $\hat{u}$ denote the element of $R$ which is constantly $u$. We will see that $\hat{x}R\hat{y}\hat{x} \subseteq Q$. Indeed, for $v \in R$, we may assume that $v \notin \bigoplus_{i \in \mathbb{N}} R_1$ and, hence, that $v$ has the form $v = (0, \ldots, 0, w, w, \ldots)$, where $w \in R_0$. Then, as in the proof of Example 3.2, $\hat{x}v\hat{y}\hat{x} \in Q$. The rest of the proof follows as in the proof of Example 3.2, showing that $\hat{y} \in Q$ and that $Q = P$.

Finally, $\hat{y}$ is neither a left nor a ring zero-divisor but is, of course, a weak zero-divisor. $\square$

It can also be seen that the ring of Example 3.4 is left and right nonsingular. See also Proposition 3.9 for more about constructions related to that in Example 3.4.

**Example 3.5.** *There is a ring $R$ with $\mathbf{N}^*(R) = \mathbf{0}$ where $M_r = M_r(\mathbf{N}(R)) = \{a \in R \mid \exists\, r \notin \mathbf{N}(R)$ with $ar \in \mathbf{N}(R)\}$ is not the union of the minimal ($\mathbf{r}$-strongly) prime ideals.*

Proof. Consider a division ring $D$ and the ring $R$ of sequences of $2\times 2$ matrices over $D$ which are eventually a constant diagonal matrix (e.g., [14, Example 5.6]). Then the von Neumann regular ring $R$ has no non-zero nil ideals and the minimal $\mathbf{r}$-strongly prime ideals are also the minimal prime ideals; they are the maximal ideals (i) $I_n$ of sequences zero in the $n$th component, and (ii) the ideals $P_i$, $i = 1, 2$, of sequences eventually a constant diagonal matrix which is zero in the $ii$ position. Consider $a \in R$ where, for $i = 1, \ldots, n$, $n \geq 1$, the $i$th component of $a$, $a_i$, is non-zero but there is $0 \neq r_i$, which is not nilpotent, with $a_i r_i = 0$, while the constant part of $a$ can be the identity matrix. Put $r \in R$ to be $r_i$ for $i = 1, \ldots, n$ and 0 beyond. Then, $ar = 0$ but $a$ is not in the union of the minimal ($\mathbf{r}$-strongly )prime ideals. For example, $a = ((\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), \ldots)$ and $r = ((\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}), \ldots)$. Hence, $a \in M_r$ but is not in the union of the prime ideals. Similarly, $a$ is in the set $M_l = M_l(\mathbf{N}(R))$.

On the other hand, the union of the prime ideals is contained in $M_r \cap M_l$. $\square$

In the ring $R$ of Example 3.5, the minimal prime ideals consist of left (and right) zero-divisors. The set of elements of $R$ with constant part 0 is a completely semiprime ideal, call it $K$. The minimal prime ideals containing $K$ are $P_1$ and $P_2$ whose union is, according to Theorem 2.10, $M_r(K) = M_l(K)$. As in any von Neumann regular ring, the set of left zero-divisors is $\{a \in R \mid Ra \neq R\} = M_r(\mathbf{0})$ and that of right zero-divisors in $\{a \in R \mid aR \neq R\} = M_l(\mathbf{0})$; these coincide if, as in our example, the ring is directly finite. However, elements of a proper ideal in a von Neumann regular ring are all left and right zero-divisors. (See also Proposition 3.8, below, for information about a related class of rings to that of the von Neumann regular ones.)

The next example illustrates the left and right versions of Proposition 2.7.

**Example 3.6.** *Let $A$ be a domain which is neither left nor right Ore and $R = (\begin{smallmatrix} A & A \\ \mathbf{0} & A \end{smallmatrix})$. Then, $R$ has two minimal prime ideals, one consists of left zero-divisors and the other of right zero-divisors; neither consists of both.*

Proof. The two minimal prime ideals are $I = (\begin{smallmatrix} A & A \\ \mathbf{0} & \mathbf{0} \end{smallmatrix})$ and $J = (\begin{smallmatrix} \mathbf{0} & A \\ \mathbf{0} & A \end{smallmatrix})$. $\square$

See also [2, Example 2.6] and its references for information on minimal prime ideals and zero-divisors of rings of the form of that of Example 3.6..

The ring $K\langle X, Y\rangle/I$, where $I = \langle XY\rangle$, of [2, Example 2.8] shows the same phenomenon as that of Example 3.6.

As in the commutative case, zero-divisors of all sorts do not behave well with respect to homomorphic images. Some information can be gleaned.

**Proposition 3.7.** *Suppose $R$ is an NI-ring. (i) If $a + \mathbf{N}(R) \in R/\mathbf{N}(R)$ is a weak zero-divisor then $a$ is a weak zero-divisor in $R$. (ii) If every element of a proper ideal of $R/\mathbf{N}(R)$ is a weak zero-divisor then every element of a proper ideal of $R$ is a weak zero-divisor.*

Proof. (i) Suppose that $a \in R$ is such that $a + \mathbf{N}(R)$ is a weak zero-divisor. Then, there are $r, s \in R$ such that $ras \in \mathbf{N}(R)$ and $rs \notin \mathbf{N}(R)$. For some minimal $m \in \mathbb{N}$, $(ras)^m = 0$. If some of the factors $a$ in $(ras)^m$ can be removed to get a non-zero element, the proof is complete. Removing all the factors $a$, if necessary, leaves $(rs)^m \neq 0$, which gives the result. (ii) This follows directly from (i). $\square$

The converse of Proposition 3.7 is false even in the commutative case. Consider a field $K$ and the ring $R = K[X, Y]/I$, where $I = \langle\{X^n, XY\}\rangle$, for some $n \geq 2$. Then $Y + I$ is a zero-divisor in $R$ but not modulo $\mathbf{N}_*(R) = \langle X + I\rangle$.

The argument in Proposition 3.7(i) does not work for left zero-divisors and, in fact, the conclusion is false for left (or right) zero-divisors. In Example 3.3, the element $a$ shown to be a weak zero-divisor but neither a left nor a right zero-divisor, is both a left and right zero-divisor modulo $\mathbf{N}(R)$.

There are various weak forms of von Neumann regularity which guarantee that elements of proper ideals are in fact zero-divisors. Recall that a ring $R$ is *right weakly $\pi$-regular* if for every $a \in R$ there is $m \in \mathbb{N}$ such that $a^m \in a^m\langle a^m\rangle$.

**Proposition 3.8.** *Let $R$ be a right weakly $\pi$-regular ring. Then, every element of a proper ideal is a left zero-divisor.*

Proof. Let $a \in R$ be in a proper ideal and we may assume that $a$ is not nilpotent. We can write, for some $m \in \mathbb{N}$, $a^m = a^m \sum_{i=1}^n r_i a^m s^i$ and $a^m(1 - \sum r_i a^m s_i) = 0$. We know that $\sum r_i a^m s_i \neq 1$ and, thus, there is a minimal $k \geq 1$ such that $a^k(1 - \sum r_i a^m s^i) = 0$. Then, $a^{k-1}(1 - \sum r_i a^m s_i) \in \operatorname{rann} a$. $\square$

Proposition 3.7 applies to rings not covered by Proposition 3.8. Using [1, Theorem 2.6], one needs to find NI-rings $R$ which do not satisfy the idempotent condition WCI ([1, Definition 2.1]), and, hence, is not right weakly $\pi$-regular, but for which $R/\mathbf{N}(R)$ is right weakly $\pi$-regular. One such is [1, Example 1.7].

For a von Neumann regular ring satisfying general comparability ([4, Definition, page 83]), the minimal prime ideals are generated by central idempotents ([4, Theorem 8.26]) and, hence, an element of a minimal prime ideal is annihilated by a non-zero central idempotent. More generally the observation applies to any ring in which the minimal primes are generated by central idempotents. We will not go into details here but the condition that each minimal prime of a ring $R$ is generated by central idempotents is equivalent to saying that the Pierce sheaf of $R$ has prime stalks (see [8, V 2] or [3]). Biregular rings have this property as do full products of prime rings.

More generally we have the following which will help in the construction of examples. The key property of a Pierce sheaf of a ring $R$ which we will use is that if for some $x \in \operatorname{Spec} \mathbf{B}(R)$ and $r, s \in R$ we have $r_x = s_x$ then there is $e \in \mathbf{B}(R) \setminus x$ such that $re = se$.

**Proposition 3.9.** *Let $R$ be a ring whose Pierce sheaf has stalks $R_x$ which have the property that each minimal prime ideal of $R_x$ consists of left or of right zero-divisors. Then each minimal prime ideal of $R$ consists of left or of right zero divisors.*

Sketch of proof. Let $R_x$ be a stalk of $R$ ($x$ refers to a maximal ideal of the boolean algebra $\mathbf{B}(R)$ of central idempotents of $R$ and $R_x = R/Rx$).

Since for any prime ideal $P$ of $R$, $P \cap \mathbf{B}(R) = x$, for some $x \in \operatorname{Spec} \mathbf{B}(R)$ and $R \to R_x = R/Rx$ is surjective, a minimal prime ideal $P$ of $R$ has the following form. For $x = P \cap \mathbf{B}(R)$ and $Q = P_x = P/Rx$, $P = \{r \in R \mid r_x \in Q\}$. Moreover, each such pair $(x, Q)$ yields a minimal prime ideal of $R$.

Then, if $Q$, a minimal prime ideal of $R_x$, consists, say, of left zero-divisors, for $u \in P$, as constructed above, there is $r \in R$ with $r_x \neq 0_x$ and $u_x r_x = 0$. For some $e \in \mathbf{B}(R) \setminus x$, $ure = 0$. Since $re \neq 0$, $u$ is a left zero-divisor. □

The converse is true in a ring like that in Example 3.4 but a small change in that example shows that it is false in general.

**Example 3.10.** *There is a ring $R$ which has a Pierce stalk $R_x$ so that $R_x$ has a minimal prime ideal with an element which is neither a left nor a right zero-divisor but the corresponding minimal prime ideal of $R$ consists of zero-divisors.*

Proof. Let $R_0$ be the ring of Example 3.2 and $S = K\langle x, y, z\rangle/I$, where $I$ is generated by $\{xy^i x \mid i \geq 1\}$ and $\{zxy, zx^2\}$. Then, let $R$ be the ring of sequences from $S$ which are eventually constant and in $R_0$. The Pierce stalks of $R$ are $R_n = S$, $n \in \mathbb{N}$, and $R_\infty = R_0$. Let $P = \{r \in R \mid r_\infty \in \langle y\rangle\}$; $P$ is a minimal prime ideal of $R$. It can be seen that the elements of $P$ are all right zero-divisors even though $P_\infty$ has an element which is not a right zero-divisor. Indeed, any monomial in $\langle y\rangle$ is annihilated on the left by $zx \neq 0$. Now let $r \in P$ be such that $r_\infty = u \in R_0$, $u \neq 0$. Then, for some $n \in \mathbb{N}$, $r_n = u$. Let $e \in R$ be such that $e_m = 0$ if $m \neq n$ and $e_n = 1$. Then, $zxer = 0$, while $zxe \neq 0$. □

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