MAT 2384 C Midterm Solutions

1. $$\frac{\partial M}{\partial y} = 12xy - 9x^2y^2 + 2$$

$$\frac{\partial N}{\partial x} = 18xy - 12x^2y^2 + 4$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{not exact.}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -6xy + 3x^2y^2 - 2 = \frac{-M}{y}$$

i.e. $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = -\frac{1}{y}$$

Integrating factor: $\mu(y) = e^{-\int\left(-\frac{1}{y}\right)dy} = |y|$

Drop the absolute value, by considering only $y > 0$ (as in the I.C.)

$\mu = y$. Multiply original ODE:

$$\underbrace{(6xy^3 - 3x^2y^4 + 2y^2)}_{M'}dx + \underbrace{(9x^2y^2 - 4x^3y^3 + 4xy)}_{N'}dy = 0$$

$$u(x,y) := \int M'\,dx = 3x^2y^3 - x^3y^4 + 2y^2x$$

(optional check: $\frac{\partial u}{\partial y} = N'$ ✓)

GS: $3x^2y^3 - x^3y^4 + 2y^2x = c$

IC: $y(1) = 1 \rightarrow 3 - 1 + 2 = c \Rightarrow c = 4$.

unique solution: $\boxed{3x^2y^3 - x^3y^4 + 2y^2x = 4}$

2. $y' - \frac{1}{x} y = x y^2, \quad x > 0, \quad y(1) = 1.$

This is a Bernoulli equation with $a = 2$.

Substitute $u = y^{1-a} = \frac{1}{y}$ (assume $y \neq 0$)

$$y = \frac{1}{u}$$

$$y' = -\frac{u'}{u^2}$$

The original ODE is equivalent to:

$$\left(-\frac{u'}{u^2}\right) - \frac{1}{x}\frac{1}{u} = x \frac{1}{u^2}$$

$$\Leftrightarrow \quad u' + \underbrace{\frac{1}{x}}_{f(x)} u = \underbrace{-x}_{r(x)}$$

This has general solution:

$$u = \frac{\int e^{\int f(x)\,dx} r(x)\,dx + C}{e^{\int f(x)\,dx}} = \frac{\int (x)(-x)\,dx + c}{x}$$

$$= -\frac{x^2}{3} + \frac{c}{x}$$

$$y = \frac{1}{u} = \frac{1}{\left(\frac{-x^3 + 3c}{3x}\right)} = \frac{3x}{3c - x^3}$$

I.C.: $y(1) = 1 \Rightarrow 1 = \frac{3}{3c-1} \Rightarrow 3c = 4$

$$\therefore \quad y = \frac{3x}{4 - x^3}$$

3. Step 1 hom eq: $y'' - 4y' + 4y = 0$

char eq: $\lambda^2 - 4\lambda + 4 = 0$

$\Leftrightarrow (\lambda - 2)^2 = 0$

double root: $\lambda = 2$.

G.S.: $y_h = c_1 e^{2x} + c_2 x e^{2x}$

Step 2 $y_p = ax^2 e^{2x} + b\cos x + c\sin x$

(modification rule used)

$y_p' = 2axe^{2x} + 2ax^2e^{2x} - b\sin x + c\cos x$

$y_p'' = 2ae^{2x} + 8axe^{2x} + 4ax^2e^{2x} - b\cos x - c\sin x$

$$y_p'' - 4y_p' + 4y_p = e^{2x}(2a + \cancel{8ax} + \cancel{4ax^2} - \cancel{8ax} - \cancel{8ax^2} + \cancel{4ax^2})$$
$$- b\cos x - c\sin x + 4b\sin x - 4c\cos x$$
$$+ 4b\cos x + 4c\sin x$$

$$= e^{2x}(2a) + \cos x(3b - 4c) + \sin x(3c + 4b)$$

This equals $6e^{2x} + 25\cos x$ if and only if

$a = 3$ and $\begin{cases} 3b - 4c = 25 \\ 3c + 4b = 0 \end{cases} \Leftrightarrow b = 3,\ c = -4$

$\therefore\ y = y_h + y_p = c_1 e^{2x} + c_2 x e^{2x} + 3x^2 e^{2x} + 3\cos x - 4\sin x$

4a) $L_0 = \dfrac{(x-0.5)(x-0.75)}{(0.3-0.5)(0.3-0.75)} = \dfrac{(x-0.5)(x-0.75)}{0.09}$

$= 11.1111\,(x-0.5)(x-0.75) = 11.1111\,(x^2 - 1.25x + 0.37$

$= 11.1111\,x^2 - 13.8889x + 4.1667$

$L_1 = \dfrac{(x-0.3)(x-0.75)}{(0.5-0.3)(0.5-0.75)} = \dfrac{(x-0.3)(x-0.75)}{-0.05}$

$= -20\,(x^2 - 1.05x + 0.225)$

$= -20x^2 + 21x - 4.5$

$L_2 = \dfrac{(x-0.3)(x-0.5)}{(0.75-0.3)(0.75-0.5)} = \dfrac{(x-0.3)(x-0.5)}{0.1125}$

$= 8.8889\,[x^2 - 0.8x + 0.15]$

$= 8.8889\,x^2 - 7.1111\,x + 1.3333$

$$p_2(x) = 1.0782 L_0 + 1.1797 L_1 + 1.2854 L_2$$

$$= 11.9800(x-0.5)(x-0.75) - 23.5940(x-0.3)(x-0.75)$$

$$+ 11.4258(x-0.3)(x-0.5)$$

$$= -0.1882x^2 + 0.6581x + 0.8977$$

b) $p_2(0.6) = 1.2248$

c) $|\epsilon_2(0.6)| := |f(0.6) - p_2(0.6)|$

$$|\epsilon_2(0.6)| = |(0.6-0.3)(0.6-0.5)(0.6-0.75)| \left|\frac{f'''(\xi)}{3!}\right|$$

$$= \frac{(0.3)(0.1)(0.15)}{6} |f'''(\xi)|$$

$$= 0.00075 \, |f'''(\xi)| \quad \text{for some } \xi \in (0.3, 0.75)$$

Since $2.7 \leq f'''(\xi) \leq 42.0$,

$$(2.7)(0.00075) \leq |\epsilon_2(0.6)| \leq (42)(0.00075)$$

$$0.002025 \leq |\epsilon_2(0.6)| \leq 0.0315$$