MAT 1348/1748 SUPPLEMENTAL EXERCISES

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1 Propositional Logic

1. The table below is a truth table for six mystery compound propositions $P_1, \ldots, P_6$. Each of these consists of propositional variables p and q, and logical connectives.

p	q	P_1	P_2	P_3	P_4	P_5	P_6
T	T	T	T	T	F	F	F
T	F	T	F	F	T	F	F
F	T	T	F	T	F	T	F
F	F	F	T	T	T	F	T

   (a) Write a DNF formula (Disjunctive Normal Form) for each of the compound propositions $P_1, \ldots, P_6$.

   (b) Write a formula for each of the compound propositions $P_1, \ldots, P_6$ using only logical connectives $\neg$ and $\wedge$.

   (c) Write a formula for each of the compound propositions $P_1, \ldots, P_6$ using only logical connectives $\neg$ and $\rightarrow$.

2. Recall the definition of the "exclusive or":

$$p \oplus q \text{ is true if an only if exactly one of } p \text{ and } q \text{ is true.}$$

   (a) Construct a truth table for $p \oplus q$.

   (b) Show that "exclusive or" is associative; that is, show that

$$(p \oplus q) \oplus r \equiv p \oplus (q \oplus r).$$

   (c) Using truth tables, show that

$$p \oplus q \equiv \neg(p \leftrightarrow q).$$

3. Find a compound proposition equivalent to

$$\neg(a \rightarrow b) \rightarrow c$$

   which contains only logical connectives $\neg$ and $\vee$.

4. Using (a) truth tables, (b) truth trees, determine whether each of the following propositions is a tautology. If you claim that the proposition is not a tautology, give all counterexamples.

(i) $((x \vee y) \wedge (\neg x \vee z) \wedge (y \rightarrow z)) \rightarrow z$
(ii) $(x \rightarrow (y \vee z)) \rightarrow ((x \rightarrow y) \wedge (x \rightarrow z))$

5. (a) Rewrite the argument

"If Peter goes to the party only if John goes to the party, then Mary will be sad unless the party is cancelled. If John isn't going to the party, then Peter is not going either. In order for the party not to be cancelled, it suffices that John does not come to the party and that Peter leaves early. In order for Mary not to be sad, it is necessary that Peter go to the party. Therefore, the party will be cancelled unless John goes to the party and Peter does not."

using the following atomic propositions

P: "Peter is going to the party."
M: "Mary is sad."
J: "John is going to the party."
E: "Peter is leaving early."
C: "The party is cancelled."

(b) Use any method you know to determine whether or not the above argument is valid. If you claim it is invalid, give a counterexample.

6. (a) Rewrite the argument

"If the dog is barking, then there is someone at the door only if it is snowing. The dog is playing outside unless there is someone at the door. For the dog to be barking, it suffices that there be someone at the door or that it snows. For the dog to play outside, it is necessary that there be no one at the door and that it does not snow. Therefore the dog is barking but there is someone at the door."

using the following atomic propositions

B: "The dog is barking." D: "There is someone at the door." P: "The dog is playing outside." S: "It is snowing."

(b) Use any method you know to determine whether or not the above argument is valid. If you claim it is invalid, give a counterexample.

7. Use (a) truth tables, (b) truth trees, (c) algebraic manipulation, to find a DNF for each of the following compound propositions.

(i) $(A \rightarrow B) \rightarrow (B \rightarrow A)$
(ii) $(P \rightarrow Q) \leftrightarrow (P \rightarrow R \vee Q)$
(iii) $\neg A \rightarrow (B \rightarrow (A \rightarrow (B \wedge C)))$

8. Let x, y, z be atomic propositions and F a compound proposition with the truth table given below. Give a DNF for F.

x	y	z	F
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	T	T	F

2 Knights-and-Knaves Questions

These problems are quoted (or sometimes modified from) the wonderful logic puzzle books of Raymond Smullyan, in particular his book *What is the name of this book?*.

There is an island far off in the Pacific, called the island of Knights and Knaves. On this island, there are people called knights (who always tell the truth) and knaves (who always lie). They may be either male or female.

1. We have three people A, B, and C on the Island of Knights and Knaves. Suppose A and B say the following:

   A: All of us are knaves.

   B: Exactly one of us is a knave.

   Can it be determined what B is? Can it be determined what C is?

2. Suppose A says, "I am a knave but B isn't." What are A and B?

3. We again have three inhabitants, A, B and C, each of whom is a knight or a knave. Two people are said to be of the same type if they are both knights or both knaves. A and B make the following statements:

   A: B is a knave.

   B: A and C are of the same type.

   What is C?

4. Again three people A, B and C. A says "B and C are of the same type." Someone then asks C, "Are A and B of the same type?" What does C answer?

5. We have two people A, B, each of whom is either a knight or a knave. Suppose A makes the following statement: "If I am a knight, then so is B." Can it be determined what A and B are?

6. Someone asks A, "Are you a knight?" He replies, "If I'm a knight, then I'll eat my hat!" Prove that A has to eat his hat.

7. A says, “If I’m a knight, then two plus two equals four.” Is A a knight or a knave?

8. A says, “If I’m a knight, then two plus two equals five.” What would you conclude?

9. Given two people, A, B, both of whom are knights or knaves. A says, “If B is a knight then I am a knave.” What are A and B?

10. Two individuals, X and Y, were being tried for participation in a robbery. A and B were court witnesses, and each of A, B is either a knight or a knave. The witnesses make the following statement:

    A: If X is guilty, so is Y.

    B: Either X is innocent or Y is guilty.

    Are A and B necessarily of the same type? (i.e. either both knights or both knaves.)

11. On the island of knights and knives, three inhabitants A,B,C are being interviewed. A and B make the following statements:

    A: B is a knight.
    B: If A is a knight so is C.

    Can it be determined what any of A, B, C are?

12. Suppose the following two statements are true: (1) I love Betty or I love Jane. (2) If I love Betty then I love Jane. Does it necessarily follow that I love Betty? Does it necessarily follow that I love Jane?

13. Suppose someone asks me, “Is it really true that if you love Betty then you also love Jane?” I reply, “If it is true, then I love Betty.” Does it follow that I love Betty? Does it follow that I love Jane?

14. This problem, though simple, is a bit surprising. Suppose it is given that I am either a knight or a knave. I make the following two statements:

    (a) I love Linda.
    (b) If I love Linda then I love Kathy.

    Am I a knight or a knave?

15. Is There Gold on This Island? On a certain island of knights and knaves, it is rumored that there is gold buried on the island. You arrive on the island and ask one of the natives, A, whether there is gold on this island. He makes the following response: “There is gold on this island if and only if I am a knight.” Our problem has two parts:

    (a) Can it be determined whether A is a knight or a knave?

    (b) Can it be determined whether there is gold on the island?

16. Suppose, instead of A having volunteered this information, you had asked A, "Is the statement that you are a knight equivalent to the statement that there is gold on this island?" Had he answered "Yes," the problem would have reduced to the preceding one. Suppose he had answered "No." Could you then tell whether or not there is gold on the island?

17. The First Island. On the first Island he tried, he met two natives A, B, who made the following statements:

    A: B is a knight and this is the island of Maya.
    B: A is a knave and this is the island of Maya.

    Is this the island of Maya?

18. The Second Island. On this Island, two natives A, B, make the following statements:

    A: We are both knaves, and this is the island of Maya.
    B: That is true.

    Is this the island of Maya?

19. The Third Island. On this island, A and B said the following:

    A: At least one of us is a knave, and this is the island of Maya. B: That is true.

    Is this the island of Maya?

20. Here is a bit of an off-beat question. One day, on the island of Knights and Knaves, you see an inhabitant. You go up to her and ask: "Are you a knight or are you a knave?" She says: "I won't tell you" and walks away. Is it possible to decide if she is a knight or a knave?

3 Proofs

1. Use a proof by contradiction to show that the sum of a rational number and an irrational number is irrational.

2. Show that $\sqrt[3]{3}$ is irrational.

3. Let x and y be real numbers. Using a proof by cases, show that $\max(x, y)+\min(x, y) = x + y$.

4. Let $a = \sqrt{2}$. Show that at least one of the numbers a^a and $(a^a)^a$ is rational.

5. Show that at least one of the real numbers a_1, a_2, ..., a_n is greater than or equal to the average of these numbers.

6. Using a proof by contradiction, show that between any two distinct rational numbers there are infinitely many rational numbers.

7. Use an indirect proof to show the following: If $0 < a < 1$, then $a > a^2$.

8. Use an indirect proof to show the following: For all integers n, if $n^5 + 7$ is even, then n is odd.

9. Use a proof by cases to show that $|x + y| \leq |x| + |y|$ for all real numbers x and y.

10. Show that, for a natural number n, if 3 divides n^2, then 3 divides n.

11. Let x, a, b be three positive integers. Give an indirect proof of the following theorem: If x^2 does not divide $a^2 + b^2$, then x does not divide a or x does not divide b.

12. Prove the following theorem: Equation $x^3 + 3x + 5 = 0$ has no rational roots.

4 Sets

1. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 4, 6, 9\}$. Determine the following sets:

(1) $A \cap B$	(2) $A \cup B$	(3) $A - B$	(4) $B - A$
(5) $A \oplus B$	(6) $(A \oplus B) \cap A$	(7) $(A \oplus B) \cup B$	

2. Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. For each of the following statements, determine whether it is true or false.

$(a)\ \emptyset \subseteq A$	$(b)\ \emptyset \in A$	$(c)\ \{\emptyset\} \in A$
$(d)\ \{\emptyset\} \subseteq A$	$(e)\ \{\emptyset, \{\emptyset\}\} \in A$	$(f)\ \{\{\emptyset, \{\emptyset\}\}\} \in A$
$(g)\ \{\{\emptyset\}\} \in A$	$(h)\ \{\{\emptyset\}\} \subseteq A$	$(i)\ \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq A$
$(j)\ \{\emptyset, \{\emptyset\}, \{\{\{\emptyset\}\}\}\} \subseteq A$	$(k)\ \{\emptyset, \{\{\emptyset\}\}\} \in \mathcal{P}(A)$	$(l)\ \{\{\{\{\emptyset\}\}\}\} \subseteq \mathcal{P}(A)$

3. For each of the following sets, determine its cardinality and its power set:

$$(i)\ A = \{a, b, \{a, b\}\} \qquad (ii)\ B = \{\emptyset, \{\emptyset\}\}$$

4. Let A, B, and C be three subsets of the universal set U such that $A \oplus B = A \oplus C$. Show that $B = C$.

5. Let A, B, and C be three subsets of the universal set U. Using properties of set operations, show that

   (a) $\overline{(\overline{A} - \overline{B})} \cap B = (A - \overline{B})$.

   (b) $\overline{(\overline{A} - B)} \cap C = (C \cap A) \cup (C \cap B)$

   (c) $(A - B) - C = (A - C) - (B - C)$

   (d) $A \cap (B - A) = \emptyset$

   (e) $(B \cup C) - A = (B - A) \cup (C - A)$

6. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

7. As usual, if a, b are real numbers such that $a \leq b$, then $[a,\, b] = \{x \in \mathbb{R};\, a \leq x \leq b\}$ and $(a,\, b) = \{x \in \mathbb{R};\, a < x < b\}$.

   Determine the following sets:

$$(i)\,[-3,\, 6] \cap (-2,\, 7] \qquad (ii)\,(-5, 7] \cap \mathbb{Z}$$

8. Show that there do not exist subsets A and B of $\mathbb{N}$ such that $A \times B = \{(0,0),\, (1,1)\}$.

9. Let A, B, and C be three subsets of the universal set U. Is it true that

$$(A - B) - C = A - (B - C)?$$

   If so, give a proof. Otherwise, give a counterexample.

10. Let A and B be sets such that $|A| = 6$, $|B| = 8$, and $|A \cap B| = 3$. Find the cardinality of the power set of $A \cup B$.

11. Let A, B, and C be three subsets of the universal set U. For each of the following statements, determine whether it is true or false. Justify your answer.

    (a) If $A \cap B = \emptyset$, then $A = B$.
    (b) If $A - B = \emptyset$, then $A = B$.
    (c) If $A - B = \emptyset$, then $A \subseteq B$.
    (d) If $A \cup B = \emptyset$, then $A = B$.
    (e) If $A \oplus B = \emptyset$, then $A = B$.
    (f) If $A \times B = \emptyset$, then $A = B$.
    (g) If $\overline{A} - \overline{B} = \emptyset$, then $A = B$.
    (h) If $A - B = A$, then $A \subseteq B$.
    (i) If $A \cup B = A$, then $B = \emptyset$.
    (j) If $\overline{B} \subseteq \overline{A}$, then $A \subseteq B$.

12. Let $S = \{x \in \mathbb{R} :\ -x^2 + x + 2 \geq 0\}$. What is the cardinality of $S \cap \mathbb{Z}$?

13. For any positive integer n we define a set $A_n = \{-n,\, \ldots, -1,\, 0,\, 1,\, 2,\, 3,\, \ldots\}$. Determine the following sets:

$$(i)\, A_{100} \backslash A_{96} \qquad (ii)\, \cup_{n=1}^{\infty} A_n \qquad (iii)\, \cap_{n=1}^{\infty} A_n$$

5 Functions

1. Give an example of a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ that is:

   (a) one-to-one but not onto;

   (b) onto but not one-to-one;

   (c) a bijection but is not the identity;

   (d) neither one-to-one nor onto.

2. Let $g : A \longrightarrow B$ and $f : B \longrightarrow C$ be two functions.

   (a) If f and $f \circ g$ are one-to-one, is g necessarily one-to-one?

(b) If f and $f \circ g$ are onto, is g necessarily onto?

3. Let $f : A \longrightarrow B$ be a function, and S and T two subsets of A. Show that:

   (a) $f(S \cup T) = f(S) \cup f(T)$

   (b) $f(S \cap T) \subseteq f(S) \cap f(T)$

4. Let $f : \mathbb{Z} \times (\mathbb{Z} - \{0\}) \longrightarrow \mathbb{Q}$ be defined by $f\,(m,\, n) = \frac{m}{n}$. Is f one-to-one? Is it onto? Justify your answer.

5. For each of the following assignments, determine whether it is a function or not. If it is a function, is it one-to-one? Is it onto?

   (a) $f_1 : \mathbb{R} \longrightarrow \mathbb{R}, \quad f_1(x) = -x^3 + 1;$

   (b) $f_2 : \mathbb{N} \longrightarrow \mathbb{Z}, \quad f_2(n) = n^2 + 3;$

   (c) $f_3 : \mathbb{R} \longrightarrow [0,\, +\infty), \quad f_3(x) = 2^x;$

   (d) $f_4 : \mathbb{N} \longrightarrow \mathbb{N}, \quad f_2(n) = \sqrt{n} + 1;$

   (e) $f_5 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad f_5(x, y) = x + y;$

   (f) $f_5 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{N}, \quad f_6(x, y) = (2x, 0);$

6. Let $\phi : A \longrightarrow \mathbb{N}$ and $\psi : B \longrightarrow \mathbb{N}$ be two one-to-one functions. Show that the function $\lambda : A \times B \longrightarrow \mathbb{N}$ defined by $\lambda(a,\, b) = 2^{\phi(a)}3^{\psi(b)}$ is also one-to-one.

7. Let $f : (0,\, +\infty) \longrightarrow (0,\, +\infty)$ be a function defined by $f(x) = 2x^2 + 3$.

   (a) Determine whether or not f is one-to-one.

   (b) Find the image of f. From this, determine whether or not f is onto.

8. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \sqrt[3]{x+2}$ and $g(x) = x^3 - 2$. Find an expression for each of the functions $f \circ g$ and $g \circ f$. Deduce that f and g are both bijections.

9. Let A be a set with 4 elements and B a set with 6 elements. For each of the following statements, determine whether it is true or false. Justify your answers.

   (a) There is no onto function from A to B.

   (b) There is no one-to-one function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.

   (c) There is no one-to-one function from $\mathcal{P}(A)$ to B.

   (d) There is no onto function from $A \times B$ to $\mathcal{P}(B)$.

   (e) Theer is not one-to-one function from $\mathcal{P}(\mathcal{P}(A))$ to $\mathcal{P}(A \times B)$.

10. For a real number x, we define $\lfloor x \rfloor$ as the largest integer that is less than or equal to x, and $\lceil x \rceil$ as the smallest integer that is greater than or equal to x. For example, $\lfloor -1.2 \rfloor = -2$ and $\lceil 3.1 \rceil = 4$. Consider the following two functions:

$$f : \mathbb{R} \longrightarrow \mathbb{Z}, \quad f(x) = \lfloor x \rfloor \qquad \text{and} \qquad g : \mathbb{R} \longrightarrow \mathbb{Z}, \quad g(x) = \lceil x \rceil.$$

Determine whether f and g are one-to-one or onto.

11. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions.

    (a) Show that if f and g are bijections, then so are $g \circ f$ and $(g \circ f)^{-1}$. (Note that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.)

    (b) Give an example of f and g such that $g \circ f$ is a bijection but neither f nor g is a bijection.

6 Relations

1. Let $A = \{1, 2, 3, 4\}$. For each of the following relations on the set A, determine whether it is reflexive, symmetric, antisymmetric, or transitive. Justify your answer.

$$\begin{aligned} R_1 &= \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\} \\ R_2 &= \{(1,1), (2,2), (3,3), (4,4)\} \\ R_3 &= \{(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)\} \end{aligned}$$

2. Let $A = \{a, b, c\}$. Consider the following three relations on A:

$$\begin{aligned} R_1 &= \emptyset \\ R_2 &= \{(a,b), (b,c), (a,c))\} \\ R_3 &= \{(a,a), (b,c), (c,b)\} \\ R_4 &= \{(a,a), (b,b), (c,c), (a,c)\} \end{aligned}$$

For each of these relations, determine whether it is reflexive, symmetric, antisymmetric, or transitive. Justify your answer.

3. For each of the following relations on the set $\mathbb{Z}$, determine whether it is reflexive, symmetric, antisymmetric, or transitive. Justify your answer.

$$\begin{aligned} R_1 &= \{(a,b) \in \mathbb{Z} \times \mathbb{Z};\ a < b\} \\ R_2 &= \{(a,b) \in \mathbb{Z} \times \mathbb{Z};\ 2b - a < 3\} \end{aligned}$$

4. Let $A = \{a, b, c\}$ and $B = \{0, 1, 2, 3, 4\}$. Determine whether each of the following statements is true or false. Justify your asnwer.

    (a) There are 32768 binary relations from A to B.

(b) There are 1024 binary relations from B to A.

(c) A binary relation from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ is a subset of $\mathcal{P}(A \times B)$.

(d) The number of binary relation from A to B that contain the subset $\{(x,\, 0);\, x \in A\}$ of $A \times B$ is 4096.

(e) The number of binary relation from A to B that contain the subset $\{(a,\, y);\, y \in B\}$ of $A \times B$ is 4096.

5. Let $A = \{0,\, 1,\, 2,\, 3\}$ and $B = \{a,\, b,\, c,\, e\}$. Which of the following relations from A to B are functions? For the functions among them, determine whether they are one-to-one or onto.

$$\begin{array}{ll} R_1 = \{(0,a),(1,a),(2,c),(1,b),(3,a)\}, & R_2 = \{(0,c),(1,a),(2,c)\} \\ R_3 = \{(3,c),(2,c),(1,c),(0,c)\}, & R_4 = \{(0,b),(1,a),(3,a),(2,c)\} \\ R_5 = \{(2,a),\ (0,e),(2,b),(3,c)\}, & R_6 = \{(2,a),\ (0,e),(1,b),(3,c)\} \end{array}$$

6. For each of these relations, determine whether it is reflexive, symmetric, or transitive. Justify your answer.

(a) $x\, R_1\, y \Leftrightarrow x + y$ is odd

(b) $x\, R_2\, y \Leftrightarrow xy$ is odd

(c) $x\, R_3\, y \Leftrightarrow x + xy$ is even

7. Let $\mathcal{U}$ be the universal set. For each of the following relations on the power set $\mathcal{P}(\mathcal{U})$ of $\mathcal{U}$, determine whether it is reflexive, symmetric, antisymmetric, or transitive. Justify your answer.

(a) $A\, R_1\, B \Leftrightarrow A \subseteq B$

(b) $A\, R_2\, B \Leftrightarrow A \cap B = \emptyset$

(c) $A\, R_3\, B \Leftrightarrow A - B = \emptyset$
Hint: First show that $A - B = \emptyset$ if and only if $A \subseteq B$.

(d) $A\, R_4\, B \Leftrightarrow A \oplus B = \emptyset$
Hint: First show that $A \oplus B = \emptyset$ if and only if $A = B$.

8. A binary relation R on a set A will be called *cyclic* if the following condition is satisfied:

$$(a,\, b) \in R \text{ and } (b,\, c) \in R \text{ implies } (c,\, a) \in R.$$

(a) Show that if R is reflexive and cyclic, then it is symmetric and transitive.

(b) Show that if R is symmetric and transitive, then it is cyclic.

9. (a) Let $A = \{1,\, 2,\, 3\}$. Give an example of a nonempty binary relation on A that is symmetric and transitive, but not reflexive.

(b) Let A be a set containing at least two elements and let R be a nonempty binary relation on A. By (a), we know that the following "theorem" is false:

"If R is symmetric and transitive, then R is reflexive."

Hence, what is wrong with the following "proof"?

"Take any $a \in A$. Choose $b \in A$ such that $(a, b) \in R$. This is possible because $R \neq \emptyset$ and A contains two elements. Hence $(b, a) \in R$ since R symmetric. As R is also transitive, $(a, b) \in R$ and $(b, a) \in R$ implies that $(a, a) \in R$. Therefore R is reflexive."

7 Equivalence Relations

1. Consider the following binary relation on the set $A = \{1, \frac{1}{3}, \frac{1}{27}, \frac{1}{4}, 3, \frac{1}{36}, 2, \frac{2}{9}, \frac{9}{4}, 5\}$: for all $x, y \in A$,

$$xR\,y \Leftrightarrow \frac{x}{y} = 3^k \text{ for some } k \in \mathbb{Z}.$$

   (a) Show that R is an equivalence relation on A.

   (b) Determine the partition of A defined by R.

2. Let $A = \{f : \mathbb{Z} \longrightarrow \mathbb{R}; f \text{ is a function}\}$ be the set of all functions from $\mathbb{Z}$ to $\mathbb{R}$. WE define the following binary relation on A: for all $f, g \in A$,

$$f R\, g \Leftrightarrow \text{ for all } x \in \mathbb{Z},\ f(x) - g(x) = c \text{ for some constant } c \in \mathbb{Z}.$$

   (a) Show that R is an equivalence relation on A.

   (b) Which of the following functions are in the equivalence class of the function $f(x) = 2x + 3$?

   $f_1(x) = -2x + 1,\ f_2(x) = 2x - 1,\ f_3(x) = 1,\ f_4(x) = 4x + 8,\ f_5(x) = 2x,$ $f_6(x) = x^2 + 2x + 3.$

3. Let $\mathbb{W} = \{(x, y) \in \mathbb{R}^2;\ x \neq 0, \text{et}\, y \neq 0\}$. Define the following binary relation on $\mathbb{W}$:

$$(x, y)R\,(a, b) \Leftrightarrow xa > 0 \text{ and } \ yb > 0.$$

   (a) Show that R is an equivalence relation on $\mathbb{W}$.

   (b) How many equivalence classes are there on the set $\mathbb{W}$?

4. Let

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

   On $\mathcal{A}$, we define a binary relation R as follows:

$$AR\,B \Leftrightarrow \det A = \det B$$

   where $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$

(a) Show that R is an equivalence relation on $\mathcal{A}$.

(b) Determine the partition of $\mathcal{A}$ defined by R.

5. Recall that the dot product of vectors $\vec{u} = (x, y)$ and $\vec{v} = (x', y')$ in $\mathbb{R}^2$ is defined as $\vec{u} \cdot \vec{v} = xx' + yy'$. Let $\vec{u}_0 = (1, -1)$.

   Define a binary relation R on $\mathbb{R}^2$ as follows:

$$\vec{u} R \, \vec{v} \Leftrightarrow \vec{u} \cdot \vec{u}_0 = \vec{v} \cdot \vec{u}_0.$$

   (a) Show that R is an equivalence relation on $\mathbb{R}^2$.

   (b) Geometrically, describe the equivalence class of the vector $(1, 1)$.

6. We define a binary relation R on $\mathbb{R}$ as follows:

$$a R \, b \Leftrightarrow b = ac \text{ for some real number } c > 0.$$

   (a) Show that R is an equivalence relation on $\mathbb{R}$.

   (b) Determine the partition of $\mathbb{R}$ defined by R.

7. For a positive integer n, define $\kappa(n)$ to be the smallest prime divisor of n. For example, $\kappa(6) = 2$, $\kappa(9) = 3$, and $\kappa(35) = 5$.

   Define the following binary relation on the set $\mathbb{Z}^+$ of positive integers:

$$a R \, b \Leftrightarrow \kappa(a) = \kappa(b).$$

   Observe, for example, that $6 R \, 10$ and $9 R \, 15$.

   (a) Show that R is an equivalence relation on $\mathbb{Z}^+$.

   (b) Determine the equivalence class of 2.

8. Let A be the set of all binary strings of length 3. Define the following binary relation on A:

$$x R \, y \Leftrightarrow \text{ the last two bits of } x \text{ and } y \text{ are the same.}$$

   (a) Show that R is an equivalence relation on A.

   (b) Determine the partition of A defined by R.

9. Define the following binary relation on the set $\mathbb{R}^2$:

$$(x, y) R \, (a, b) \Leftrightarrow x^2 + y^2 = a^2 + b^2.$$

   (a) Show that R is an equivalence relation on $\mathbb{R}^2$.

   (b) Give a geometric description of the equivalence class of the element $(0, 2)$. In addition, list three elements of the equivalence class of $(0, 2)$ other than $(0, 2)$ itself.

10. Recall that $\mathbb{M}_{22}$ denotes the set of all 2×2 matrices with real entries. Define the following binary relation on $\mathbb{M}_{22}$:

$$A R B \Leftrightarrow A = \lambda B \text{ for some } \lambda \in \mathbb{R} - \{0\}.$$

Recall that, if $\lambda \in \mathbb{R}$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{M}_{22}$, then $\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$.

(a) Show that R is an equivalence relation on $\mathbb{M}$.

(b) Let $S = \{A_1, A_2, \ldots, A_8\}$ be the set containing the following matrices:

$$A_1 = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} -3 & 3 \\ -6 & -9 \end{pmatrix} \quad A_6 = \begin{pmatrix} -1 & -3 \\ 1 & 0 \end{pmatrix} \quad A_7 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_8 = \begin{pmatrix} 2 & -2 \\ 4 & 6 \end{pmatrix}$$

Determine the partition of S defined by the equivalence relation R.

11. Define a binary relation on the set $\mathbb{R}^* = \mathbb{R} - \{0\}$ as follows:

$$a R b \Leftrightarrow |a| b = a |b|.$$

Here, $|x|$ denotes the absolute value of x.

(a) Show that R is an equivalence relation on $\mathbb{R}^*$.

(b) What is the equivalence class of 1?

8 Basic Counting Techniques

1. How many distinct licence plates can be made that consist of three digits followed by three letters, or three letters followed by three digits?

2. A palindrome is a binary string that reads the same in reverse. How many binary strings of length n are palindromes? (Consider two cases: n is even and n is odd.)

3. Consider a country with postal codes of the following form: LDL-DLD (where "L" stands for "letter" and "D" for "digit"). How many distinct postal codes are there such that all digits are odd, and no symbol (letter of digit) is repeated?

4. Let A and B be sets with $|A| = 7$and $|B| = 4$. Determine:

(a) the number of functions from A to B;

(b) the number of functions from B to A;

(c) the number of one-to-one functions from A to B;

(d) the number of one-to-one functions from B to A;

(e) the number of onto functions from A to B;

(f) the number of onto functions from B to A.

5. Find the number of integers between 10000 and 99999 (inclusive) that consist of 5 distinct digits.

6. How many integers between 7 and 2125 (inclusive):

   (a) are divisible by 3 or 11?

   (b) are relatively prime with 11?

   (c) are divisible by 3 but not by 11?

7. How many binary strings of length 13 begin with the string 0110 or end with the string 1000?

8. How many positive integers not exceeding 250 are divisible either by 4 or by 6?

9. Let $n \in \mathbb{N}$. Determine the number of binary strings of length less than or equal to n.

10. How many licence plates are there that have either two or three letters followed by either 2 or three digits?

11. A password must contain between 6 and 9 characters. A character is an uppercase letter, a lower case letter, or a digit. Each password must contain at least 2 distinct characters. How many such passwords are there?

12. A photographer at a wedding is lining up 6 people, including the wedding couple, for a picture. How many ways can he do that so that

    (a) the bride stands next to the groom?

    (b) the bride does not stand next to the groom?

    (c) the bride stands anywhere to the left of the groom?

9 The Pigeonhole Principle

1. Show that no matter how we choose 7 points in the interior or on the perimeter of a regular hexagon with a side of length 1, at least two of these points will be at distance at most 1.

2. A network consists of 6 computers. Each computer is directly linked to at least one other computer in this network. Show that at least two of these computers are linked to the same number of computers in the network.

3. How many people do we need to guarantee that at least two among them were born on the same day of the week and in the same month (but possibly a different year).

4. A package of baseball cards contains 20 cards. How many packages should you buy to be sure that you have two identical cards, if you know that there are in total 550 distinct baseball cards?

5. Show that any set of 15 natural numbers contains a pair of numbers whose difference is divisible by 14.

6. What is the smallest number n such that if any n natural numbers are chosen at random, at least 6 among them give the same remainder when divided by 9?

7. In a certain country, every birth certificate has a number of the form DLDLDLD (where "D" stands for a digit and "L" stands for a letter). For some reason, the authorities want to make sure that at least 26 birth certificates carry the same number. What is the smallest number of people that this country can have?

8. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For each subset X of A, we define $\sigma(X)$ to be the sum of the elements in X, with the convention that $\sigma(\emptyset) = 0$. For example, $\sigma(\{2, 4, 9\}) = 2+4+9 = 15$. Show that we can find 6 subsets A_i, $i = 1, \ldots, 6$, of A such that $|A_i| \leq 3$ for all i, and $\sigma(A_i) = \sigma(A_j)$ for all i and j.

9. 500 points are placed in the interior of a rectangle with sides 2m and 1m. Show that there exist three points among these 500 such that the area of the triangle these three points define is at most 50cm^2.

10. Show that there exists a positive integer all of whose digits are 1 (such as 1, 11, 111, ..., 1111111, ...) that is divisible by 7777. (Hint: Consider the integers 1, 11, 111, ..., up to the integer with 7778 repeated ones, and their remainders when divided by 7777.)

11. On a test in a class with 30 students, a student called *Ace* made 12 mistakes, while every other student made fewer than 12 mistakes. Show that at least 3 students in this class made the same number of mistakes.

12. Let (x_i, y_i, z_i), for $i = 1, 2, \ldots, 9$, be 9 distinct points in a 3-dimensional space, all of whose coordinates are integer. Show that for at least one pair of these points, the midpoint of the line segment joining the two points in the pair has integer coordinates.

10 Permutations and Combinations

1. A group of people consists of n men and n women. How many ways can these $2n$ people be arranged in a line if men and women should alternate? (Hint: the line may begin with a man or a woman.)

2. An exam consists of 40 true-false questions, of which exactly 17 are true. If the questions may be arranged in any order, how many possible answer keys are there?

3. Give the coefficient of x^k (for $k \in \{0, 1, \dots, 100\}$) in the expansion of $\left(x + \frac{1}{x}\right)^{100}$.

4. How many binary strings contain exactly five 0s and fourteen 1s if each 0 is followed by two 1s?

5. (a) What is the coefficient of $x^{99}y^{101}$ in the expansion of $(2x - 3y)^{200}$?
   (b) What is the coefficient of x^{16} in the expansion of $(x - 2x^{-3})^{24}$?
   (c) What are the coefficients of x^4 and x^6 in the expansion of $(2x^{-2} - x^3)^8$?

6. (a) How many binary strings of length 9 contain exactly four 0s?
   (b) How many binary strings of length 9 contain at most four 0s?
   (c) How many binary strings of length 9 contain at least four 0s?

7. Let $S = \{1, 2, \dots, 50\}$.
   (a) How many subsets A of S are such that $\{1, 2, 5, 11, 23, 49\} \cap A = \emptyset$?
   (b) How many subsets A of S are such that $|A| = 20$ and $\{1, 2, 5, 11, 23, 49\} \subseteq A$?
   (c) How many subsets A of S are such that $|A| = 12$ and $|A \cap \{1, 2, 5, 11, 23, 49\}| = 3$?

8. (a) Let $S = \{1, 2, \dots, 25\}$. Find the number of subsets of S that contain exactly five elements, of which two are odd and three are even.
   (b) Let $H = \{1, 2, \dots, 19\}$. Find the number of subsets of H that contain the same number of odd and even elements.

9. A 5-member committee is to be selected from a group of 6 men and 8 women (which includes the people named below). How many ways can this be done if:
   (a) Joseph should be on the committee?
   (b) Katie should be on the committee, but Joseph should not?
   (c) At most three men should be on the committee, but Joseph should not be on the committee?
   (d) Joseph should be on the committee, but none of Caroline, Rachelle, and Steve should be on?

10. A 6-member research group is to be formed from a group of 10 chemists, 5 politicians, 8 economists, and 15 biologists. How many ways are there to form a committee that:
    (a) contains at least 3 chemists?
    (b) contains exactly 3 economists?
    (c) contains exactly 4 chemists but no economist?
    (d) contains exactly 2 politicians and 2 biologists but Steven (a politician) and Rebecca (a biologist) are not both on the committee?

11. How many licence plates consisting of three letters followed by 3 digits contain three distinct letters and three distinct digits?

12. How many ways can the letters in ACCESSORTATTIA be arranged so that neither the two Ss nor the two Cs are adjacent?

13. A standard deck of cards consists of 52 (distinct) cards. At the beginning of a game, each of the 6 players receives 5 cards. How many ways can this be done?

11 Mathematical Induction

1. Show that for all integers $n \geq 1$,
$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Let $h > -1$ be a real number. Prove Bernoulli's inequality:
$$1 + nh \leq (1+h)^n \text{ for all integers } n \geq 0.$$

3. Show that for all odd integers $n \geq 1$, the integer $n^2 - 1$ is divisible by 8.

4. Show that for all integers $n \geq 1$, the integer $4^{n+1} + 5^{2n-1}$ is divisible by 21.

5. For which natural numbers n is it true that $2^n > n^3$? Justify your answer.

6. Consider the sequence defined recursively as follows:
$$f_0 = 1,\ f_1 = 1,\ \text{and } f_n = f_{n-1} + f_{n-2} \ \text{ for all } n \geq 2.$$
Show that $f_n > \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}$ for all $n \geq 3$.

7. Show that $n^2 \geq 2n + 3$ for all $n \geq 3$.

8. Show that $7^n - 2^n$ is divisible by 5 for all $n \geq 0$.

9. Show that for all integers $n \geq 1$,
$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

10. Show that for all integers $n \geq 1$,
$$1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1.$$

11. Consider the sequence defined recursively as follows:
$$a_0 = 2,\ a_1 = 1,\ \text{and } a_n = a_{n-1} + 2a_{n-2} \text{ for all } n \geq 2.$$
Show that $a_n = 2^n + (-1)^n$ for all integers $n \geq 0$.

12. Consider the sequence defined recursively as follows:
$$a_n = \begin{cases} 2 & \text{if } n \text{ is odd} \\ a_{\frac{n}{2}}^2 & \text{if } n \text{ is even} \end{cases}$$
(1) Determine $a_1, \ldots, a_8$.
(2) Use mathematical induction to show that $a_n \leq 2^n$ for all $n \geq 1$.

13. Show that every integer $n \geq 2$ is the product of (one or more) prime numbers. (Note that a positive integer p is called *prime* if p and 1 are its only divisors.)

14. Show that $\sum_{i=1}^{n}(3i^2 - i) = n^2(n+1)$ for all integers $n \geq 1$.

12 Graphs

1. For each of the following sequences, determine whether or not there exists a simple graph with this degree sequence. If you claim such a graph exists, draw a picture. Otherwise, explain why such a graph does not exists.

   (a) (2, 3, 3, 4, 4, 5)
   (b) (2, 3, 4, 4, 5)
   (c) (1, 1, 1, 1, 4)
   (d) (1, 3, 3, 3)
   (e) (1, 2, 2, 3, 4, 4)
   (f) (1, 3, 3, 4, 5, 6, 6)
   (g) (2, 2, 2)
   (h) (4, 4, 4, 6, 6, 6)

2. (i) What is the maximum number of edges that a simple graph with n vertices can have? What type of a simple graph achieves this upper bound on the number of edges?

   (ii) How many edges in a graph with degree sequence $(2, 2, 3, 3, 4)$? Draw a picture of such a graph.

   (iii) What is the minimum number of edges that a graph G with n vertices can have if $\deg(v) \geq 3$ for all vertices v of G?

3. Show that a simple graph with at least 2 vertices has at least 2 vertices of the same degree.

4. Define a graph G as follows:

   - The vertex set of G is the set of all 4-elements subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For example, $\{2, 4, 6, 7\}$ is one vertex of G.
   - two vertices A and B are adjacent in G if and only if $|A \cap B| = 1$.

   (1) Determine the order (=number of vertices) of G.
   (2) Determine the number of edges of G. (Hint: what is the degree of each vertex?)

5. Use Mathematical Induction to show that K_n (the complete graph on n vertices) has $\frac{n(n-1)}{2}$ edges.

6. A graph is called *regular of degree k* if every vertex has degree k. Let G be a regular graph of degree k for an odd integer k.

   (1) Show that G has an even number of vertices.
   (2) Show that the number of edges of G is a multiple of k.

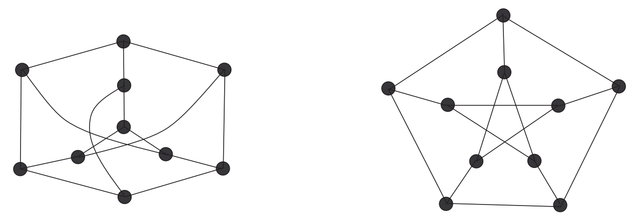
7. Consider the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

(1) If possible, draw a picture of a simple graph that has A as an adjacency matrix.
(2) If possible, draw a picture of a simple graph that has A as an incidence matrix.

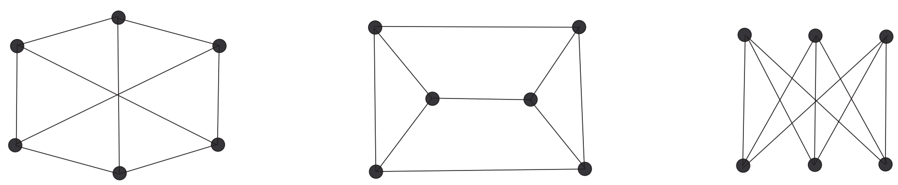
8. If possible, draw a picture of a simple graph whose adjacency matrix and incidence matrix are the same.

9. Are the following two graphs isomorphic? If so, give an isomorphism. If not, explain why.



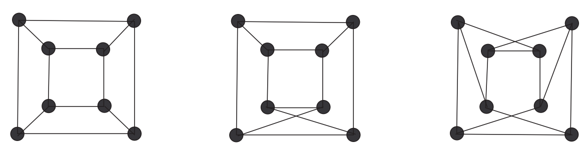
10. Consider the following three graphs:

(1) Two of these are isomorphic – which two? Give an isomorphism.
(2) One of these graphs is not isomorphic to the other two – which one? Justify your answer.

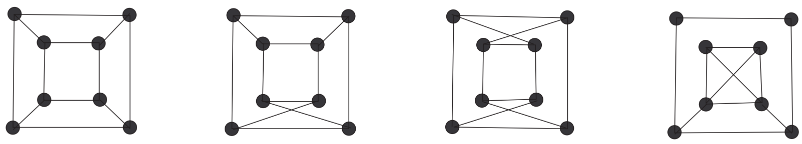


11. Consider the following three graphs:

(1) Two of these are isomorphic – which two? Give an isomorphism.
(2) One of these graphs is not isomorphic to the other two – which one? Justify your answer.



12. Which of the following graphs are bipartite? If a graph is bipartite, give a 2-colouring. If not, justify.



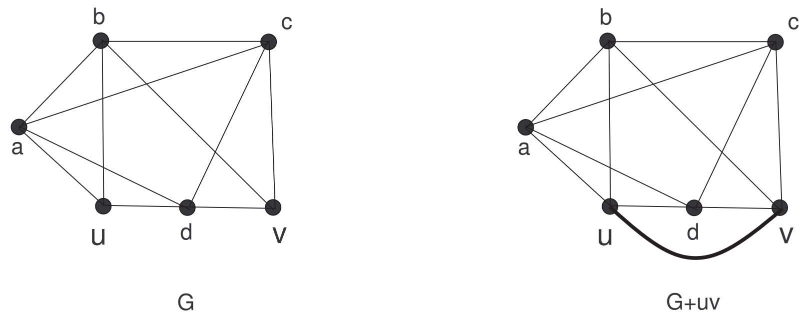
13. Does the graph G below (left) have

    (a) an Euler tour? (If so, give such an Euler tour as a sequence of vertices.)

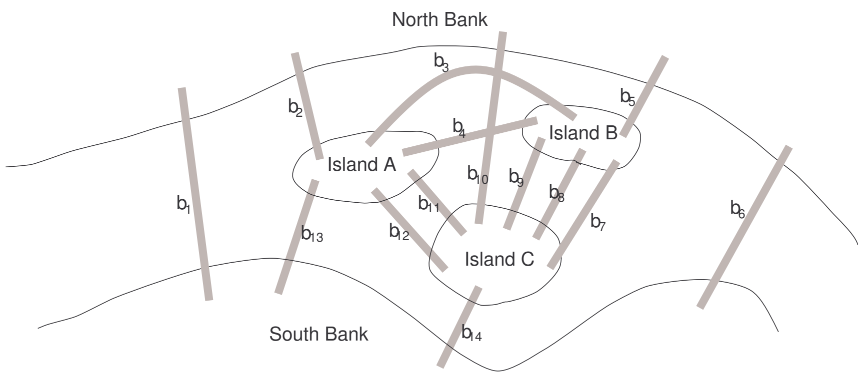
    (b) an open Euler trail? (If so, give such an open Euler trail as a sequence of vertices.)

    (c) Let $G + uv$ denote the graph G with the edge $\{u, v\}$ added to it (figure below, right). Answer the two questions above for the graph $G + uv$.

    Fully justify your answers referring to appropriate theorems in graph theory.



14. The picture below shows a town on a river. The town consists of five quarters: three islands (A, B, and C), and two river banks (North and South). Fourteen bridges (labelled $b_1, b_2, \ldots, b_{14}$) connect the five quarters as shown in the figure.



    (a) Is it possible to walk through the town, crossing each bridge exactly once before returning to the starting point? If so, give a sequence of bridges that describes such a walk. If not, explain why.

(b) Is it possible to walk through the town, crossing each bridge exactly once but ending in a quarter distinct from the starting point? If the answer is positive, give a sequence of bridges that describes such a walk. Is such a walk possible no matter where it starts and ends? — If such a walk is not possible, explain why.

Use appropriate theorems from graph theory to justify your answers.

15. The diagram below represents a plan of a house. Each large rectangle (labelled A, B, C, D, E, F) represents a room, and each small black rectangle represents a door.

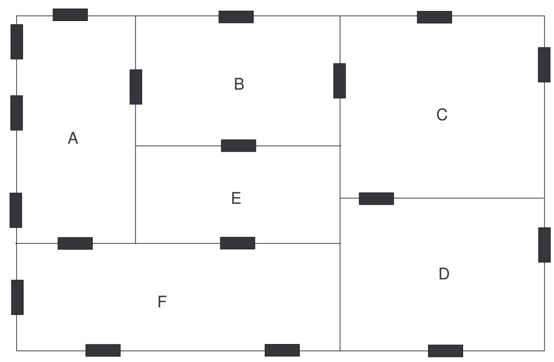
    (a) Suppose you wish to walk through the house (starting either outside or in a room) so that you walk through each door exactly once before returning to your starting point. Is this possible?

    (b) Now suppose that you want to walk through each door exactly once but wish to end the trip at a point other than your starting point (which was again either a room or the outside of the house). Is this possible? If so:

    i. Can such a trip start and end in a room? If so, which room(s) would that be?

    ii. Can such a trip start in a room and end outside of the house? If so, which room(s) can you start in?

    Fully justify your answers using appropriate theorems from graph theory. Clearly state these theorems.

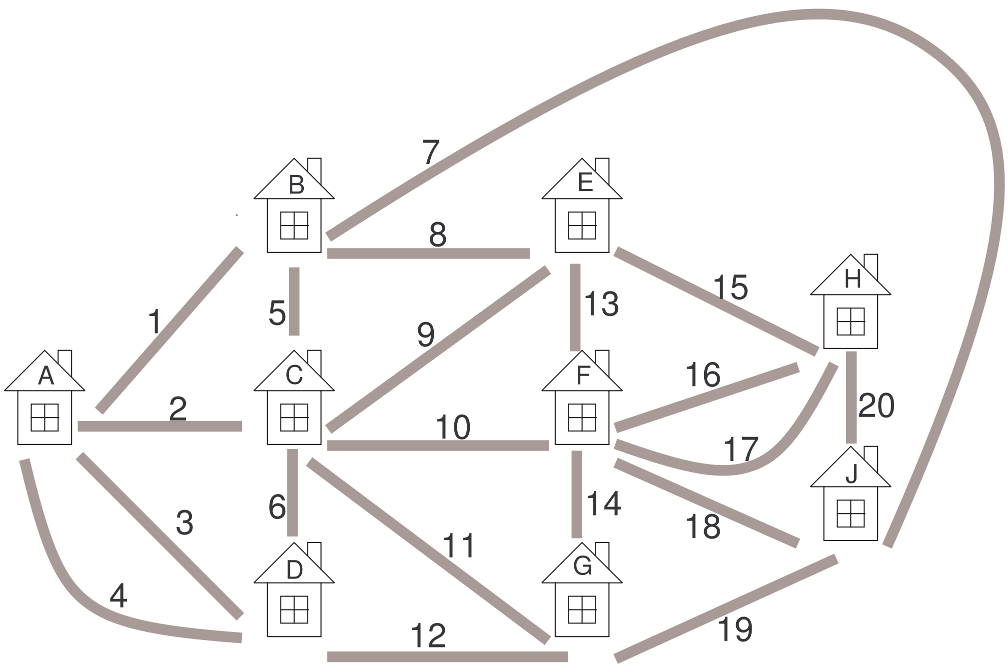


16. The picture below shows a village where Mateja's friends live. The gray lines show the 20 roads you can take to get from one house to another.

    (a) One Sunday afternoon, Mateja decides to visit all her friends. She would like to start her trip at one house, visit all her friends' houses and traverse each of the 20 roads exactly once before finishing at the same house she started at. Is this possible? State appropriate theorems from graph theory to justify your answer.

(b) The following Sunday, Mateja decides to start her trip at one house, traverse each of the 20 roads exactly once, and finish at a different house. Is such a trip possible? If the answer is yes, is such a trip possible no matter where the trip starts and ends? State appropriate theorems from graph theory to justify your answer.

(c) A month later, Mateja finds that road number 7 is closed for construction work. Answer questions (a) and (b) for this case.



17. (a) A full 5-ary tree T has 101 leaves. How many internal vertices and how many edges does T have?

(b) A full m-ary tree has 51 leaves and height 3. Determine all possible values of m. For each possible m, draw an example of such a tree.