MAT 2384 C Winter 2019 Assignment 3 Solutions

1. $x^2y'' - 4xy' + 6y = 0$, $y(1)=4$, $y'(1)=10$

the characteristic equation is $m(m-1)-4m+6=0$

or $m^2-5m+6=(m-2)(m-3)=0$ and then the roots

are $m_1=2$ and $m_2=3$

and the general solution is $y(x)=c_1x^2+c_2x^3$

$$\left.\begin{aligned} &y(1)=4 \Rightarrow 4=c_1(1)^2+c_2(1)^3 \Rightarrow c_1+c_2=4 \\ &y'(x)=2c_1x+3c_2x^2 \\ &y'(1)=10 \Rightarrow 10=2c_1(1)+3c_2(1)^2 \Rightarrow 2c_1+3c_2=10 \end{aligned}\right\} \begin{aligned} c_1=2 \\ c_2=2 \end{aligned}$$

$\therefore$ the unique solution is $\boxed{y(x)=2x^2+2x^3}$

2. $x^2y''-5xy'+13y=0$, $x>0$, $y(1)=0$; $y'(1)=6$

the char. eq. is $m(m-1)-5m+13=m^2-6m+13=0$

the roots are $\lambda_{1,2}=\dfrac{6\pm\sqrt{(-6)^2-4(1)(13)}}{2}=\dfrac{6\pm\sqrt{-16}}{2}=3\pm 2i$

and the general solution is $y(x)=c_1x^3\cos(2\ln x)+c_2x^3\sin(2\ln x)$

$y(1)=0 \Rightarrow 0=c_1(1)^3\cos(0)+c_2(1)^3\sin(0) \Rightarrow c_1=0$

$$y'(x)=3c_1x^2\cos(2\ln x)-2c_1x^2\sin(2\ln x)+3c_2x^2\sin(2\ln x)$$
$$+2c_2x^2\cos(2\ln x)$$

$$y'(1)=6 \Rightarrow 6=3c_1(1)^2\cos(0)-2c_1(1)^2\sin(0)+3c_2(1)^2\sin(0)$$
$$+2c_2(1)^2\cos(0) \Rightarrow 3c_1+2c_2=6 \Rightarrow c_2=3$$

$\therefore$ the unique solution is $\boxed{y(x)=3x^3\sin(2\ln x)}$

3. $y''' - 6y'' + 11y' - 6y = 0, \quad y(0)=6,\ y'(0)=14,\ y''(0)=36$

the characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

by inspection, $\lambda=1$ is a root, so $\lambda^3-6\lambda^2+11\lambda-6 = (\lambda-1)(\lambda^2-5\lambda+6)$

so $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$ $\qquad = (\lambda-1)(\lambda-2)(\lambda-3)$

the general solution is $y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

$y(0) = 6 \Rightarrow 6 = c_1 e^0 + c_2 e^0 + c_3 e^0 \Rightarrow c_1 + c_2 + c_3 = 6$ ①

$y'(x) = c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x}$

$y'(0) = 14 \Rightarrow 14 = c_1 e^0 + 2c_2 e^0 + 3c_3 e^0 \Rightarrow c_1 + 2c_2 + 3c_3 = 14$ ②

$y''(x) = c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}$

$y''(0) = 36 \Rightarrow 36 = c_1 e^0 + 4c_2 e^0 + 9c_3 e^0 \Rightarrow c_1 + 4c_2 + 9c_3 = 36$ ③

$\left.\begin{array}{ll} ③ - ① & 3c_2 + 8c_3 = 30 \quad ④ \\ ② - ① & c_2 + 2c_3 = 8 \quad ⑤ \end{array}\right\}$ ④ − 3×⑤ $\Rightarrow 2c_3 = 6 \Rightarrow c_3 = 3$

then $c_2 = 2$ and $c_1 = 1$

∴ the unique solution is $\boxed{y(x) = e^x + 2e^{2x} + 3e^{3x}}$

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4. $x^3 y''' - x^2 y'' + 2xy' - 2y = 0, \quad x>0, \quad y(1) = -3,\ y'(1) = -7,\ y''(1) = -9$

the characteristic equation is $m(m-1)(m-2) - m(m-1) + 2m - 2 = 0$

or $m(m-1)(m-2) - m(m-1) + 2(m-1) = 0$

or $(m-1)\big(m(m-2) - (m-2)\big) = 0$

so $(m-1)^2(m-2) = 0 \Rightarrow m_{1,2} = 1,\ m_3 = 2$

so the general solution is $y(x) = c_1 x + c_2 x \ln x + c_3 x^2$

$y(1) = -3 \Rightarrow -3 = c_1(1) + c_2(1)\ln(1) + c_3(1)^2 \Rightarrow c_1 + c_3 = -3$ ①

$y'(x) = c_1 + c_2 \ln x + c_2 + 2c_3 x$

$y'(1) = -7 \Rightarrow -7 = c_1 + c_2 \ln(1) + c_2 + 2c_3(1) \Rightarrow c_1 + c_2 + 2c_3 = -7$ ②

$y''(x) = c_2/x + 2c_3$

$y''(1) = -9 \Rightarrow -9 = c_2/(1) + 2c_3 \Rightarrow c_2 + 2c_3 = -9$ ③

② − ③ $\Rightarrow c_1 = 2 \Rightarrow c_3 = -5 \Rightarrow c_2 = 1$

∴ the unique solution is $\boxed{y(x) = 2x + x\ln x - 5x^2}$

5. $y'' - 4y = 12e^{2x} + 8x - 16, \quad y(0)=10, \; y'(0)=1$

the corresponding homogeneous equation is $y''-4y=0$, which has characteristic equation $\lambda^2-4=(\lambda+2)(\lambda-2)=0$
and so $y_h(x) = C_1e^{2x} + C_2e^{-2x}$

(Mod Rule needed)

$r(x) = 12e^{2x} + 8x - 16 \Rightarrow y_p(x) = axe^{2x} + bx + c$

then $y_p'(x) = ae^{2x} + 2axe^{2x} + b$

and $y_p''(x) = 4ae^{2x} + 4axe^{2x}$

so $y_p'' - 4y_p = 4ae^{2x} + 4axe^{2x} - 4(axe^{2x} + bx + c)$

$= 4ae^{2x} - 4bx - 4c$

$= r(x) = 12e^{2x} + 8x - 16$

so $a=3$, $b=-2$ and $c=4$ and the particular solution is $y_p(x) = 3xe^{2x} - 2x + 4$

the general solution is $y_g(x) = y_h(x) + y_p(x) = C_1e^{2x} + C_2e^{-2x} + 3xe^{2x} - 2x + 4$

$y(0)=10 \Rightarrow 10 = C_1e^0 + C_2e^0 - 3(0)e^0 - 2(0) + 4 \Rightarrow C_1 + C_2 = 6$

$y_g'(x) = 2C_1e^{2x} - 2C_2e^{-2x} + 3e^{2x} + 6xe^{2x} - 2$

$y'(0)=1 \Rightarrow 1 = 2C_1e^0 - 2C_2e^0 + 3e^0 + 6(0)e^0 - 2 \Rightarrow 2C_1 - 2C_2 = 0$

so $C_1 = C_2 = 3$

$\therefore$ the unique solution is $\boxed{y(x) = 3e^{2x} + 3e^{-2x} + 3xe^{2x} - 2x + 4}$

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6. $y''' - 4y'' + y' + 6y = 10\cos x + 18x + 3, \quad y(0)=5, \; y'(0)=17, \; y''(0)=33$

the corresp. homog. DE is $y''' - 4y'' + y' + 6y = 0$, which has char. eq. $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$

by inspection, $\lambda=-1$ is a root, so

$\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda+1)(\lambda^2 - 5\lambda + 6) = (\lambda+1)(\lambda-2)(\lambda-3) = 0$

ie $\lambda_1 = -1$, $\lambda_2 = 2$ and $\lambda_3 = 3$, so we have

$y_h(x) = C_1e^{-x} + C_2e^{2x} + C_3e^{3x}$

$r(x) = 10\cos x + 18x + 3 \Rightarrow y_p(x) = a\cos x + b\sin x + cx + d$

$y_p'(x) = -a\sin x + b\cos x + c$ and $y_p''(x) = -a\cos x - b\sin x$

and $y_p'''(x) = a\sin x - b\cos x$

so $y_p''' - 4y_p'' + y_p' + 6y_p = a\sin x - b\cos x - 4(-a\cos x - b\sin x)$
$- a\sin x + b\cos x + c + 6(a\cos x + b\sin x + cx + d)$

$= 10a\cos x + 10b\sin x + 6cx + 6d + c$

$= r(x) = 10\cos x + 18x + 3$

so $a = 1$, $b = 0$, $c = 3$ and $d = 0$

then $y_p(x) = \cos x + 3x$

the general solution is $y_g(x) = c_1e^{-x} + c_2e^{2x} + c_3e^{3x} + \cos x + 3x$

$y(0) = 5 \Rightarrow 5 = c_1e^0 + c_2e^0 + c_3e^0 + \cos(0) + 3(0) \Rightarrow c_1 + c_2 + c_3 = 4$ ①

$y_g'(x) = -c_1e^{-x} + 2c_2e^{2x} + 3c_3e^{3x} - \sin x + 3$

$y'(0) = 17 \Rightarrow 17 = -c_1e^0 + 2c_2e^0 + 3c_3e^0 - \sin(0) + 3 \Rightarrow -c_1 + 2c_2 + 3c_3 = 14$ ②

$y_g''(x) = c_1e^{-x} + 4c_2e^{2x} + 9c_3e^0 - \cos x$

$y''(0) = 33 \Rightarrow 33 = c_1e^0 + 4c_2e^0 + 9c_3e^0 - \cos(0) \Rightarrow c_1 + 4c_2 + 9c_3 = 34$ ③

then ① + ② $\quad 3c_2 + 4c_3 = 18$
③ − ① $\quad 3c_2 + 8c_3 = 30$ $\Big\}$ $c_3 = 3 \Rightarrow c_2 = 2 \Rightarrow c_1 = -1$

$\therefore$ the unique solution is $\boxed{y(x) = -e^{-x} + 2e^{2x} + 3e^{3x} + \cos x + 3x}$

7. $y'' + 9y = \csc(3x) \quad , \quad y(\frac{\pi}{2}) = 1 \, , \; y'(\frac{\pi}{2}) = 0$

<u>Step 1</u> $y'' + 9y = 0$ has char eq. $\lambda^2 + 9 = 0$

with roots $\lambda = \pm 3i$

G.S. to hom. eq: $y_h = c_1 \cos 3x + c_2 \sin 3x$

<u>Step 2</u> Variation of parameters:

we seek $y_p = u(x) \cos 3x + v(x) \sin 3x$

$$\underbrace{\begin{bmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{bmatrix}}_{M} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ \csc(3x) \end{bmatrix}$$

$|M| = 3$.

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ \csc 3x \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3\cos 3x & -\sin 3x \\ 3\sin 3x & \cos 3x \end{bmatrix} \begin{bmatrix} 0 \\ \csc 3x \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \cot 3x \end{bmatrix}$$

$u(x) = -\frac{x}{3}$.

$v(x) = \frac{1}{9} \ln |\sin 3x|$

G.S.: $y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{3} \cos 3x$

$+ \frac{1}{9} \ln |\sin 3x| \sin 3x$

7 (cont.)

check:
$$y_p = -\frac{x}{3}\cos 3x + \frac{1}{9}\ln|\sin 3x|\sin 3x$$
$$y_p' = x\sin 3x + \frac{1}{3}\ln|\sin 3x|\cos 3x$$
$$y_p'' = \sin 3x + 3x\cos 3x$$
$$+ \cot 3x\cos 3x - (\ln|\sin 3x|)\sin 3x$$
$$= -9y_p + \underbrace{\left(\sin 3x + \frac{\cos^2 3x}{\sin 3x}\right)}_{= \frac{1}{\sin 3x} = \csc 3x} \quad ✓$$

I.C.:
$$y\left(\tfrac{\pi}{2}\right) = c_2 \sin\frac{3\pi}{2} + \frac{1}{9}\underbrace{\ln\left|\sin\frac{3\pi}{2}\right|}_{=0}\sin\frac{3\pi}{2}$$
$$= -c_2$$
$$= 1 \quad \text{(given)}$$
$$\Rightarrow c_2 = -1$$

$$y'\left(\tfrac{\pi}{2}\right) = -3c_1\sin\frac{3\pi}{2} + \frac{\pi}{2}\sin\frac{3\pi}{2}$$
$$= 3c_1 - \frac{\pi}{2}$$
$$= 0 \quad \text{(given)}$$
$$\Rightarrow c_1 = \frac{\pi}{6}$$

So the unique solution is:
$$y = \frac{\pi}{6}\cos 3x - \sin 3x - \frac{x}{3}\cos 3x + \frac{1}{9}\ln|\sin 3x|\sin 3x$$

8. This is a nonhomogeneous Euler-Cauchy equation.

Step 1 The associated homogeneous equation is

$$x^2y'' - 2xy' + 2y = 0$$

char eq: $m(m-1) - 2m + 2 = 0$

$\Leftrightarrow m^2 - 3m + 2 = 0$

$\Leftrightarrow m = 1$ or 2

G.S: $y_h = c_1 x + c_2 x^2$

Step 2 We didn't see nonhom. E-C. equations in class. But we did see that E-C equations can be transformed to linear ones with the substitution $t = \ln x$, which in this case leads to:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{2t}$$

This can be solved by either Undet. Coeffs or Var of Params.

Undet Coeffs method: $y_h = c_1e^t + c_2e^{2t}$, $r(t) = e^{2t}$.

Since e^{2t} is part of y_h already,

we try $y_p = ate^{2t}$.

$y_p' = ae^{2t} + 2ate^{2t}$

$y_p'' = 2ae^{2t} + 2ae^{2t} + 4ate^{2t}$

(writing $y' = \frac{dy}{dt}$ now)

8 (cont.) $y_p'' - 3y_p' + 2y_p$

$= 4ae^{2t} + \cancel{4ate^{2t}} - 3ae^{2t} - \cancel{6ate^{2t}} + \cancel{2ate^{2t}}$

$= ae^{2t}$

This equals $r(t)$ if $a = 1$.

Thus $y_p = te^{2t}$.

The ~~unique~~ general solution to the non-hom eq. is

$y(t) = c_1 e^t + c_2 e^{2t} + te^{2t}$.

changing variables again, with $x = e^t$,

$y(x) = c_1 x + c_2 x^2 + (\ln x) x^2$

$= c_1 x + (c_2 + \ln x) x^2$

Step 3 I.C.'s: $y(1) = c_1 + c_2 = 4$ ... (1)

$y'(x) = c_1 + 2c_2 x + x + 2(\ln x) x)$

$y'(1) = c_1 + 2c_2 + 1 = 5$ ... (2)

Subst $c_1 = 4 - c_2$ from (1) into (2):

$(4 - c_2) + 2c_2 + 1 = 5$

$\Rightarrow c_2 = 0$

$\Rightarrow c_1 = 4$

$\therefore$ The unique solution is $y = 4x + x^2 \ln x$

9. Step 1 Hom eq.: $y'' - 4y' + 4y = 0$

char eq: $\lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

double root $\lambda = 2$.

GS: $y_h = c_1 e^{2x} + c_2 x e^{2x}$

Step 2 Variation of parameters:

$y_p = u(x) e^{2x} + v(x) x e^{2x}$

$$\underbrace{\begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{bmatrix}}_{M} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ 3x^{-2} e^{2x} \end{bmatrix}$$

$W = |M| = e^{4x}(1 + 2x - 2x) = e^{4x}$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ 3x^{-2} e^{2x} \end{bmatrix} = e^{-4x} \begin{bmatrix} (1+2x)e^{2x} & -xe^{2x} \\ -2e^{2x} & e^{2x} \end{bmatrix} \begin{bmatrix} 0 \\ 3x^{-2} e^{2x} \end{bmatrix}$$

$$= e^{-4x} \begin{bmatrix} -\frac{3}{x} e^{4x} \\ 3x^{-2} e^{4x} \end{bmatrix} = \begin{bmatrix} -\frac{3}{x} \\ \frac{3}{x^2} \end{bmatrix}$$

$u' = -\frac{3}{x} \Rightarrow u = -3 \ln x$

$v' = \frac{3}{x^2} \Rightarrow v = -\frac{3}{x}$

9 (cont.) $y_p = -3(\ln x)e^{2x} - 3e^{2x}$

$\therefore$ G.S.: $y = y_h + y_p = c_1 e^{2x} + c_2 x e^{2x} - 3(\ln x)e^{2x} - 3e^{2x}$

$$= e^{2x}(c_1 + c_2 x - 3\ln x - 3)$$

<u>Step 3</u> I.C.: $y(1) = e^2(c_1 + c_2 - 3)$

$$= 2e^2 \quad \text{(I.C.)}$$

$$\Rightarrow c_1 + c_2 = 5 \quad \cdots (1)$$

$$y'(x) = 2e^{2x}(c_1 + c_2 x - 3\ln x - 3) + e^{2x}\left(c_2 - \frac{3}{x}\right)$$

$$y'(1) = 2e^2(c_1 + c_2 - 3) + e^2(c_2 - 3)$$

$$= e^2(2c_1 + 3c_2 - 9)$$

$$= 2e^2 \quad \text{(I.C.)}$$

$$\Rightarrow 2c_1 + 3c_2 = 11 \quad \cdots (2)$$

From (1), $c_1 = 5 - c_2$. Subst. into (2):

$$2(5 - c_2) + 3c_2 = 11$$

$$\Rightarrow 10 + c_2 = 11$$

$$\Rightarrow c_2 = +1$$

$$\Rightarrow c_1 = 4$$

Unique solution: $y = e^{2x}(1 + x - 3\ln x)$

10. a)

x_j	f_j	$f[x_j, x_{j+1}]$	$f[x_j, x_{j+1}, x_{j+2}]$
0.2	$\boxed{0.4651}$		
		$\boxed{1.3425}$	
0.4	0.7336		$\boxed{-4.41702}$
		-1.08686	
0.75	0.3532		~~4.01953~~ 5.02441
		2.93267	
1.2	1.6729		

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{5.02441 + 4.41702}{1.2 - 0.2} = \boxed{9.4414}$$

$$p_3(x) = f_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$= 0.4651 + 1.3425(x - x_0) - 4.41702(x - x_0)(x - x_1) + 9.4414(x - x_0)(x - x_1)(x - x_2)$$

(This is an acceptable final answer.)

b) $p_3(1) = 0.4651 + 1.3425(0.8) - 4.41702(0.8)(0.6) + 9.4414(0.8)(0.6)(0.25)$

$= 0.4651 + 1.074 - 2.1202 + 1.1330$

$= 0.5519$

10 c) Since $p_3(x)$ is also the Lagrange interpolating polynomial, the error approximation for Lagrange poly's applies to Newton D.D. too.

$$\epsilon_3(x) = \frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$$\epsilon_3(1) = \frac{f^{(4)}(\xi)}{4!}(0.8)(0.6)(0.25)(-0.2)$$

$$|\epsilon_3(1)| = \frac{|f^{(4)}(\xi)|}{24}(+0.024)$$

$$= \frac{|f^{(4)}(\xi)|}{1000} \quad \text{for some } \xi \in (0.2, 1.2)$$

We are given that $1.2 \le f^{(4)}(x) \le 273$ on this interval, so

$$\frac{1.2}{1000} \le |\epsilon_3(1)| \le \frac{273}{1000}$$

i.e. $$0.0012 \le |\epsilon_3(1)| \le 0.273$$