The Regularity Degree and Epimorphisms in the Category of Commutative Rings

W.D. Burgess[1]
Department of Mathematics
University of Ottawa
Ottawa, Canada K1N 6N5

R. Raphael[1]
Department of Mathematics
Concordia University
Montreal, Canada H4B 1R6

Abstract

Elements of the universal (von Neumann) regular ring $T(R)$ of a commutative semiprime ring R can be expressed as a sum of products of elements of R and quasi-inverses of elements of R. The maximum number of terms required is called the *regularity degree*, an invariant for R measuring how R sits in $T(R)$. It is bounded below by 1 plus the Krull dimension of R. For rings with finitely many primes and integral extensions of noetherian rings of dimension 1, this number is precisely the regularity degree.

For each $n \geq 1$, one can find a ring of regularity degree $n+1$. This shows that an infinite product of epimorphisms in the category of commutative rings need not be an epimorphism.

Finite upper bounds for the regularity degree are found for noetherian rings R of finite dimension using the Wiegand dimension theory for Patch R. These bounds apply to integral extensions of such rings as well.

I. Introduction. All rings in this article are assumed to be *commutative, unitary and semiprime*. The word "regular" is only used for *von Neumann regular rings*. The *dimension* of a ring is always its Krull dimension.

Our first purpose is have another look at epimorphisms in the category of commutative rings and at the *universal regular ring functor*, T (called "$\hat{}$" in [13]). Epimorphisms have been studied by, among others, Olivier ([7]) and Storrer ([11]) and the universal regular ring, $T(R)$ of a ring R, is a major tool. It is constructed from R via the boolean space Patch R, which is a refinement of $\mathrm{Spec}(R)$; this space is used in algebraic geometry under the name "constructible topology" but we use the terminology of Hochster ([4], see also, Wiegand [13, 14]).

[1]This work was partially supported by grants A7539 and A7753 of the NSERC.
e-mail: *wdbsg@uottawa.ca* and *raphael@alcor.concordia.ca*
AMS Classification: 13A99, 13B25, 13E15

It has long been known that in the category of all rings a product of ring epimorphisms (in the categorical meaning of "epimorphism") is not necessarily an epimorphism (Storrer, [12]) but the methods used do not apply in the quite different category of commutative rings. In order to look further at epimorphisms in this latter category we were led to the idea of the *regularity degree*.

If R is embedded epimorphically in a regular ring S, each element of S can be written $s = \sum_{i=1}^{k} c_i d_i^*$, where the c_i and d_i are in R and * indicates the *quasi-inverse* $(d^2 d^* = d, (d^*)^2 d = d^*)$. The *regularity degree*, $\operatorname{reg\,deg}_R S$, of S over R is the minimum $k \in \mathbf{N}$ so that each $s \in S$ can be expressed as a sum of k terms. If no such k exists we say that $\operatorname{reg\,deg}_R S = \infty$.

The invariant, $\operatorname{reg\,deg}_R T(R)$, gives information about how R sits in its universal regular ring $T(R)$.

Theorem 3.1 shows, among other things, that $\operatorname{reg\,deg}_R T(R) \geq k+1$ when the dimension of R is k. In particular, any R of infinite dimension has $\operatorname{reg\,deg}_R T(R) = \infty$. It follows that in the category of commutative rings a product of epimorphisms is not necessarily an epimorphism. The methods of the theorem also show that if R is noetherian of dimension 1, then $\operatorname{reg\,deg}_R T(R) = 2$.

The rest of the article is devoted to finding classes of rings R where $\operatorname{reg\,deg}_R T(R)$ is finite, and to getting upper bounds on its value. The dimension theory for the space $\operatorname{Patch} R$, where R is noetherian, developed by R. Wiegand ([14]), will be used to show that for any noetherian ring R of finite (Krull) dimension, there is a finite upper bound on $\operatorname{reg\,deg}_R T(R)$. The same bound applies to integral extensions of such rings.

Other families of examples of rings R with $\operatorname{reg\,deg}_R T(R) < \infty$ are known. For example, if S is a regular ring then $\operatorname{reg\,deg}_{S[X]} T(S[X]) = 2$. Details will be published elsewhere. Related ideas in the context of rings of continuous functions are examined in [3].

The language of Pierce sheaves, [8], is used.[2] Some of it is recalled below. We will distinguish between epimorphisms in the category of commutative rings and the special case of onto homomorphisms, by calling the latter "surjections".

II. Definitions and notation. For a ring R (commutative, semiprime and unitary) the usual topology on $\operatorname{Spec} R$ can be refined by making the basic open sets closed and open ([4, page 45]) to get the so-called patch topology, $\operatorname{Patch} R$. It is a boolean space. The basic open sets of $\operatorname{Spec} R$, $D(r) = \{P \in \operatorname{Spec} R \mid r \notin P\}$, for $r \in R$, become clopen (closed and open) in $\operatorname{Patch} R$. A basic open set in $\operatorname{Patch} R$ has the form $D(r) \cap V(I)$, where $r \in R$ and I is a finitely generated ideal of R ([14, page 461]).

Notice that a maximal ideal M of a ring R is an isolated point in $\operatorname{Patch} R$ if there is a finitely generated ideal $I \subseteq M$ so that $V(I) = \{M\}$. This always occurs when R is noetherian. (This is essential in the dimension theory developed in [14].)

The Pierce sheaf of a (von Neumann) regular ring S has base space the boolean space $\operatorname{Spec} S$ (= $\operatorname{Patch} S$) and its stalks are the fields S/P, $P \in \operatorname{Spec} S$. The topology

[2] The referee has pointed out that the proofs in this paper can equally well be done without reference to sheaves.

on the *espace étalé* is given by making sets of the form $\{s+P \mid P \in N\}$ open, where $s \in S$ and N is clopen set in $\operatorname{Spec} S$. We consider $s \in S$ as a section of a sheaf; its value at $P \in \operatorname{Spec} S$ is denoted $s(P) = s + P$.

A regular ring, $T(R)$, can be built on $\operatorname{Patch} R$ by using the fields $Q_{\mathrm{cl}}(R/P)$, the field of fractions of R/P, for each $P \in \operatorname{Spec} R$, as the stalks of a Pierce sheaf over $\operatorname{Patch} R$. For $r \in R$, its quasi-inverse in $T(R)$ is the element r^* which is r^{-1} on $D(r) \subseteq \operatorname{Patch} R$, and 0 on $V(r)$.

As noted in [4, page 55], T(R) is an epimorphic extension of R, and it coincides with the ring constructed in [7] by first adjoining indeterminates to R. Furthermore, the assignment $R \mapsto T(R)$ is functorial – it is the left adjoint to the inclusion of commutative regular rings into the category of commutative rings. A typical element of $T(R)$ has the form $\sum_{i=1}^{k} c_i d_i^*$, where $c_i, d_i \in R$. (See also the formula in [5, Proposition 4.2].) By the universal property of $T(R)$, for any ring epimorphism $\varepsilon\colon R \to S$ there is a canonical ring *surjection* $T(\varepsilon)\colon T(R) \to T(S)$. This gives the following lemma ([7, Proposition] or [9, Theorem 2]).

Lemma 2.1. *Given a ring epimorphism $\varepsilon\colon R \to S$, where S is regular, each element of S can be expressed as $s = \sum_{i=1}^{k} \varepsilon(c_i)\varepsilon(d_i)^*$, for some $k \geq 1$ and $c_i, d_i \in R$.*

Definition 2.2. *For a ring epimorphism $\varepsilon\colon R \to S$, as in the lemma, the* regularity degree *of S over R, denoted* $\operatorname{reg\,deg}_R S$*, is the smallest k, if it exists, such that each element of $s \in S$ can be expressed $s = \sum_{i=1}^{k} \varepsilon(c_i)\varepsilon(d_i)^*$ with $c_i, d_i \in R$. If no such integer exists, we say* $\operatorname{reg\,deg}_R S = \infty$.

If $R \to S$ is a ring epimorphism with S regular then $\operatorname{reg\,deg}_R S \leq \operatorname{reg\,deg}_R T(R)$. When $R \to S$ is surjective, $\operatorname{reg\,deg}_R S = 1$ and when $Q_{\mathrm{cl}}(R)$ is regular, then again $\operatorname{reg\,deg}_R Q_{\mathrm{cl}}(R) = 1$. This latter occurs exactly when R has the property that for each $r \in R$, $\operatorname{ann}(r)$ contains a principal ideal which is essential in it ([10, Corollary 1.6]), for example when R is p.p. ([1, Lemma 3.1]). It will be seen in Theorem 3.1 that $\operatorname{reg\,deg}_R T(R) = 1$ is not possible if R is not regular. As would be expected, the regularity degree behaves well with respect to ring epimorphisms.

Remark 2.3. *Let $\varepsilon\colon R \to R'$ be an epimorphism of rings, then*

$$\operatorname{reg\,deg}_{R'} T(R') \leq \operatorname{reg\,deg}_R T(R).$$

Proof. The ring epimorphism $R \to R' \to T(R')$ factors through $R \to T(R)$ via a surjection. □

Olivier ([7, Propositions 3 and 4]) shows that for any subset X of R there is an extension $T_X(R)$ which is universal with respect to the property that elements of X have quasi-inverses. Then $T_X(R)$ embeds naturally in $T(R)$. By what we have seen, $T(R) = \bigcup_{|X|<\infty} T_X(R)$. The case of finite regularity degree is different.

Proposition 2.4. *Let R be a ring such that* $\operatorname{reg\,deg}_R T(R) = k < \infty$*. Then $T(R) = \bigcup_{|X|=k} T_X(R)$.*

We will exhibit rings R_n where $\operatorname{reg\,deg}_{R_n} T(R_n) = n + 1$, for all $n \geq 1$, as well as rings R with $\operatorname{reg\,deg}_R T(R) = \infty$. These rings may be chosen to be domains.

The next proposition will be used later.

Proposition 2.5. *Let R be any ring. (i) If I is a semiprime ideal of R, the restriction of the sheaf of $T(R)$ to the closed set $V(I)$ yields the sheaf of $T(R/I)$. (ii) If F is any closed set of* Patch R*, the restriction of the sheaf of $T(R)$ to F yields a regular ring.*

Proof. (i) Consider a basic open set of Patch R/I of the form $D(\bar{r})\cap V(\bar{J})$, where $\bar{r} \in R/I$ and $\bar{J}$ is a finitely generated ideal of R/I (bars mean reduction modulo I). But this is $D(r)\cap V(J)\cap V(I)$. This shows that the restriction of the sheaf of $T(R)$ to $V(I)$ is the sheaf of $T(R/I)$. Alternatively, it can be seen that the ring obtained by restricting the sheaf of $T(R)$ to $V(I)$ satisfies the universal property of $T(R/I)$.

(ii) This is obvious since the restriction is a homomorphic image of $T(R)$. □

III. Regularity degree and dimension. In this section we start by considering a ring R and a finite subset F of Patch R. If S is the regular ring obtained by restricting the sheaf of $T(R)$ to F, we will be able to compute $\operatorname{reg\,deg}_R S$. The method will show, among other things, that if R is a ring of dimension k, then $\operatorname{reg\,deg}_R T(R) \geq k+1$.

Every subset X of Spec R may be viewed as a poset using set inclusion. We keep the name "dimension" for the length of a longest chain in X.

Theorem 3.1. *Let R be a semiprime ring and F a finite subset of dimension k in* Patch R*. Define S to be the regular ring obtained by restricting the sheaf of $T(R)$ to F. Then* $\operatorname{reg\,deg}_R S = k+1$ *and, consequently,* $\operatorname{reg\,deg}_R T(R) \geq k+1$*. In particular, if the dimension of R is k,* $\operatorname{reg\,deg}_R T(R) \geq k+1$*.*

Proof. Notice that since F is discrete, S is the full product $S = \prod_{P\in F} Q_{\mathrm{cl}}(R/P)$. We label the primes in F as follows: $\{P_{ij}\}$ where $i = 0, \ldots, k$ with P_{ij} of height i in F and, for each i, $j = 1, \ldots, l_i$. The actual inclusion relations among the primes do not have to be specified. The height of a prime in F, *as an element of* F, will be denoted by the *F-height*.

For $i = 0, \ldots, k-1$, pick $a_i \in R$ to be in all primes of F of F-height $> i$ and not in any prime in F of F-height $\leq i$. Put $a_k = 1$. Then create (non-primitive) idempotents in S by $e_0 = a_0 a_0^*$ and for i