MAT 2384 Assignment # 2 Solutions

1. $(12xy^2 - 21\ln y + 5x^2)dx + (6x^2y - 7x/y)dy = 0$, $y(1) = 1$

$M(x,y) = 12xy^2 - 21\ln y + 5x^2 \Rightarrow M_y = 24xy - 21/y$
$N(x,y) = 6x^2y - 7x/y \Rightarrow N_x = 12xy - 7/y$ } $M_y \neq N_x$ DE not exact

$$\frac{M_y - N_x}{N} = \frac{12xy - 14/y}{6x^2y - 7x/y} = \frac{2}{x} \quad \text{(a function of } x \text{ only)}$$

the integrating factor is $\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$
and the DE becomes

$(12x^3y^2 - 21x^2\ln y + 5x^4)dx + (6x^4y - 7x^3/y)dy = 0$

$M^*(x,y) = 12x^3y^2 - 21x^2\ln y + 5x^4 \Rightarrow M_y^* = 24x^3y - 21x^2/y$
$N^*(x,y) = 6x^4y - 7x^3/y \Rightarrow N_x^* = 24x^3y - 21x^2/y$ } $M_y^* = N_x^*$ DE now exact

$F(x,y) = \int N^*(x,y)\,dy + g(x)$ (or $\int M^*(x,y)dx + g(y)$)
$= \int (6x^4y - 7x^3/y)dy + g(x)$
$= 3x^4y^2 - 7x^3\ln y + g(x)$

then $\frac{\partial F}{\partial x} = 12x^3y^2 - 21x^2\ln y + g'(x)$
$= M^*(x,y) = 12x^3y^2 - 21x^2\ln y + 5x^4$

thus $g'(x) = 5x^4 \Rightarrow g(x) = x^5$

and so $F(x,y) = 3x^4y^2 - 7x^3\ln y + x^5$ and the general solution is $3x^4y^2 - 7x^3\ln y + x^5 = C$

$y(1) = 1 \Rightarrow 3(1)^4(1)^2 - 7(1)^3\ln(1) + (1)^5 = C \Rightarrow C = 4$

$\therefore$ the unique solution is $\boxed{3x^4y^2 - 7x^3\ln y + x^5 = 4}$

---

2. $(-2y\sin x + 2xy^3 - e^xy^2)dx + (4\cos x + 4x^2y^2 - 3e^xy)dy = 0$, $y(0) = 2$

$M(x,y) = -2y\sin x + 2xy^3 - e^xy^2 \Rightarrow M_y = -2\sin x + 6xy^2 - 2e^xy$
$N(x,y) = 4\cos x + 4x^2y^2 - 3e^xy \Rightarrow N_x = -4\cos x + 8xy^2 - 3e^xy$ } $M_y \neq N_x$ DE not exact

$$\frac{M_y - N_x}{M} = \frac{2\sin x - 2xy^2 + e^xy}{-2y\sin x + 2xy^3 - e^xy^2} = \frac{-1}{y} \quad \text{(a function of } y \text{ only)}$$

and so $\mu(y) = e^{-\int \frac{-1}{y}dy} = e^{\ln y} = y$ and the DE becomes

$$(-2y^2\sin x + 2xy^4 - e^x y^3)\,dx + (4y\cos x + 4x^2y^3 - 3e^x y^2)\,dy = 0$$

$$M^*(x,y) = -2y^2\sin x + 2xy^4 - e^x y^3 \Rightarrow M^*_y = -4y\sin x + 8xy^3 - 3e^x y^2$$

$$N^*(x,y) = 4y\cos x + 4x^2y^3 - 3e^x y^2 \Rightarrow N^*_x = -4y\sin x + 8xy^3 - 3e^x y^2$$

$\Big\}\ M_y = N_x$ DE now exact

$$F(x,y) = \int M^*(x,y)\,dx + g(y) = \int(-2y^2\sin x + 2xy^4 - e^x y^3)\,dx + g(y)$$

$$= 2y^2\cos x + x^2y^4 - e^x y^3 + g(y)$$

then $\dfrac{\partial F}{\partial y} = 4y\cos x + 4x^2y^3 - 3e^x y^2 + g'(y)$

$= N^*(x,y) = 4y\cos x + 4x^2y^3 - 3e^x y^2 \Rightarrow g'(y) = 0$ take $g(y) = 0$

then $F(x,y) = 2y^2\cos x + x^2y^4 - e^x y^3$ and the general solution is $2y^2\cos x + x^2y^4 - e^x y^3 = C$

$y(0) = 2 \Rightarrow 2(2)^2\cos(0) + (0)^2(2)^4 - e^0(2)^3 = C \Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{2y^2\cos x + x^2y^4 - e^x y^3 = 0}$

---

3. $y' + \frac{2}{x}y = x, \quad y(4) = 3$, linear with $f(x) = \frac{2}{x}$, $r(x) = x$

$\mu(x) = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

the general solution is $y(x) = \frac{1}{\mu(x)}\left[\int \mu(x) r(x)\,dx\right] + C$

$$= x^{-2}\left[\int (x^2)(x)\,dx + C\right]$$

$$= x^{-2}\left(\int x^3\,dx + C\right)$$

$$= x^{-2}\left(\tfrac{1}{4}x^4 + C\right) = \tfrac{1}{4}x^2 + Cx^{-2}$$

then $y(4) = 3 \Rightarrow 3 = \frac{1}{4}(4)^2 + C(4)^{-2} = 4 + \frac{1}{16}C \Rightarrow C = -16$

$\therefore$ the unique solution is $\boxed{y(x) = \frac{1}{4}x^2 - 16x^{-2}}$

---

4. $y' - 2y = 3e^x, \quad y(0) = 4$, linear with $f(x) = -2$, $r(x) = 3e^x$

$\mu(x) = e^{\int -2dx} = e^{-2x}$

general solution $y(x) = e^{2x}\left[\int e^{-2x} 3e^{x}\,dx + C\right]$

$$= e^{2x}\left(3\int e^{-x}\,dx + C\right)$$

$$= e^{2x}\left(-3e^{-x} + C\right) = Ce^{2x} - 3e^{x}$$

$y(0) = 4 \Rightarrow 4 = Ce^{0} - 3e^{0} \Rightarrow C = 7$

$\therefore$ the unique solution is $\boxed{y(x) = 7e^{2x} - 3e^{x}}$

---

5. $y' + y\tan x = y^2, \quad y(0) = 1/2$

this is a Bernoulli equation with $p(x) = \tan x$, $g(x) = 1$, $a = 2$

so let $u = y^{1-a} = y^{-1}$ and the DE becomes

$$u' + (1-a)\,p\,u = (1-a)\,g$$

or $u' - \tan x\; u = -1$

which is linear with $f(x) = -\tan x$, $r(x) = -1$

so $\mu(x) = e^{\int -\tan x\,dx} = e^{\int -\sin x/\cos x\,dx} = e^{\ln\cos x} = \cos x$

so $u(x) = \sec x\left[\int -\cos x\,dx + C\right]$

$$= \sec x\,(-\sin x + C) = C\sec x - \tan x$$

but $y(x) = u(x)^{-1} \Rightarrow y(x) = \dfrac{1}{C\sec x - \tan x}$ (general solution)

then $y(0) = 1/2 \Rightarrow \dfrac{1}{2} = \dfrac{1}{C\sec(0) - \tan(0)} = \dfrac{1}{C} \Rightarrow C = 2$

$\therefore$ the unique solution is $\boxed{y(x) = \dfrac{1}{2\sec x - \tan x}}$

Q6. $y'' - 9y = 0$

ch. eq.: $\lambda^2 - 9 = 0$

$\lambda = \pm 3$

G.S.: $y = c_1 e^{3x} + c_2 e^{-3x}$

I.C.'s: $y(0) = c_1 + c_2 = 2$ ... (1)

$y' = 3c_1 e^{3x} - 3c_2 e^{-3x}$

$y'(0) = 3c_1 - 3c_2 = 0$ ...(2)

~~(1) & (2) ⇒~~ $\cancel{\begin{cases} 3c_1 + 3c_2 = 6 \\ 3c_1 - 3c_2 = 0 \end{cases}}$

From (2), $c_1 = c_2$

From (1), $c_1 = c_2 = 1$

∴ The unique solution of the IVP is

$$\boxed{y = e^{3x} + e^{-3x}}$$

check: $y' = 3e^{3x} - 3e^{-3x}$

$y'' = 9e^{3x} + 9e^{-3x} = 9y$

$\therefore y'' - 9y = 0$ ✓

$y(0) = 2$ ✓ $\quad y'(0) = 0$ ✓

Q7. $y'' + \pi^2 y = 0 \qquad y(0) = 1, \; y'(0) = \pi$

char-eq: $\lambda^2 + \pi^2 = 0$

$\lambda = \pm i\pi$

GS: $y = c_1 \cos \pi x + c_2 \sin \pi x$

IC's: $y(0) = c_1 = 1$

$y' = \pi c_2 \cos \pi x$

$y'(0) = \pi c_2 = \pi \quad \Rightarrow c_2 = 1$

$\therefore$ unique solution to I.V.P.:

$$\boxed{y = \cos \pi x + \sin \pi x}$$

check: $y' = -\pi \sin \pi x + \pi \cos \pi x$

$y'' = -\pi^2 (\cos \pi x + \sin \pi x) = -\pi^2 y$

$\therefore y'' + \pi^2 y = 0$ ✓

$y(0) = 1, \quad y'(0) = \pi$ ✓

Q8. $y'' + 4y' + 4y = 0$, $y(0) = 7$, $y'(0) = -18$

char. eq: $\lambda^2 + 4\lambda + 4 = 0$

$\Leftrightarrow (\lambda+2)^2 = 0$

$\Leftrightarrow \lambda = -2$ (double root)

G.S.: $y = c_1 e^{-2x} + c_2 x e^{-2x}$

I.C.'s: $y(0) = c_1 = 7$

$y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$

$y'(0) = -2c_1 + c_2 = -18$

$\Rightarrow -2(7) + c_2 = -18$

$\Rightarrow c_2 = -4$

$\therefore$ unique solution to I.V.P. is:

$$\boxed{y = 7e^{-2x} - 4xe^{-2x}}$$

Check: $y' = -14e^{-2x} - 4e^{-2x} + 8x e^{-2x}$

$= -18e^{-2x} + 8xe^{-2x}$

$y'' = 36e^{-2x} + 8e^{-2x} - 16xe^{-2x}$

$= 44e^{-2x} - 16xe^{-2x}$

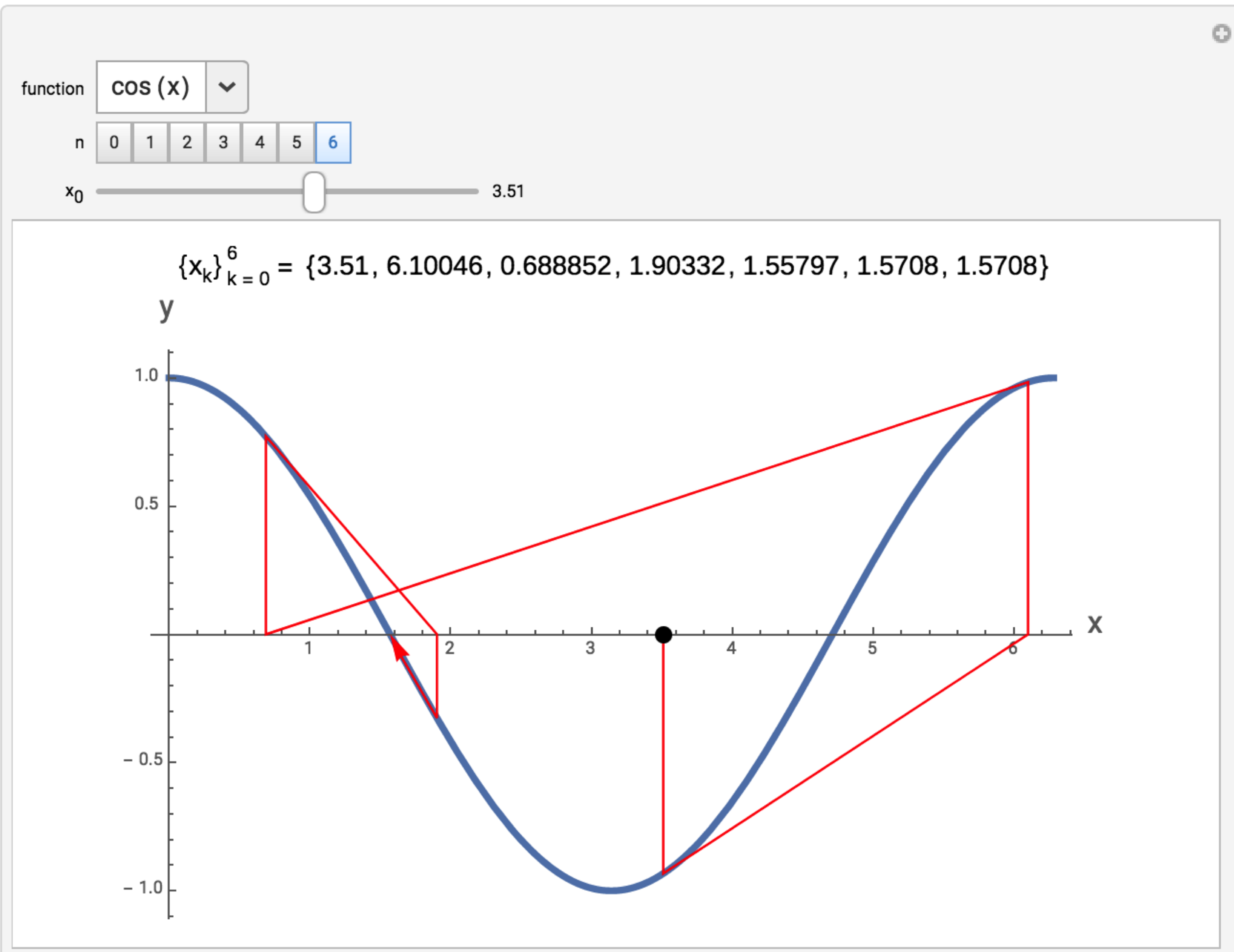
$y'' + 4y' + 4y = e^{-2x}[44 - 16x + 4(-18 + 8x) + 4(7 - 4x)]$

$= e^{-2x}[44 - 72 + 28 - 16x + 32x - 16x]$

$= 0$ ✓

$y(0) = 7$ ✓ $\qquad y'(0) = -18$ ✓

Q 9. a)



function
cos (x)
n
0
1
2
3
4
5
6
x_0
3.51
$\{x_k\}_{k=0}^{6} = \{3.51, 6.10046, 0.688852, 1.90332, 1.55797, 1.5708, 1.5708\}$
y
1.0
0.5
− 0.5
− 1.0
x
1
2
3
4
5
6

9 b) $f(x) = \cos x$, $f'(x) = -\sin x$

$$\frac{f(x)}{f'(x)} = -\cot x$$

$a_0 = \pi$, $x_0 = 3.51$, $b_0 = 2\pi$

Step 1 $c := x_0 - \dfrac{f(x_0)}{f'(x_0)} = 3.51 + \cot(3.51)$

≈ 6.100459 (to 6 d.p.)

$c \in (a_0, b_0)$, so $x_1 := c$.

$f(a_0) = -1$, $f(x_1) \approx 0.983352$, $f(b_0) = 1$

opposite signs, i.e. $f(a_0)f(x_0) < 0$, so:

$a_1 := \pi$, $x_1 = 6.100459$, $b_1 := x_1$

Step 2 $c := x_1 - \dfrac{f(x_1)}{f'(x_1)} = x_1 + \cot(x_1)$

≈ 0.688838

$c \notin (a_1, b_1)$, so $x_2 := \frac{1}{2}(a_1 + b_1)$

$= 4.621026$

$f(a_1) = -1$, $f(x_2) = -0.091236$, $f(b_1) = 0.983352$

opposite signs, so:

$a_2 := x_2 = 4.621026$, $b_2 = b_1 = 6.100459$

9 b) cont.:

<u>Step 3</u> $\quad c := x_2 - \dfrac{f(x_2)}{f'(x_2)} = x_2 + \cot(x_2)$

≈ 4.712644

$c \in (a_2, b_2)$, so $x_3 := c$.

$$\underbrace{f(a_2) < 0 \,,\; f(x_3)}_{\text{opposite signs, so:}} > 0 \,,\; f(b_2) > 0$$

$a_3 := a_2 = 4.621026 \,,\quad x_3 = b_3 = 4.712644$

<u>Step 4</u> $\quad c := x_3 + \cot(x_3) \approx 4.712389$

$c \in (a_3, b_3)$, so $x_4 := c$

$$\underline{\underline{x_4 = 4.712389}}$$

10 a) $L_0(x) = \dfrac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$

$$= \frac{x^3-(x_1+x_2+x_3)x^2+(x_1x_2+x_1x_3+x_2x_3)-x_1x_2x_3}{(-0.5)(-1.1)(-1.4)}$$

$$= (-1.2987)(x^3-6.3x^2+\underbrace{(3.52+4+5.5)}_{=13.02}x-8.8)$$

$$= -1.2987x^3+8.1818x^2-16.9091x+11.4286$$

Check: $L_0(x_0) = -1.729+9.900-18.600+11.429 \approx 1$ ✓

(could also check $L_0(x_1)=0$, $L_0(x_2)=0$ and/or $L_0(x_3)=0$

$L_1(x) = \dfrac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$

$$= \frac{x^3-(x_0+x_2+x_3)x^2+(x_0x_2+x_0x_3+x_2x_3)x-x_0x_2x_3}{(0.5)(-0.6)(-0.9)}$$

$$= 3.7037(x^3-5.8x^2+10.67x-6.05)$$

$$= 3.7037x^3-21.4815+39.5185x-22.4074$$

$L_2(x) = \dfrac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$

$$= \frac{x^3-(x_0+x_1+x_3)x^2+(x_0x_1+x_0x_3+x_1x_3)x-x_0x_1x_3}{(1.1)(0.6)(-0.3)}$$

$$= (-5.0505)(x^3-5.2x^2+8.51x-4.4)$$

$$= -5.0505x^3+26.2626x^2-42.9798x+22.2222$$

$L_3(x) = \dfrac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$

$$= 2.6455x^3-12.9630x^2+20.3704x-10.2434$$

10 a) (cont.)

$$p_3(x) = f_0 L_0(x) + f_1 L_1(x) + f_2 L_2(x) + f_3 L_2(x)$$

$$= (3.2)(-1.2987x^3 + 8.1818x^2 - 16.9091x + 11.4286)$$

$$+ (5.5)(3.7037x^3 - 21.4815x^2 + 39.5185x - 22.4074)$$

$$+ (8.0)(-5.0505x^3 + 26.2626x^2 - 42.9798x + 22.2222)$$

$$+ (9.1)(2.6455x^3 - 12.9630x^2 + 20.3704x - 10.2434)$$

$$= -0.1154x^3 + 0.1710x^2 + 4.7749x - 2.1065$$

check: $p_3(1.1) \approx 3.2$ ✓

However, I should have kept 5 d.p. of intermediate results, to have final results accurate to 4 d.p., since I multiplied by eg. 9.1. Indeed wolframalpha.com (see next page) gives:

$$-0.11544x^3 + 0.171717x^2 + 4.77475x - 2.10635$$

10 b) $p_3(1.25) = 3.9049$

$p_3(2.0) = 7.2065$

c) $$\epsilon_{min}(1.25) = \frac{0.5}{4!}\left|(1.25-1.1)(1.25-1.6)(1.25-2.2)(1.25-2.5)\right|$$

$$= \frac{0.5}{24}(0.06234) \approx 0.0013$$

$$\epsilon_{max}(1.25) = \frac{3}{24}(0.06234) \approx 0.0078$$

$$\epsilon_{min}(2) = \frac{0.5}{24}(0.036) \approx \cancel{0.0045}\ 0.00075$$

$$\epsilon_{max}(2) = \frac{3}{24}(0.036) = 0.0045$$

WolframAlpha computational intelligence.

InterpolatingPolynomial[[[1.1,3.2],[1.6,5.5],[2.2,8.0],[2.5,9.1]],x]

Browse Examples | Surprise Me

Assuming "InterpolatingPolynomial" is a math function | Use as referring to a data fit instead

Input:

$$\text{InterpolatingPolynomial}\left[\begin{pmatrix} 1.1 & 3.2 \\ 1.6 & 5.5 \\ 2.2 & 8 \\ 2.5 & 9.1 \end{pmatrix}, x\right]$$

Open code

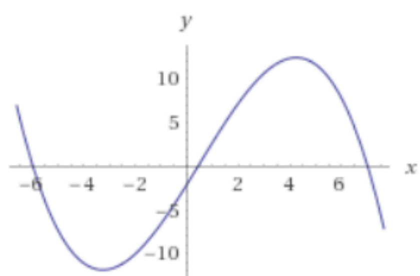
Enlarge | Data | Customize | Plaintext | Interactive

Result:

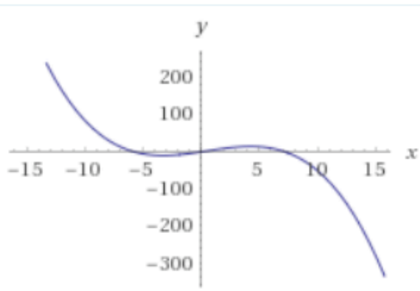
Copyable Plaintext:

$((-0.11544\,(x-1.6) - 0.4285$ … $21429)\,(x-2.5) + 9.1$

Plots:



(x from −6.6 to 7.6)



(x from −15.8 to 15.8)

Alternate forms:

More

$$x\,(x\,(0.171717 - 0.11544\,x) + 4.77475) - 2.10635$$

$$-0.11544\,x^3 + 0.171717\,x^2 + 4.77475\,x - 2.10635$$