MAT2384C Winter2019 Assignment 1 Solutions

1. $\cot x \sin y \, y' = 1 \quad , \quad y(0) = 0$

this DE is also separable: $\quad \sin y \, y' = \tan x$

and so $\quad \int \sin y \, dy = \int \frac{\sin x}{\cos x} \, dx + C$

which gives $\quad -\cos y = -\ln|\cos x| + C$

or $\quad \cos y = \ln|\cos x| + C$

and so the general solution is $y = \arccos(\ln|\cos x| + C)$

$y(0) = 0 \quad \Rightarrow \quad 0 = \arccos(\ln(\cos(0)) + C)$

$0 = \arccos(\ln(1) + C) = \arccos(C)$

thus $C = 1$. and the unique solution is

$$\boxed{y = \arccos(\ln|\cos x| + 1)}$$

---

2. $xy' = y + x\sec(y/x) \quad , \quad y(1) = \pi/2$

rewrite as $\quad (y + x\sec(y/x))\,dx - x\,dy = 0$

then $\left.\begin{aligned} M(x,y) &= y + x\sec(y/x) \\ N(x,y) &= -x \end{aligned}\right\}$ both homogeneous of degree 1

(continued next page)

so let $y = ux$, $dy = u\,dx + x\,du$ and the DE will become

$$(ux + x\sec u)\,dx - x\,(u\,dx + x\,du) = 0$$

$$\cancel{ux\,dx} + x\sec u\,dx - \cancel{xu\,dx} - x^2\,du = 0$$

so $\quad x\sec u\,dx - x^2\,du = 0$

or $\quad \cos u\,du = \frac{1}{x}\,dx \quad$ (separable)

thus $\quad \int \cos u\,du = \int \frac{1}{x}\,dx + C$

we get $\quad \sin u = \ln|x| + C$

and so $\quad u = \arcsin(\ln|x| + C)$

but $u = y/x$, so $\quad y = x\arcsin(\ln|x| + C)$

$y(1) = \pi/2 \Rightarrow \quad \pi/2 = (1)\arcsin(\ln(1) + C) = \arcsin C \Rightarrow C = 1$

$\therefore$ the unique solution is $\boxed{y = x\arcsin(\ln|x| + 1)}$

---

3. $(\cos y + y^2\cos x + 1)\,dx + (2y\sin x - x\sin y)\,dy = 0$, $y(\pi) = \pi$

$M(x,y) = \cos y + y^2\cos x + 1 \quad \Rightarrow M_y = -\sin y + 2y\cos x$

$N(x,y) = 2y\sin x - x\sin y \quad \Rightarrow N_x = 2y\cos x - \sin y$

$\Big\}$ $M_y = N_x$ DE exact

$F(x,y) = \int M(x,y)\,dx + g(y) \quad$ (or $\int N(x,y)\,dy + g(x)$)

$\quad = \int (\cos y + y^2\cos x + 1)\,dx + g(y)$

$\quad = x\cos y + y^2\sin x + x + g(y)$

then $\frac{\partial F}{\partial y} = -x\sin y + 2y\sin x + g'(y)$

$\quad = N(x,y) = 2y\sin x - x\sin y \quad \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0$

so $F(x,y) = x\cos y + y^2\sin x + x$

and the general solution $\quad x\cos y + y^2\sin x + x = C$

$y(\pi) = \pi \quad \Rightarrow \quad \pi\cos(\pi) + (\pi)^2\sin(\pi) + \pi = C \quad \Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{x\cos y + y^2\sin x + x = 0}$

---

4. $(4xy^4 + 3y^2)\,dx + (8x^2y^3 + 6xy + 3y^2)\,dy = 0, \quad y(1) = -1$

$M(x,y) = 4xy^4 + 3y^2 \Rightarrow M_y = 16xy^3 + 6y$
$N(x,y) = 8x^2y^3 + 6xy + 3y^2 \Rightarrow N_x = 16xy^3 + 6y$
$\Big\}$ $M_y = N_x$ DE is exact

then $F(x,y) = \int M(x,y)\,dx + g(y) \quad \left(\underline{\text{or}} \int N(x,y)\,dy + g(x)\right)$

$= \int (4xy^4 + 3y^2)\,dx + g(y)$

$= 2x^2y^4 + 3xy^2 + g(y)$

so $\frac{dF}{dy} = 8x^2y^3 + 6xy + g'(y) = N(x,y) = 8x^2y^3 + 6xy + 3y^2$

then $g'(y) = 3y^2 \Rightarrow g(y) = y^3$

then $F(x,y) = 2x^2y^4 + 3xy^2 + y^3$

and the general solution is $2x^2y^4 + 3xy^2 + y^3 = C$

$y(1) = -1 \Rightarrow 2(1)^2(-1)^4 + 3(1)(-1)^2 + (-1)^3 = C \Rightarrow C = 4$

$\therefore$ the unique solution is $\boxed{2x^2y^4 + 3xy^2 + y^3 = 4}$

5. $\underbrace{(x \ln y^4 + 4\ln y - y^2)}_{M} dx + \underbrace{\left(-2y + \frac{4x}{y}\right)}_{N} dy = 0$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{4x}{y} + \frac{4}{y} - 2y \\ \frac{\partial N}{\partial x} &= \frac{4}{y} \end{aligned}\right\} \text{not equal} \;\therefore\; \text{not exact}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = 1 =: f(x)$$

Integrating factor (see DV1.5)

$$\mu(x) = e^{\int f(x)\,dx} = e^x \qquad \text{(no "+c" needed)}$$

$M^* := \mu(x) M \quad , \quad N^* := \mu(x) N$

~~Check:~~ $\frac{\partial M^*}{\partial y} = e^x \frac{\partial M}{\partial y} = e^x\left(\frac{4x}{y} + \frac{4}{y} - 2y\right)$

$\frac{\partial N^*}{\partial x} = e^x\left(N + \frac{\partial N}{\partial x}\right) = e^x\left(-2y + \frac{4x}{y} + \frac{4}{y}\right)$

(check: exact ✓)

$$\begin{aligned} u(x,y) &:= \int N^* \, dy + g(x) \\ &= e^x \int \left(-2y + \frac{4x}{y}\right) dy + g(x) \\ &= e^x(-y^2 + 4x \ln y) + g(x) \end{aligned}$$

5 (cont.)

$$\frac{\partial u}{\partial x} = e^x(-y^2 + 4x\ln y + 4\ln y) + g(x)$$

$$= M^* + g(x)$$

So take $g(x) \equiv 0$

G.S.: $u = c$

i.e. $e^x(x\ln y^4 - y^2) = c$

Particular solution with I.C.:

$y(0) = 1 \Rightarrow e^0(0\ln 1 - 1^2) = c$

$\Rightarrow c = -1$

$$\boxed{e^x(x\ln y^4 - y^2) = -1}$$

6. Let $M(t)$ be the moisture in the laundry at time t. Note: $M(t) \geq 0$ for all t.

$$\frac{dM}{dt} \propto M \qquad \text{("}\propto\text{" means "proportional")}$$

i.e. $$\frac{dM}{dt} = kM \qquad \text{for some constant } k$$

This is separable. $$\int \frac{dM}{M} = \int k\,dt$$

GS: $$\ln|M| = kt + c_0$$

$$\Leftrightarrow \quad M = e^{kt+c_0}$$

$$\Leftrightarrow \quad M = Ce^{kt}$$

Let $t=0$ when the dryer starts.

We know $M(10) = \frac{1}{2}M(0)$.

So $$Ce^{10k} = \frac{C}{2}$$

$$\Rightarrow \quad k = -\frac{1}{10}\log 2 \approx -0.0693$$

We want T s.t. $M(T) = \frac{1}{10}M(0)$,

i.e. $$\cancel{C}e^{kT} = \frac{1}{10}\cancel{C}$$

$$\Leftrightarrow \quad T = \frac{1}{k}\log\left(\frac{1}{10}\right) \approx 33.2$$

The laundry will be 90% dry in 33 minutes.

7. $f(x) = x^3 - 4x - 2$

a) $f(0) = 2 > 0$, $f(1) = -1 < 0$.

f is continuous, so by the Intermediate Value Thm, it has a root in $[0,1]$.

b) $f(x) = 0 \Leftrightarrow 4x = x^3 + 2 \Leftrightarrow x = \frac{1}{4}(x^3+2)$

Let $g(x) = \frac{1}{4}(x^3+2)$.

c) $g'(x) = \frac{3}{4}x^2$

$|g'(x)| \leq \frac{3}{4} < 1$ for all $x \in [0,1]$.

Also, for all $x \in [0,1]$,

$0 < g(x) \leq \frac{3}{4}$, so $g(x) \in [0,1]$.

Hence iteration of g with $x_0 = 0.5$ will converge to a fixed point.

(Note: if you picked a different function g, this proof attempt might fail, in which case, you should go back and try a different g.)

d) $x_0 = 0.5$

$x_1 = \frac{1}{4}((0.5)^3 + 2) = 0.53125$

$x_2 = 0.5374832$

$x_3 = 0.5388181$

$x_4 = 0.5391081$

$x_5 = 0.539173$

$x_6 = 0.5391850$

$x_7 = 0.5391880$

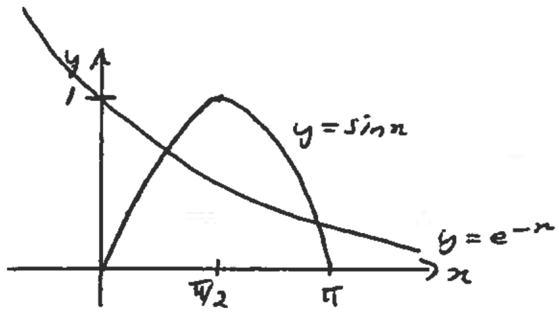
$x_8 = 0.5391887$

$x_9 = 0.5391888$

$\boxed{\text{The root is } 0.539189}$ (to 6d.p.

check: $f(0.539189) \simeq -4 \times 10^{-7}$ ✓

8.



let $f(x) = e^{-x} - \sin x$

then $f'(x) = -e^{-x} - \cos x$

Newton's Method : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{e^{-x_n} - \sin(x_n)}{-e^{-x_n} - \cos(x_n)}$$

$$= x_n + \frac{e^{-x_n} - \sin(x_n)}{e^{-x_n} + \cos(x_n)}$$

$x_0 = \pi/4$, $x_1 = \pi/4 + \dfrac{e^{-\pi/4} - \sin(\pi/4)}{e^{-\pi/4} + \cos(\pi/4)} = 0.569440$

$x_2 = 0.588389$

$x_3 = 0.588533$

$x_4 = 0.588533 = x_3$: stop

(check: $e^{-0.588533} = 0.555141$

$\sin(0.588533) = 0.555141$ okay!)

∴ the solution to 6 decimal places is $\boxed{0.588533}$