MAT 2384 C Winter 2019 Assignment 5 Solutions

1. $A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$. $\quad |A-\lambda I| = \begin{vmatrix} 2-\lambda & -4 \\ 4 & -6-\lambda \end{vmatrix}$

$$= \lambda^2 + 4\lambda + 4$$

$$= (\lambda+2)^2$$

$\lambda = -2$ (double root)

$$(A+2I)\underline{v} = \underline{0} \iff \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff v_1 = v_2$$

eval -2 has evec $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Unfortunately, completely solving this problem requires generalised eigenvectors, which were not covered in class!

Generalised eigenvectors will not be on the exam But here is the method:

We seek $\underline{u}$ s.t. $(A+2I)\underline{u} = \underline{v}$,

i.e. $\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, e.g. $\underline{u} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$

GS to hom eq:

$$y_h = c_1 e^{-2x}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-2x}\left(x\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1/4\\0\end{bmatrix}\right)$$

(see DV. pp 86-87)

Particular solution:

$$y_p = x^2\underline{a} + x\underline{b} + \underline{c}$$

$$y_p' = 2x\underline{a} + \underline{b}$$

$$A y_p + \begin{bmatrix}-2x^2+14x+14\\-4x^2+18x+23\end{bmatrix} = x^2\left[A\underline{a} + \begin{bmatrix}-2\\-4\end{bmatrix}\right]$$

$$+ x\left[A\underline{b} + \begin{bmatrix}14\\18\end{bmatrix}\right] + \left[A\underline{c} + \begin{bmatrix}14\\23\end{bmatrix}\right]$$

This equals y_p' if and only if:

$$A\underline{a} + \begin{bmatrix}-2\\-4\end{bmatrix} = 0 \; ; \quad A\underline{b} + \begin{bmatrix}14\\18\end{bmatrix} = 2\underline{a};$$

and $A\underline{c} + \begin{bmatrix}14\\23\end{bmatrix} = \underline{b}$. Since $A^{-1} = \frac{1}{4}\begin{bmatrix}-6 & 4\\-4 & 2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-3 & 2\\-2 & 1\end{bmatrix}$,

$$\underline{a} = \frac{1}{2}\begin{bmatrix}-3 & 2\\-2 & 1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix},$$

$$\underline{b} = \frac{1}{2}\begin{bmatrix}-3 & 2\\-2 & 1\end{bmatrix}\begin{bmatrix}-12\\-18\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}, \quad \underline{c} = \frac{1}{2}\begin{bmatrix}-3 & 2\\-2 & 1\end{bmatrix}\begin{bmatrix}-14\\-20\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$$

GS: $\underline{y} = c_1 e^{-2x}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-2x}\begin{bmatrix}x+1/4\\x\end{bmatrix} + \begin{bmatrix}x^2+1\\3x+4\end{bmatrix}$

This problem was marked, however the entire assignment was considered to be out of 58 instead of 60, so a perfect mark was possible without knowing generalised eigenvectors.

Note that the method for finding a particular solution <u>is</u> examinable, and does not require generalised eigenvectors.

2 a) $f(t) = e^t - u(t-\frac{\pi}{2})e^t$

$$= e^t - e^{\pi/2} e^{t-\frac{\pi}{2}} u(t-\tfrac{\pi}{2})$$

$$F(s) = \frac{1}{s-1} - e^{\frac{\pi}{2}} e^{-\frac{\pi}{2}s} \frac{1}{s-1}$$

$$= \frac{1 - e^{\frac{\pi}{2}(1-s)}}{s-1}$$

b) $\mathcal{L}\{e^{-4t} u(t-3)\} = \mathcal{L}\{e^{-12} e^{-4(t-3)} u(t-3)\}$

$$= e^{-12} \mathcal{L}\{e^{-4(t-3)} u(t-3)\}$$

$$= e^{-12} e^{-3s} \frac{1}{s+4}$$

$$\mathcal{L}\{t e^{-4t} u(t-3)\} = -\frac{d}{ds} \mathcal{L}\{e^{-4t} u(t-3)\}$$

$$= -e^{-12} \frac{d}{ds} \left\{e^{-3s} \frac{1}{s+4}\right\}$$

$$= e^{-12} \left(\frac{3e^{-3s}}{s+4} + \frac{e^{-3s}}{(s+4)^2} \right)$$

$$= e^{-3s-12} \frac{3s+13}{(s+4)^2}$$

(See similar problem: Prof Khoury's #13, for other methods.)

c) $f(t) = \sin(\omega t + \theta) = \sin(\omega t)\cos\theta + \cos(\omega t)\sin\theta$

$$F(s) = \cos\theta \frac{\omega}{s^2+\omega^2} + \sin\theta \frac{s}{s^2+\omega^2}$$

$$= \frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}$$

3 a) $\displaystyle\int_0^t e^{\tau} e^{-(t-\tau)}\, d\tau = e^{-t}\int_0^t e^{2\tau}\, d\tau$

$$= e^{-t}\left[\tfrac{1}{2}e^{2\tau}\right]_0^t$$

$$= e^{-t}\left(\tfrac{1}{2}e^{2t} - \tfrac{1}{2}\right) = \tfrac{1}{2}e^{-t}(e^{2t}-1)$$

$$= \tfrac{1}{2}(e^{t}-e^{-t}) = \sinh t$$

Note: in this course, always use $\int_0^t$ in the definition of convolution (not $\int_{-\infty}^{\infty}$).

b) $\mathcal{L}^{-1}\{6(1-e^{-\pi s})/s^2+9\}$

$$= 2\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} - 2\,\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{3}{s^2+3^2}\right\}$$

$$= 2\sin 3t - 2\sin 3(t-\pi)\,u(t-\pi)$$

$$= 2\sin 3t + 2\sin 3t\,u(t-\pi)$$

$$= \begin{cases} 2\sin 3t, & 0<t<\pi, \\ 4\sin 3t, & t>\pi \end{cases}.$$

(Kreyszig)

Hence the final answer is

$$y = 8\cos 2t + (\sin 2t)\tfrac{1}{2}u(t-\pi).$$

given on p. A14 and shown in the accompanying figure, where the effect of sin $2t$ (beginning at $t = \pi$) is hardly visible.

Q4

~~6.~~ **Initial value problem.** This is an undamped forced motion with two impulses (at $t = \pi$ and 2π) as the driving force:

$$y'' + y = \delta(t-\pi) - \delta(t-2\pi), \qquad y(0) = 0, \qquad y'(0) = 1.$$

By Theorem 1, Sec. 6.2, and (5), Sec. 6.4, we have

$$(s^2Y - s\cdot 0 - 1) + Y = e^{-\pi s} - e^{-2\pi s}$$

so that

$$(s^2+1)Y = e^{-\pi s} - e^{-2\pi s} + 1.$$

Hence,

$$Y = \frac{1}{s^2+1}(e^{-\pi s} - e^{-2\pi s} + 1).$$

Using linearity and applying the inverse Laplace transform to each term we get

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) &= \sin(t-\pi)\cdot u(t-\pi) \\
&= -\sin t\cdot u(t-\pi) \qquad \text{(by periodicity of sine)} \\
\mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+1}\right) &= \sin(t-2\pi)\cdot u(t-2\pi) \\
&= \sin t\cdot u(t-2\pi) \\
\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) &= \sin t \qquad \text{(Table 6.1, Sec. 6.1).}
\end{aligned}$$

Together

$$y = -\sin t\cdot u(t-\pi) - \sin t\cdot u(t-2\pi) + \sin t.$$

Thus, from the effects of the unit step function,

$$y = \begin{cases} \sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \\ -\sin t & \text{if } t > 2\pi. \end{cases}$$

5. $y' = x + 2\cos y$, $y(0) = 0$.

a) Improved Euler, $h = 0.1$: $x_0 = 0$, $y_0^c = 0$.

$$y_{n+1}^p = y_n^c + h f(x_n, y_n^c)$$

$$y_{n+1}^c = y_n^c + \frac{h}{2}\left[f(x_n, y_n^c) + f(x_{n+1}, y_{n+1}^p)\right]$$

$x_1 = 0.1$

$$y_1^p = 0 + 0.1(0 + 2\cos 0) = 0.2$$

$$y_1^c = 0 + 0.05[2 + (0.1 + 2\cos 0.2)]$$

$$= 0.05[2 + 2.06133] = 0.203007$$

n	x_n	y_n^c	$f(x_n, y_n^c)$	y_{n+1}^p	$f(x_{n+1}, y_{n+1}^p)$
1	0.1	0.203007	2.058929	0.408900	2.035117
2	0.2	0.407709	2.036063	0.611315	1.937788
3	0.3	0.606402	1.943408	0.800743	1.792348
4	0.4	0.793189			

(We can't calculate the absolute error, since we can't solve the ODE exactly)

(But we can estimate the error by comparing with a more accurate method, eg. Imp. Euler with $h = 0.01$, which gives $y_{40}^c = 0.795606$. So the absolute error of the earlier calculation ($h = 0.1$) is approximately $|0.795606 - 0.793189| = 0.002417$.)

b) 4th order Runge-Kutta ("RK4"), $h = 0.2$:

At every step,

$$k_1 = 0.2\,(x_n + 2\cos y_n)$$

$$k_2 = 0.2\,((x_n + 0.1) + 2\cos(y_n + \tfrac{1}{2}k_1))$$

$$k_3 = 0.2\,((x_n + 0.1) + 2\cos(y_n + \tfrac{1}{2}k_2))$$

$$k_4 = 0.2\,((x_n + 0.2) + 2\cos(y_n + k_3))$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

n	x_n	y_n	k_1	k_2	k_3	k_4
0	0	0	0.4	0.4120266	0.4115417	0.406602
1	0.2	0.408956	0.4070146	0.3872938	0.3895452	0.3591124
2	0.4	0.795591				

Using as an approximate "true value" $y(0.4) \doteq 0.795606$, as above in (a), gives approx. absolute error:

$$|0.795606 - 0.795591| = 0.000015$$

which is much smaller than for the Improved Euler Method in (a).