MAT 2384 C Assignment 4 Solutions

1. $A = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 6$

$= (\lambda - 6)(\lambda + 1)$

Eigenvalues: $\lambda = 6, -1$.

Case $\lambda = 6$ $\quad (A - 6I)\underline{v} = \underline{0}$

$\Leftrightarrow \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Leftrightarrow 4v_1 = 3v_2$

An eigenvector is $\underline{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Case $\lambda = -1$ $\quad \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow u_1 = -u_2$

An eigenvector is $\underline{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

G.S.: $\underline{y} = c_1 e^{6t} \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

I.C.: $\underline{y}(0) = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\Leftrightarrow \begin{cases} 3c_1 + c_2 = 2 \\ 4c_1 - c_2 = 5 \end{cases}$

$\Rightarrow$ (add) $7c_1 = 7 \Rightarrow c_1 = 1, \; c_2 = -1$

$\therefore \; \underline{y} = e^{6t} \begin{bmatrix} 3 \\ 4 \end{bmatrix} - e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2. $$y' = \underbrace{\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}}_{A} y$$

$$|A - \lambda I| = \begin{vmatrix} -\frac{1}{2}-\lambda & 1 \\ -1 & -\frac{1}{2}-\lambda \end{vmatrix} = \left(-\tfrac{1}{2}-\lambda\right)^2 + 1$$

$$= 0$$

$$\Leftrightarrow \left(-\tfrac{1}{2}-\lambda\right)^2 = -1$$

$$\Leftrightarrow -\tfrac{1}{2}-\lambda = \pm i$$

$$\Leftrightarrow \lambda = -\tfrac{1}{2} \pm i$$

(or use the quadratic formula)

<u>Case $\lambda = -\frac{1}{2} + i$</u> $\quad (A - \lambda I)\underline{v} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$$\Leftrightarrow i v_1 = v_2$$

an eigenvector is $\underline{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

A solution is $e^{-\frac{1}{2}t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix} = e^{-\frac{t}{2}}\begin{bmatrix} \cos t + i\sin t \\ -\sin t + i\cos t \end{bmatrix}$

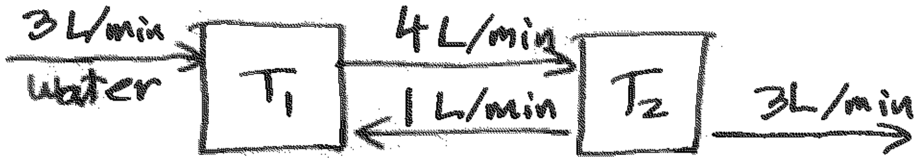
Taking real and imaginary parts gives:

GS: $\quad y = c_1 e^{-t/2}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-t/2}\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$.

(Using $\lambda = -\frac{1}{2} - i$ would lead to same answer.)

3. Let $y_i(t)$ be the amount of salt, in kg, in tank i after t minutes. So $y_1(0)=25$, $y_2(0)=0$.

Liquid flow:

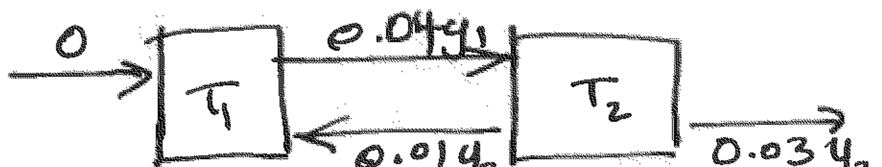


Notice that for each tank, total inflow = total outflow, so the volume in each tank remains constant: 100L.

Since each tank is well-stirred, we can assume the concentration of salt in each tank at a given time is

$$\frac{1}{100} y_i = 0.01\, y_i \text{ kg/L}$$

Thus the salt flow, in kg/min, is:



from which:

$$y_1' = -0.04y_1 + 0.01y_2$$
$$y_2' = 0.04y_1 - 0.04y_2$$

i.e. $y' = \begin{bmatrix} -0.04 & 0.01 \\ 0.04 & -0.04 \end{bmatrix} y =: Ay$

3 (cont.)

$$|A-\lambda I| = \begin{vmatrix} -0.04-\lambda & 0.01 \\ 0.04 & -0.04-\lambda \end{vmatrix} = (-0.04-\lambda)^2 - 0.0004$$

$$= 0 \iff -0.04-\lambda = \pm 0.02$$

$$\iff \lambda = -0.02 \text{ or } -0.06$$

Case $\lambda = -0.02$

$$\begin{bmatrix} -0.02 & 0.01 \\ 0.04 & -0.02 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v_2 = 2v_1$$

An eigenvector is: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Case $\lambda = -0.06$

$$\begin{bmatrix} 0.02 & 0.01 \\ 0.04 & 0.02 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v_2 = -2v_1$$

An eigenvector is: $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

G.S.: $y = c_1 e^{-0.02t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-0.06t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

IC: $y(0) = \begin{bmatrix} c_1 + c_2 \\ 2c_1 - 2c_2 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \end{bmatrix} \iff c_1 = c_2 = 12.5$

$\therefore$ The amount of salt, in kg, in each tank after t minute is: $y_1 = 12.5\left(e^{-0.02t} + e^{-0.06t}\right)$, $y_2 = 25\left(e^{-0.02t} - e^{-0.06t}\right)$,

(Note: $y_1' < 0$ for all t, but y_2 peaks at about $t \approx 27$.)

4. $y' = \underbrace{\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}}_{A} y + \begin{bmatrix} -14 \\ -2x-7 \end{bmatrix} = Ay + x\begin{bmatrix} 0 \\ -2 \end{bmatrix} + \begin{bmatrix} -14 \\ -7 \end{bmatrix}$

Step 1: For hom. eq.:

$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1),$

So eigenvalues of A are $\lambda = 1$, $\lambda = 2$.

$\lambda = 1$: $\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff 3v_1 = 2v_2$

Eigenvector: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (any multiple is good too)

$\lambda = 2$: $\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \underline{v} = \underline{0}$. Eigenvector: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

GS: $y_h = c_1 e^{x} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Step 2 Particular solution: $\underline{y}_p = \underline{a}x + \underline{b} = x\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$A\underline{y}_p + \begin{bmatrix} -14 \\ -2x-7 \end{bmatrix} = xA\underline{a} + A\underline{b} + \begin{bmatrix} -14 \\ -2x-7 \end{bmatrix}$$

$$\underline{y}_p' = \underline{a}$$

These are equal, for all x, if and only if

① $A\underline{a} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \underline{0}$

and

② $A\underline{b} + \begin{bmatrix} -14 \\ -7 \end{bmatrix} = \underline{a}$

① $\Leftrightarrow$ $\underline{a} = A^{-1}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \left(\frac{1}{2}\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Subst into ②: $A\underline{b} + \begin{bmatrix} -14 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$$\Leftrightarrow A\underline{b} = \begin{bmatrix} 16 \\ 11 \end{bmatrix}$$

$$\Leftrightarrow \underline{b} = A^{-1}\begin{bmatrix} 16 \\ 11 \end{bmatrix} = \left(\frac{1}{2}\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}\right)\begin{bmatrix} 16 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

G.S. nonhom: $\underline{y} = \underline{y}_h + \underline{y}_p = c_1 e^x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Step 3 $\underline{y}(0) = c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 + 3 \\ 3c_1 + c_2 - 2 \end{bmatrix}$

I.C.: $\underline{y}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \Rightarrow \begin{cases} 2c_1 + c_2 = 4 \\ 3c_1 + c_2 = 5 \end{cases} \Rightarrow c_1 = 1,\ c_2 = 2$

Sol. to IVP: $\underline{y} = e^x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2e^{2x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

5 a) $\int_0^3 \frac{2x}{1+x^2}\,dx = \ln(1+x^2)\Big|_0^3 = \ln 10 \approx 2.302585$

b) $\int_0^3 f(x)\,dx \approx \frac{0.5}{3}\big(f(0)+4f(0.5)+2f(1)+4f(1.5)+2f(2)$
$+4f(2.5)+f(3)\big)$

$= \frac{1}{6}(0+3.2+2+3.692308+1.6+2.758621+0.6)$

$= \frac{1}{6}(13.850929) = 2.308488$

$\epsilon_6 = 2.302585 - 2.308488 = -0.005903$

c) Error term for Simpson's (composite) is
$-\frac{(b-a)}{180}h^4 f^{(4)}(\xi)$ for some $\xi \in (a,b)$.
If we double the number of intervals, then h is replaced by $\frac{h}{2}$. If we assume $f^{(4)}(\xi)$ stays approx. constant, then the error term becomes approx $\frac{(h/2)^4}{h^4} = \frac{1}{16}$ times what it was before, so approx. $\frac{1}{16}(-0.005903) = -0.00037$.

d) To guarantee $|\epsilon_n| < 10^{-5}$, since $|\epsilon_n| = \frac{3}{180}h^4|f^{(4)}(\xi)|$,
we need $h^4|f^{(4)}(\xi)| < \frac{180}{3}10^{-5} = 6\times10^{-4}$
Since $|f^{(4)}(\xi)| < 50$ [ERROR in question: abs. value missing],
we need $h^4 < (6\times10^{-4})/50 = 1.2\times10^{-5}$.
Since $n = 3/h$, it suffices to have $n > \frac{3}{(1.2\times10^{-5})^{1/4}} \approx 50.97$

5 e) 2-point Gaussian quadrature

On [1,2] Substitute $x = mt + c$ so that

$$\left\{\begin{array}{l} t=-1 \iff x=1 \\ t=1 \iff x=2 \end{array}\right\} \iff \left\{\begin{array}{l} 1 = m(-1)+c \\ 2 = m(1)+c \end{array}\right.$$

$$\iff m = 0.5 \ , \ c = 1.5$$

or use general formula $m = \frac{b-a}{2}$, $c = \frac{b+a}{2}$.

$$\int_1^2 f(x)\,dx = \frac{1}{2}\int_{-1}^1 f(0.5t+1.5)\,dt \qquad \left(\frac{dx}{dt} = 0.5\right)$$

$$\approx \frac{1}{2}\left(f(0.5(-0.577350)+1.5) + f(0.5(0.57735)+1.5)\right)$$

nodes from table

$$= \frac{1}{2}\left(f(1.211325) + f(1.788675)\right) = (0.9819 + 0.85188)/2$$

$$= 1.83378/2 = \underline{.91689}$$

On [2,3] Similarly, $m = 0.5$, $c = 2.5$

$$\int_2^3 f(x)\,dx = \frac{1}{2}\int_{-1}^1 f(0.5t+2.5)\,dt$$

$$\approx \frac{1}{2}\left(f(0.5(-.577350)+2.5) + f(0.5(0.57735)+2.5)\right)$$

$$= \frac{1}{2}\left(f(2.211325) + f(2.788675)\right) = \frac{1}{2}(0.750879 + 0.635472)$$

$$= 1.386351/2 = \underline{.693175}$$

There was an error in the question! It should have also asked for the same calculation on [0,1].

On [0,1]: $\int_0^1 f(x)\,dx = \frac{1}{2}\int_{-1}^1 f(0.5t+0.5)\,dt = \ldots = 1.377049/2 = \underline{0.6885245}$

Total estimate: $\int_0^3 f(x)\,dx \approx 2.29859$ (sum)

Error: $2.30\cancel{2585} - 2.29859 = 0.03995$