MAT 1339 Calculus Material

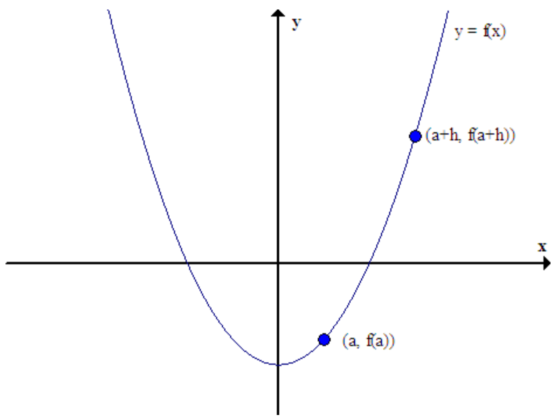
Chapter 1

Goals

- to understand average and instantaneous rates of change and their connections to the slopes of secant and tangent lines
- to be able to calculate average rates of change and estimate instantaneous rates of change from an equation, a graph or a data table
- to understand the concept of a limit and be able to use the basic limit properties to find limits of sequences and functions
- to understand the concept of continuity and be able to tell if a function is continuous or not at a point
- to be able to recognize the common discontinuities that can occur on the graph of a function
- to be able to recognize the indeterminate form $\frac{0}{0}$ when it occurs in the evaluation of a limit and know what to do to find the limit
- to understand the definition of the derivative of a function at a point and as a function and be able to use the definitions to calculate them
- to be comfortable with the different notations for the derivative

Rates of Change and the Slopes of Curves

Suppose we have a function $y = f(x)$. If we change x from $x = a$ to $x = a + \Delta x = a + h$, for a step or difference in x of $\Delta x = h$, then the value of the function (asuming it's not constant), will change from $f(a)$ to $f(a + h)$.

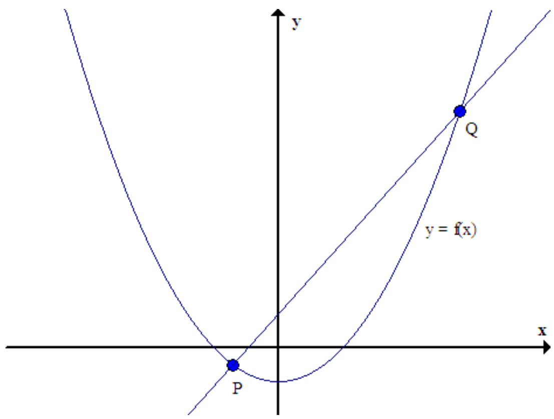


If we look at the change in the value of the function, $\Delta f = f(a + h) - f(a)$, relative to the change in the independent variable x, $\Delta x = (a + h) - a = h$, we'll have $\dfrac{\Delta f}{\Delta x} = \dfrac{f(a + h) - f(a)}{h}$, which is the average rate of change of the function on the interval $a \le x \le a + h$.

The most familiar example of this is probably the velocity of a moving object. if you drive a distance of 120 km in an hour and a half, your average speed for the trip is $\dfrac{120 \text{ km}}{1.5 \text{ h}} = 80$ km/h (and this is a change in position divided by a change in time).

Let's go back to what we had above. Notice that the expression $\dfrac{\Delta f}{\Delta x} = \dfrac{f(a + h) - f(a)}{h}$ (called the difference quotient) would represent the slope of a straight line through the two points $(a, f(a))$ and $(a + h, f(a + h))$ on the curve.

A line that passes through two points P and Q on a curve $y = f(x)$ is called a secant line.



Example:
Find the slope of the secant line passing through the points $(-1, 1)$ and $(2, 4)$ on the curve $y = x^2$.

$\dfrac{\Delta y}{\Delta x} = \dfrac{4-1}{2-(-1)} = \dfrac{3}{3} = 1$ and the line is $y = x + 2$ (*can you see that?*).

Example:
Suppose the population of a small town was measured every year from 2001 to 2010.

Year	Population
2001	6210
2002	6347
2003	6523
2004	6704
2005	6851
2006	6975
2007	7087
2008	7214
2009	7326
2010	7472

(i) What was the average rate of change of the population over the entire period?
(ii) How about over 2006 to 2010?

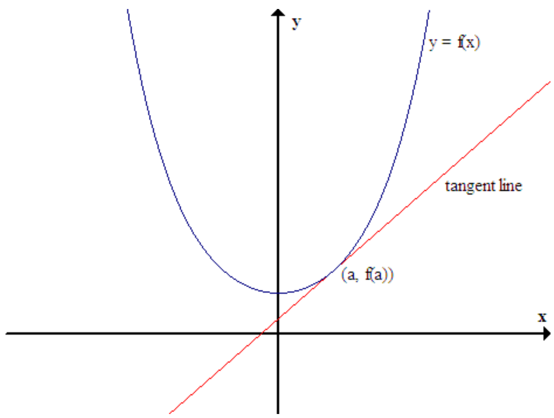
(i) $\dfrac{\Delta P}{\Delta t} = \dfrac{7472-6210}{2010-2001} \approx 140$ people/year

(ii) $\dfrac{\Delta P}{\Delta t} = \dfrac{7472-6975}{2010-2006} \approx 124$ people/year

In our driving example above, we had that the average speed over the trip was 80 km/h.

It is unlikely that we were actually driving at exactly 80 km/h the entire 1.5 hours – more likely, at times we were going faster and at other times, slower. Also, the average does not tell us what our speed was at any particular time, which would be an instantaneous velocity (speed at one particular instant).

How could we find the instantaneous rate of change of a function $y = f(x)$ at the value $x = a$? By recognizing that this would correspond to the "slope" of the curve at this point, which would have to be equal to the slope of the tangent line to the curve at the point.



The tangent line to a curve at a point $P = (a, f(a))$ is the straight line that passes through the point and best approximates the curve near the point.

So if we have the graph of the function, we could draw the tangent line and find its slope. (*But this would only be an approximation or estimate as we cannot draw the tangent line with perfect accuracy.*)

If we have a table of values, we could calculate the average rate of change over as short an interval as possible that contains the point of interest (*but again, only an estimate*).

Example:
The instantaneous rate of change m of the population of the town in 2004 is approximately
$$m \approx \frac{\Delta P}{\Delta t} = \frac{P(2005) - P(2004)}{2005 - 2004} = \frac{6851 - 6704}{1} = 147 \text{ people/year.}$$

Notice that either way (graphically or numerically), the best we can do is an estimate. Why do we have this problem? Because we only know that the tangent line passes through the point $(a, f(a))$ and nothing else. We cannot calculate the slope (or find the equation) of the line knowing only a single point. So, clearly, if we wish to calculate instantaneous rates of change (*which we do*), we need to figure out a way to do so.

Rates of Change Using Equations

If we have the equation of a function, $y = f(x)$, we can make more accurate estimates of the instantaneous rate of change when $x = a$ by finding the average rate of change over a small interval $a \leq x \leq a + h$.

Example:
The position (in meters) of a moving object is given by $s(t) = 2t^2 + 3t + 2$, where t is measured in seconds. What is the instantaneous velocity of the object at time $t = 2$ s ?

The slope of the secant line (which we are using as an approximation of the tangent line) passing through $P = (2, s(2))$ and $Q = (2 + h, s(2 + h))$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(2+h) - s(2)}{(2+h) - 2}$$

$$= \frac{2(2+h)^2 + 3(2+h) + 2 - (2(2)^2 + 3(2) + 2))}{h}$$

$$= \frac{2(4 + 4h + h^2) + 6 + 3h + 2 - 16}{h}$$

$$= \frac{8 + 8h + 2h^2 + 6 + 3h + 2 - 16}{h}$$

$$= \frac{11h + 2h^2}{h}$$

$$= 11 + 2h$$

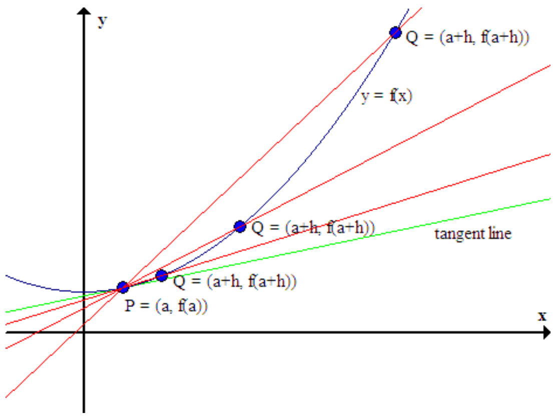
so if $h = 1$ s, $\dfrac{\Delta s}{\Delta t} = 13$ m/s

so if $h = 0.1$ s, $\dfrac{\Delta s}{\Delta t} = 11.2$ m/s

so if $h = 0.01$ s, $\dfrac{\Delta s}{\Delta t} = 11.02$ m/s

and so it appears that the instantaneous velocity of the object at time $t = 2$ s is 11 m/s.

What are we actually doing here? We're using secant lines to approximate the tangent line.



The smaller h is, the closer Q is to P and the better the secant line approximates the tangent line. And as we shrink h to zero, the difference quotient $\dfrac{f(a+h)-f(a)}{h}$, which is the slope of the secant line or the average rate of change on the interval $a \leq x \leq a+h$ becomes closer and closer to the slope of the tangent line or the instantaneous rate of change at $x = a$.

But what do we mean by shrinking h to 0 and something becoming closer and closer to something else?

Limits

The set of natural numbers is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. An infinite sequence is an infinite list of numbers generated by a function $f(n) = a_n$ whose domain is $\mathbb{N}$.

Example:
$2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \frac{2}{81}, \ldots$
The general term here is $a_n = \dfrac{2}{3^n}$ and as n gets bigger, the values will get smaller and smaller or closer to 0. There is an a_n for all n, no matter how large n is and we can denote the idea of n getting larger and larger without bound by saying that n approaches infinity, which we write as $n \to \infty$. As $n \to \infty$, $a_n \to 0$. We can write this as a limit: $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{3^n} = 0$.

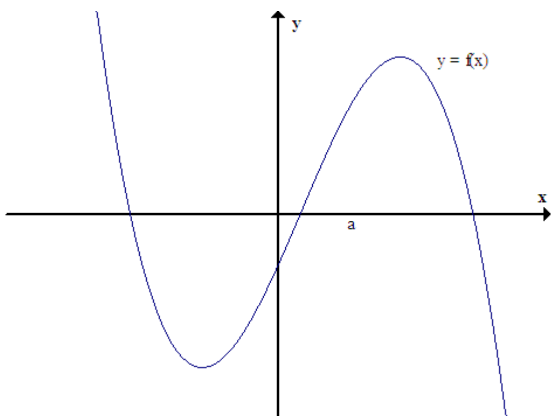
Example:
If $a_n = (-1)^n$, we have the sequence
$1, -1, 1, -1, \ldots$
Here, $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n$ does not exist because the terms in the sequence are not approaching a single value.

Example:
$1, 2, 4, 8, 16, 32, \ldots$
Here $a_n = 2^n$ and then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^n = \infty$. This limit does not exist as the terms in the sequence are growing without bound, getting larger and larger, and so they do not approach a single (finite) value.

∞ *is not a number – it represents the idea of unbounded growth.*

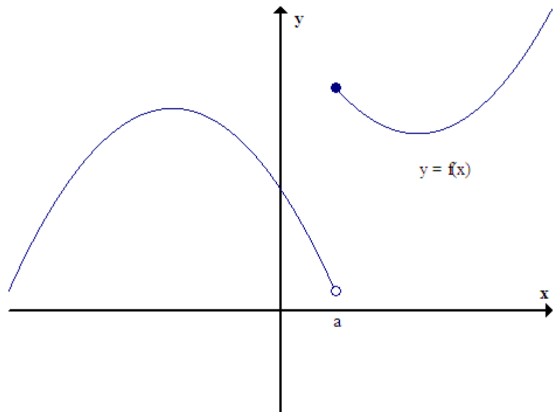
If $\lim_{n\to\infty} a_n = L$, where L is a unique and finite number, then the sequence $\{a_n\}$ has a limit as $n \to \infty$ and is said to converge to L which means that as n gets larger and larger, the values of a_n approach L.

Given a function $f(x)$, we can also look at what happens as $x \to a$, *ie* take $\lim_{x\to a} f(x)$.



We can approach a from either the left side, where $x < a$ or the right side, where $x > a$. This leads to the limit from the left $\lim_{x\to a^-} f(x)$ and the limit from the right $\lim_{x\to a^+} f(x)$.

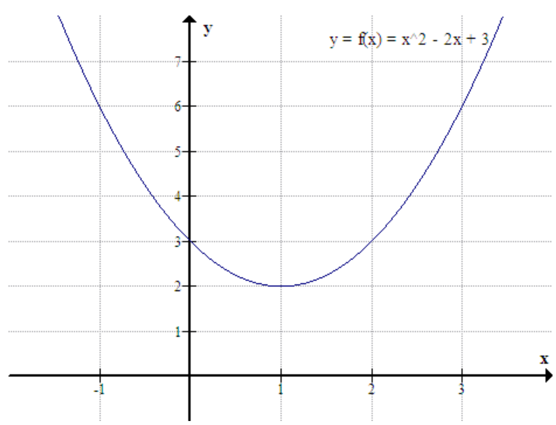
Suppose that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but are different (not equal).



Then $\lim_{x \to a} f(x)$ cannot exist as $f(x)$ is not approaching a single value as $x \to a$.

What if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$? Then $\lim_{x \to a} f(x)$ must exist and be the same value.

Example:
Consider the function $f(x) = x^2 - 2x + 3$. What is $\lim_{x \to 2} f(x)$?



We can see from the graph that as $x \to 2^-$ (*ie* from the left, $x = 1.9, 1.99, 1.999, \ldots$), the value of the function will increase up to 3. And as $x \to 2^+$ (*ie* from the right, $x = 2.1, 2.01, 2.001, \ldots$), the value of the function decreases to 3.
So we have $\lim_{x \to 2^-} f(x) = 3 = \lim_{x \to 2^+} f(x)$ and so $\lim_{x \to 2} f(x) = 3$.
We could have seen this numerically as well.

x	$f(x)$	x	$f(x)$
1.9	2.81	2.1	3.21
1.99	2.9801	2.01	3.0201
1.999	2.998001	2.001	3.002001

Can you see what these values are saying about $\lim_{x\to 2^-} f(x)$ *and* $\lim_{x\to 2^+} f(x)$ *and hence about* $\lim_{x\to 2} f(x)$ *?*

If we look at our example above, we see that $\lim_{x\to 2} f(x)$ is simply the value of $f(x)$ at $x = 2$. We can see that this must be the case because if we traced along the curve from either side of 2 towards 2, we would not experience any breaks in the graph and so we approach $f(2)$. This means that our function $f(x) = x^2 - 2x + 3$ is <u>continuous at $x = 2$</u>.

We say that <u>$f(x)$ is continuous at $x = a$</u> if three conditions are met.
(i) $f(a)$ is defined
(ii) $\lim_{x\to a} f(x)$ exists
(iii) $\lim_{x\to a} f(x) = f(a)$

A function is continuous at $x = a$ if you can draw the graph near at $x = a$ without lifting your pencil. A function is called <u>continuous</u> if it is continuous for all x in its domain. If there is a place where there is a break in the graph, we have a <u>discontinuity</u> (and at least one of the three conditions above is violated).

Limits and Continuity

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and c is any constant, we have the following <u>limit properties</u>.
(i) $\lim_{x\to a} c = c$
(ii) $\lim_{x\to a} x = a$
(iii) $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
(iv) $\lim_{x\to a}(cf(x)) = c\left(\lim_{x\to a} f(x)\right)$
(v) $\lim_{x\to a}(f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
(vi) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
(vii) $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$, if n is rational
(viii) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, provided the root on the right hand side exists

These allow us to find many limits (which are really then being calculated by substitution).

Examples:
(i) $\lim_{x\to -3} 27 = 27$

(ii) $\lim_{x \to -3} x = -3$

(iii) $\lim_{x \to 2}(2x^3 + 3x - 4) = 2\left(\lim_{x \to 2} x\right)^3 + 3\left(\lim_{x \to 2} x\right) - \lim_{x \to 2} 4 = 2(2)^3 + 3(2) - 4 = 18$

(iv) $\lim_{x \to 1} \dfrac{5x}{x-1} = \dfrac{5\left(\lim_{x \to 1} x\right)}{\lim_{x \to 1}(x-1)} = \dfrac{5}{0}$ does not exist

(v) $\lim_{x \to 4} \sqrt{\dfrac{x^2+1}{x+2}} = \sqrt{\dfrac{\left(\lim_{x \to 4} x\right)^2 + 1}{\left(\lim_{x \to 4} x\right) + 2}} = \sqrt{\dfrac{4^2+1}{4+2}} = \sqrt{\dfrac{17}{6}}$

What we can recognize is that algebraic functions, rational functions and polynomials are all continuous everywhere on their domains and hence limits (within the domains) can be calculated by simply substituting.

Examples:

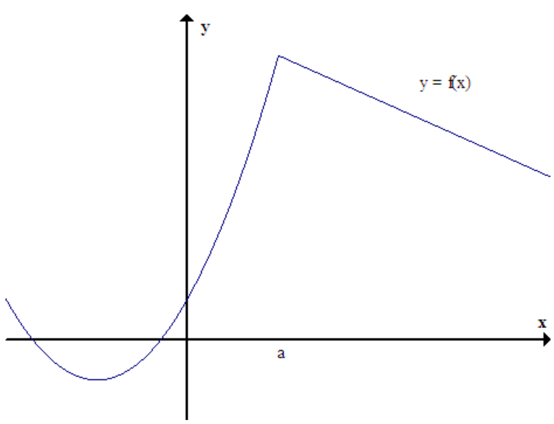
(i) $\lim_{x \to -1}(2x^2 + 7x - 2) = 2(-1)^2 + 7(-1) - 2 = -7$

(ii) $\lim_{x \to 3} \dfrac{x+7}{x-2} = \dfrac{3+7}{3-2} = \dfrac{10}{1} = 10$

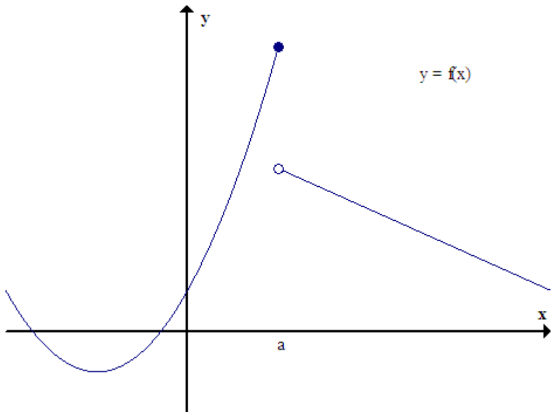
(iii) $\lim_{x \to 2} \sqrt{x-7} = \sqrt{-5}$ does not exist (2 *is not in the domain*)

(iv) $\lim_{x \to 2} \sqrt{2x+1} = \sqrt{2(2)+1} = \sqrt{5}$

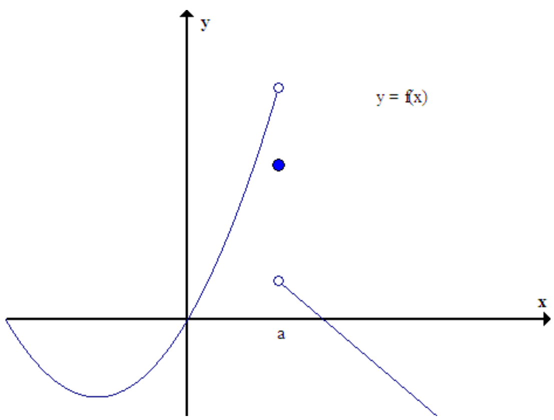
We said that $f(x)$ is continuous at $x = a$ if $f(a)$ is defined, $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$ and we said that this would correspond to being able to draw the graph of $f(x)$ without lifting our pencil or without there being a break in the graph at $x = a$. But what if there is a discontinuity at $x = a$? Consider the following graphs.



$f(a)$ defined, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ so $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$ and thus $f(x)$ is continuous at $x = a$.



$f(a)$ is defined, $\lim_{x \to a^-} f(x)$ exists and $\lim_{x \to a^+} f(x)$ exists but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ so $\lim_{x \to a} f(x)$ does not exist and thus $f(x)$ is discontinuous at $x = a$. This is called a jump discontinuity.



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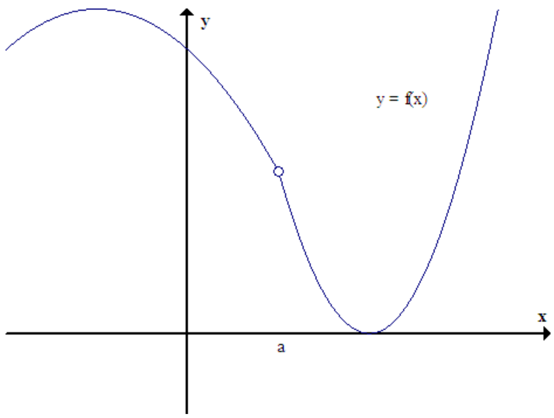
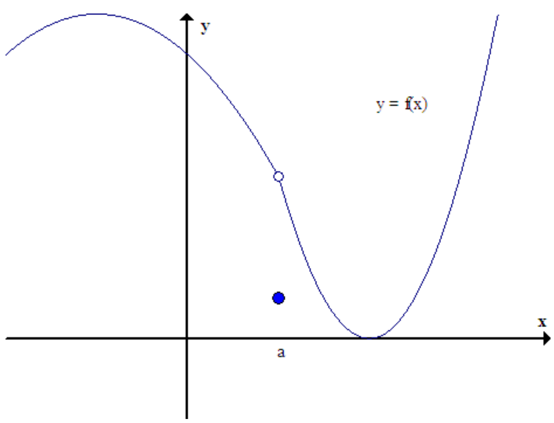
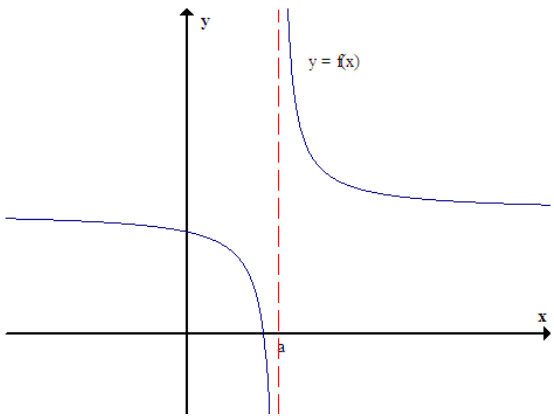


$f(a)$ is not defined, but $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ so $\lim\limits_{x\to a} f(x)$ exists, but $f(x)$ is discontinuous at $x = a$. This is called a hole or removable discontinuity.



$f(a)$ is defined, $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$ so $\lim\limits_{x\to a} f(x)$ exists, but $\lim\limits_{x\to a} f(x) \neq f(a)$, so $f(x)$ is discontinuous at $x = a$. This is called a hole or removable discontinuity.

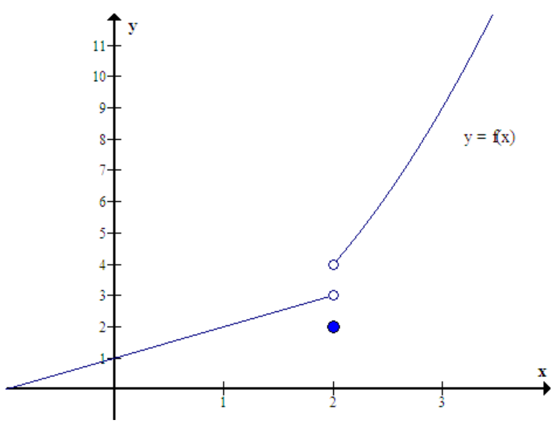


$f(a)$ is not defined, $\lim_{x\to a^-} f(x)$ does not exist, $\lim_{x\to a^+} f(x)$ does not exist and $\lim_{x\to a} f(x)$ does not exist. $f(x)$ is discontinuous at $x = a$. This is called a vertical asymptote.

Example:

Consider the function $f(x) = \begin{cases} 1+x & x<2 \\ 2 & x=2 \\ x^2 & x>2 \end{cases}$.

This is an example of a piecewise defined function.



$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (1+x) = 3$,
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 4$,
so $\lim_{x\to 2} f(x)$ does not exist (so $f(x)$ cannot be continuous at $x = 2$),
while $f(2) = 2$,

but $f(x)$ has a jump discontinuity at $x = 2$.

Consider the following limits.

(i) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x-3}$

(ii) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

(iii) $\displaystyle\lim_{x\to 2} \frac{(x-1)^2-1}{x-2}$

Substituting into all of these will yield the indeterminate form $\frac{0}{0}$ (which is undefined). We can evaluate limits like these by doing certain manipulations.

(i) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x-3}$

$\displaystyle= \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3}$ *(factor)*

$\displaystyle= \lim_{x\to 3}(x+3)$ *(cancel common factor – allowed because $x \neq 3$)*

$= 6$

(ii) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

$\displaystyle= \lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$ *(rationalize the numerator)*

$\displaystyle= \lim_{x\to 1} \frac{(x+3)-4}{(x-1)(\sqrt{x+3}+2)}$

$\displaystyle= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x+3}+2)}$

$\displaystyle= \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2}$ *(cancel common factor)*

$= 1/4$

(iii) $\displaystyle\lim_{x\to 2} \frac{(x-1)^2-1}{x-2}$

$\displaystyle= \lim_{x\to 2} \frac{x^2-2x+1-1}{x-2}$ *(expand)*

$\displaystyle= \lim_{x\to 2} \frac{x^2-2x}{x-2}$

$\displaystyle= \lim_{x\to 2} \frac{x(x-2)}{x-2}$ *(factor)*

$\displaystyle= \lim_{x\to 2} x$ *(cancel common factor)*

$= 2$

(All of these discontinuities are removable.)

Example:

Is $f(x) = \dfrac{x^2 - 2x - 3}{x^2 + 5x + 4}$ continuous at $x = -1$? Does the limit exist at $x = -1$?

$f(-1) = \frac{0}{0}$, so $f(x)$ is not defined at $x = -1$ and so it cannot be continuous there.

$\displaystyle\lim_{x \to -1} \frac{x^2 - 2x - 3}{x^2 + 5x + 4}$
$\displaystyle\lim_{x \to -1} \frac{(x-3)(x+1)}{(x+1)(x+4)}$
$\displaystyle\lim_{x \to -1} \frac{x-3}{x+4}$
$= -4/3$
so yes, $\displaystyle\lim_{x \to -1} f(x)$ exists.
And this discontinuity is removable – so the graph of the function would have a hole at point $(-1, -4/3)$.

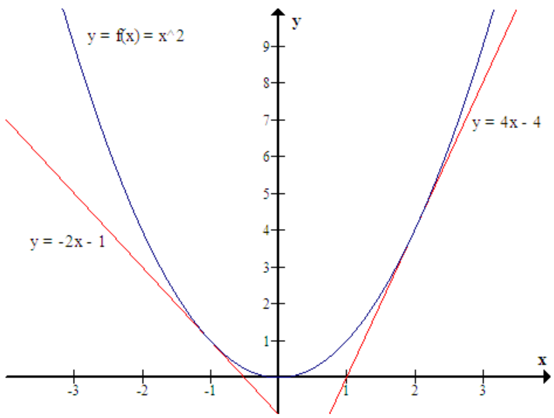
Introduction to Derviatives

If we go back to what we had previously, we had said that the instantaneous rate of change of a function $y = f(x)$ at point $P = (a, f(a))$, which is the slope m of the tangent line to the curve at that point, could be found by taking the slope of the secant line through P and $Q = (a + h, f(a + h))$, which would be the difference quotient $\dfrac{f(a+h) - f(a)}{h} = \dfrac{\Delta f}{\Delta x} = \dfrac{\Delta y}{\Delta x}$, and letting h shrink to 0. We can now understand that we mean take the limit as $h \to 0$. And so $m = \displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$. Note that this will always be an indeterminate form $\frac{0}{0}$ on substitution of $h = 0$, so we will have to do some manipulations to calculate these limits.

The instantaneous rate of change of $y = f(x)$ at $P = (a, f(a))$ is equal to the slope of the tangent line to the curve $y = f(x)$ at $x = a$ and is also called <u>the derivative of $y = f(x)$ at $x = a$</u> and is written $f'(a)$.
So $f'(a) = \displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ (which is called the <u>first principles definition</u>).

Example:
Consider the function $y = f(x) = x^2$. Find the derivative and the equations of the tangent

lines at $x = -1$ and $x = 2$.



$$
\begin{aligned}
f'(-1) &= \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} \\
&= \lim_{h\to 0} \frac{(-1+h)^2 - (-1)^2}{h} \\
&= \lim_{h\to 0} \frac{(1 - 2h + h^2) - (1)}{h} \\
&= \lim_{h\to 0} \frac{-2h + h^2}{h} \\
&= \lim_{h\to 0} -2 + h \\
&= -2
\end{aligned}
$$

so the tangent line passes through $(-1, 1)$ with slope $m = -2$ and so its equation is $y - y_0 = m(x - x_0)$ (*slope-point formula*) or $y - 1 = (-2)(x - (-1))$ or $y = -2x - 1$.

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^2 - (2)^2}{h} \\
&= \lim_{h\to 0} \frac{(4 + 4h + h^2) - (4)}{h} \\
&= \lim_{h\to 0} \frac{4h + h^2}{h} \\
&= \lim_{h\to 0} 4 + h \\
&= 4
\end{aligned}
$$

so the tangent line is $y - 4 = (4)(x - 2)$ or $y = 4x - 4$.

If we look at the graph of $y = f(x) = x^2$, we can see that we would be able to draw a tangent line to the curve at any point but its slope would depend on the value of x. So the derivative of a function is itself a function of x and we can find it as such

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

So, for $f(x) = x^2$,
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$= \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h}$
$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - (x^2)}{h}$
$= \lim_{h \to 0} \frac{2xh + h^2}{h}$
$= \lim_{h \to 0} 2x + h$
$= 2x$ *(can you see that this is what we have in the example above?)*

There is another notation for the derivative: if $y = f(x)$, $f'(x) = y'$ and $f'(x) = \frac{df}{dx} = \frac{dy}{dx}$ (from $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$) and $f'(x) = \frac{d}{dx}(f(x))$ (operator notation). If we are evaluating the derivative at $x = a$, $f'(a) = y'(a) = \frac{dy}{dx}|_{x=a} = \frac{df}{dx}|_{x=a}$. $\frac{dy}{dx}$ is read as "dee y by dee x" and it represents the derivative of y with respect to (wrt) x.

Example:
If $f(x) = x^3$,
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$= \lim_{h \to 0} \frac{(x+h)^3 - (x)^3}{h}$
$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - (x^3)}{h}$
$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$
$= \lim_{h \to 0} 3x^2 + 3xh + h^2$
$= 3x^2$.

Example:
If $f(x) = \frac{1}{x} = x^{-1}$,
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$
$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x - (x+h)}{x(x+h)} \right]$
$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-h}{x(x+h)} \right]$

$= \lim_{h \to 0} \dfrac{-1}{x(x+h)}$
$= \dfrac{-1}{x^2}.$

Do you notice something? We've just seen this:

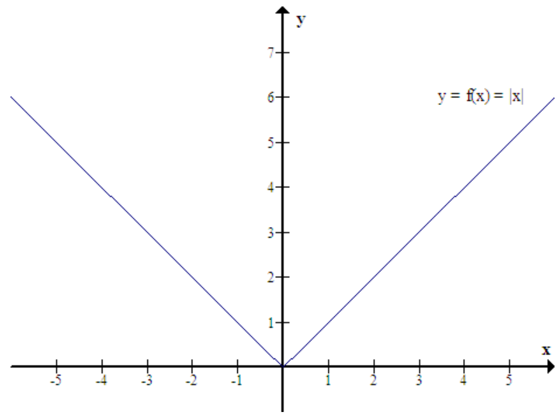
$f(x)$	$f'(x)$
x^3	$3x^2$
x^2	$2x$
x^{-1}	$-x^{-2}$

Can you see a pattern?

What if $y = f(x) = mx + b$, a straight line? Then the line would be tangent to itself everywhere and hence the slope of the tangent would always be m. So we must have that $f'(x) = m$. Let's verify:
$\dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
$= \lim_{h \to 0} \dfrac{(m(x+h)+b) - (mx+b)}{h}$
$= \lim_{h \to 0} \dfrac{mh}{h}$
$= \lim_{h \to 0} m$
$= m$ *(as expected).*

Example:
Let's go back to our moving object with position function $s(t) = 2t^2 + 3t + 2$ (where s is in m and t in s). The velocity $v(t)$ of the object is the instantaneous rate of change of position with respect to time, or the derivative of $s(t)$ with respect to t.
So $v(t) = s'(t) = \dfrac{ds}{dt} = \lim_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$
$= \lim_{h \to 0} \dfrac{(2(t+h)^2 + 3(t+h) + 2) - (2t^2 + 3t + 2)}{h}$
$= \lim_{h \to 0} \dfrac{(2(t^2 + 2th + h^2) + 3t + 3h + 2) - (2t^2 + 3t + 2)}{h}$
$= \lim_{h \to 0} \dfrac{4th + 2h^2 + 3h}{h}$
$= \lim_{h \to 0} 4t + 2h + 3$
$= 4t + 3$
so, in particular, the velocity at time $t = 2$ s is $v(2) = s'(2) = 4(2) + 3 = 11$ m/s.

Since the derivative (or instantaneous rate of change or slope of the tangent line) is defined by a limit, we should be careful to realize that it does not have to exist for all x. We say that $f(x)$ is differentiable at $x = a$ if $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. How could $f'(a)$ fail to exist? If a is not in the domain of f, $f(a)$ is undefined and $f'(a)$ cannot exist. If f is discontinuous at $x = a$, $f'(a)$ will also fail to exist (*we can't draw a tangent to the curve at* *that point*). But, it is also possible for $f(x)$ to be continuous at $x = a$ and for $f'(a)$ to fail to exist. Consider $y = f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$.



Notice that $f(x)$ is continuous at $x = 0$ as $f(0) = 0$ and $\lim\limits_{x \to 0^-} f(x) = 0 = \lim\limits_{x \to 0^+} f(x)$. But, if we try to find $f'(0)$, $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$, we'll have to consider the two one-sided limits separately as the definition of the function is different for positive and negative values.

$\lim\limits_{h \to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{|0+h| - |0|}{h} = \lim\limits_{h \to 0^-} \dfrac{|h|}{h} = \lim\limits_{h \to 0^-} \dfrac{-h}{h} = \lim\limits_{h \to 0^-} -1 = -1.$

And $\lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{|0+h| - |0|}{h} = \lim\limits_{h \to 0^+} \dfrac{|h|}{h} = \lim\limits_{h \to 0^+} \dfrac{h}{h} = \lim\limits_{h \to 0^+} 1 = 1.$

Thus $\lim\limits_{h \to 0^-} \dfrac{f(0+h) - f(0)}{h} \neq \lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h}$ and so $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist. And hence, $y = f(x) = |x|$ is not differentiable at 0. The graph of $y = f(x) = |x|$ is said to have a <u>corner</u> at $x = 0$.

$f'(a)$ will also fail to exist if there is a <u>cusp</u> or <u>vertical tangent</u> at $x = a$.

Practice Problems

1. Given the data in the table, find the average rate of change of the function $g(t)$ on the following intervals.
(a) $0 \leq t \leq 2$
(b) $0 \leq t \leq 4$
(c) $4 \leq t \leq 7$

t	$g(t)$
0	1
1	3
2	6
3	9
4	14
5	20
6	28
7	37
8	49

2. Use rates of change to estimate the instantaneous velocity of a moving object at time $t = 3$ s if the position of the object is given by the function $s(t) = 4t^2 - 2t + 1$ (in m).

3. Find the limits of the sequences.
(a) $1, \frac{5}{2}, \frac{5}{3}, \frac{9}{4}, \frac{9}{5}, \frac{13}{6}, \frac{13}{7}, \ldots, 2 + \frac{(-1)^n}{n}, \ldots$
(b) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots, 3^{1-n}, \ldots$
(c) $0.6, 0.66, 0.666, 0.6666, 0.66666, \ldots$
(d) $1, -2, 3, -4, 5, -6, \ldots, (-1)^{n-1}n, \ldots$

4. Find the limits.
(a) $\lim\limits_{x \to -2} x^2 + 3x - 7$
(b) $\lim\limits_{x \to 3} \frac{x^2 + 1}{x + 7}$
(c) $\lim\limits_{x \to 1} \sqrt{\frac{x + 2}{x^2 + 4}}$
(d) $\lim\limits_{x \to 0} \frac{2x^2 - 5x + 2}{\sqrt{x + 4}}$

5. Consider the function $f(x) = \begin{cases} x + 4 & x < -2 \\ 2 & -2 \leq x < 1 \\ 1 - x^2 & 1 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$.
Determine if the function has any discontinuities. If so, what type(s)?

6. Consider $g(x) = \dfrac{x^2 - 4}{x - 2}$. Is $g(x)$ defined for $x = 2$? Is $g(x)$ continuous at $x = 2$? Does $\lim\limits_{x \to 2} g(x)$ exist?

7. Find the limits.

(a) $\lim\limits_{x \to 0} \dfrac{2x^2 + 3x}{x + x^2}$

(b) $\lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{9 - x}$

(c) $\lim\limits_{x \to 4} \dfrac{\frac{1}{4} - \frac{1}{x}}{4 - x}$

(d) $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x - 3}$

8. What is the derivative of $f(x) = \sqrt{x} = x^{1/2}$?

9. If a moving object has position function $s(t) = 4t^2 + 2t$ (in m), what is the velocity at time $t = 5$ s?

10. What is the equation of the tangent line to the curve $y = \dfrac{1}{x^2}$ at the point $(1, 1)$?

Practice Problems Solutions

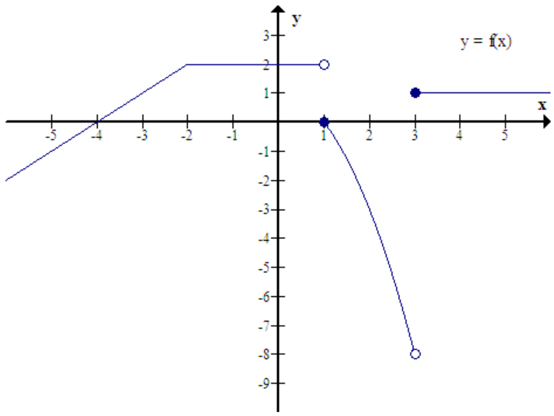
1. (a) $\dfrac{g(2)-g(0)}{2-0} = \dfrac{6-1}{2} = 5/2$
(b) $\dfrac{g(4)-g(0)}{4-0} = \dfrac{14-1}{4} = 13/4$
(c) $\dfrac{g(7)-g(4)}{7-4} = \dfrac{37-14}{3} = 23/3$

2. $v(3) \approx \dfrac{s(3+h)-s(3)}{h}$
$= \dfrac{(4(3+h)^2-2(3+h)+1)-(4(3)^2-2(3)+1)}{h}$
$= \dfrac{4(9+6h+h^2)-6-2h+1-4(9)+6-1}{h}$
$= \dfrac{24h+4h^2-2h}{h} = \dfrac{22h+4h^2}{h} = 22+4h$
so if $h=0.1$, $v(3) \approx 22.4$ m/s
if $h=0.01$, $v(3) \approx 22.04$ m/s
if $h=0.001$, $v(3) \approx 22.004$ m/s
and thus it looks like $v(3) = 22$ m/s

3. (a) since $\dfrac{(-1)^n}{n} \to 0$ as $n \to \infty$, $\displaystyle\lim_{n\to\infty} 2 + \frac{(-1)^n}{n} = 2$
(b) clearly $\displaystyle\lim_{n\to\infty} 3^{1-n} = \lim_{n\to\infty} \frac{3}{3^n} = 0$
(c) here the sequence is converging to 2/3, so $\displaystyle\lim_{n\to\infty} a_n = 2/3$
(d) no limit or limit does not exist

4. (a) $4-6-7=-9$
(b) $10/10 = 1$
(c) $\sqrt{3/5}$
(d) $2/\sqrt{4} = 1$

5.



from the graph, we see that the function is continuous at $x = -2$, but there are jump discontinuities at $x = 1$ and $x = 3$

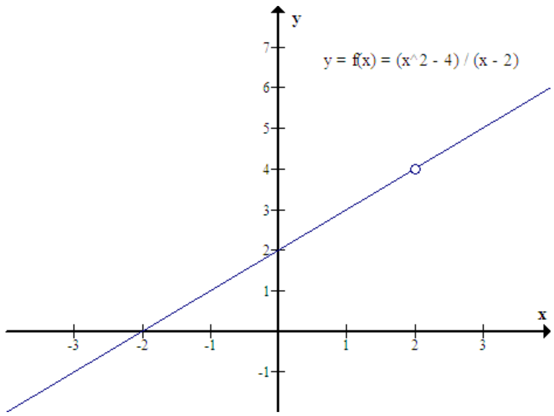
OR

since the definition of $f(x)$ changes at $x = -2, 1, 3$, we check what happens there:
$\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-} x + 4 = 2$ and $\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+} 2 = 2$
so $\lim\limits_{x\to -2} f(x) = 2$ and since $f(-2) = 2$, the function is continuous at $x = -2$
$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 2 = 2$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} 1 - x^2 = 0$
so $\lim\limits_{x\to 1} f(x)$ does not exist, the function has a jump discontinuity at $x = 1$
$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} 1 - x^2 = -8$ and $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} 1 = 1$
so $\lim\limits_{x\to 3} f(x)$ does not exist, the function has a jump discontinuity at $x = 3$

6. no, $g(2)$ is not defined (dividing by 0 or have $\frac{0}{0}$ indeterminate form)
no, since $g(2)$ is not defined, $g(x)$ cannot be continuous there
yes, since $\lim\limits_{x\to 2} g(x) = \lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+2)}{x-2} = \lim\limits_{x\to 2} x + 2 = 4$

$\langle$notice that $g(x) = \dfrac{x^2-4}{x-2} = x + 2,\ (x \neq 2)$ – *ie* the graph of $g(x)$ looks like the straight line

$y = x + 2$ with the point $(2, 4)$ removed and so the discontinuity is a hole (or removable)⟩



7. (a) $\lim_{x\to 0} \frac{2x^2+3x}{x+x^2} = \lim_{x\to 0} \frac{x(2x+3)}{x(1+x)} = \lim_{x\to 0} \frac{2x+3}{1+x} = 3$

(b) $\lim_{x\to 9} \frac{\sqrt{x}-3}{9-x} = \lim_{x\to 9} \frac{\sqrt{x}-3}{9-x}\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right) = \lim_{x\to 9} \frac{x-9}{(9-x)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{-1}{\sqrt{x}+3} = -1/6$

(c) $\lim_{x\to 4} \frac{\frac{1}{4}-\frac{1}{x}}{4-x} = \lim_{x\to 4} \frac{\frac{x-4}{4x}}{4-x} = \lim_{x\to 4} \frac{-1}{4x} = -1/16$

(d) $\lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x-3} = \lim_{x\to 3} x^2+3x+9 = 27$

8. $f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$

$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$

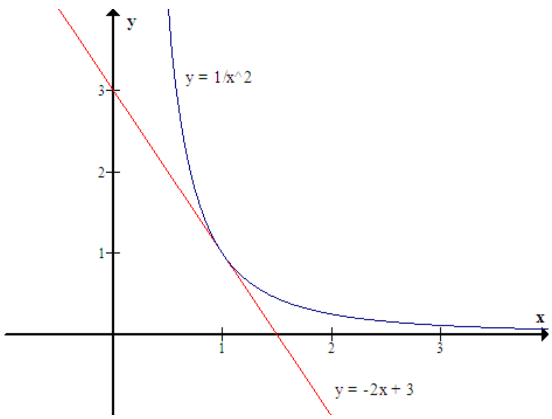
9. $v(5) = s'(5) = \lim_{h\to 0} \frac{s(5+h)-s(5)}{h}$

$= \lim_{h\to 0} \frac{4(5+h)^2+2(5+h)-110}{h}$

$= \lim_{h\to 0} \frac{4(25+10h+h^2)+10+2h-110}{h}$

$= \lim_{h\to 0} \frac{42h+4h^2}{h} = \lim_{h\to 0} 42+4h = 42$ m/s

10. $m = y'(1) = \lim_{h\to 0} \dfrac{\frac{1}{(1+h)^2} - \frac{1}{(1)^2}}{h} = \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{1-1-2h-h^2}{(1+h)^2}\right) = \lim_{h\to 0} \dfrac{-2-h}{(1+h)^2} = -2$
so $y - 1 = (-2)(x-1) \implies y = -2x = 3$



Chapter 2

Goals

- to be familiar with the basic rules of differentiation (constant, constant multiple, sum and difference, power, product, quotient and chain) and be able to use them to find the derivatives of given functions
- to be able to use derivatives to solve problems involving rates of change, including real-world applications from physics and business
- to be able to find second (and potentially higher) derivatives of functions using the rules

Derivatives of Polynomials

Let's look some more at differentiation (we're assuming that all functions below are differentiable).

Suppose that $f(x) = c$, a constant function, then
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$
ie $\frac{d}{dx}(c) = 0$, which is called the Constant Rule.
The rate of change of a constant is 0, which makes sense as a constant does not change.

Suppose $f(x) = cg(x)$, then
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
$= \lim_{h \to 0} \frac{cg(x+h) - cg(x)}{h} = \lim_{h \to 0} \frac{c(g(x+h) - g(x))}{h} = c\left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right) = cg'(x)$
ie $\frac{d}{dx}(cg(x)) = c\frac{d}{dx}(g(x))$, which is called the Constant Multiple Rule.

And if $f(x) = p(x) \pm q(x)$, then
$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{(p(x+h) \pm q(x+h)) - (p(x) \pm q(x))}{h}$
$= \lim_{h\to 0} \frac{p(x+h) - p(x)}{h} \pm \lim_{h\to 0} \frac{q(x+h) - q(x)}{h} = p'(x) \pm q'(x)$
ie $\frac{d}{dx}(p(x) \pm q(x)) = p'(x) \pm q'(x)$, which is called the Sum and Difference Rule.

Now, what if $f(x) = x^n$? The examples we have seen suggest that $f'(x) = nx^{n-1}$ or $\frac{d}{dx}(x^n) = nx^{n-1}$. If $n \in \mathbb{N}$, this is easily proven to be true:
$\frac{d}{dx}(x^n) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$
$= \lim_{h\to 0} \frac{1}{h}\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n - x^n\right]$
$= \lim_{h\to 0} \frac{1}{h}\left[nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n\right]$
$= \lim_{h\to 0} nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \frac{n(n-1)(n-2)}{3!}x^{n-3}h^2 + \cdots + nxh^{n-2} + h^{n-1}$
$= nx^{n-1}$

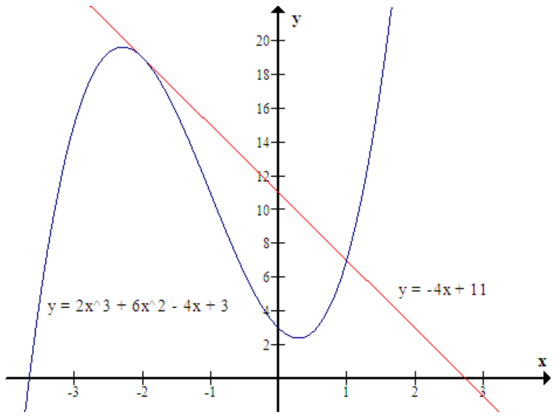
But we have also seen examples that suggest the rule is true for negative integer and rational powers. In fact, the Power Rule $\frac{d}{dx}(x^n) = nx^{n-1}$ is true for all real n (*we just can't prove that yet*).

Examples:
(i) $\frac{d}{dx}\left(\sqrt[3]{x}\right) = \frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$
(ii) $\frac{d}{dx}\left(4x^2\right) = 4\frac{d}{dx}\left(x^2\right) = 4(2x) = 8x$
(iii) If $f(x) = 2x^4 - 3x^2 + x$, then $f'(x) = 2(4x^3) - 3(2x) + 1 = 8x^3 - 6x + 1$.
(iv) If $y = 3t^2 - \sqrt{t}$, then $\frac{dy}{dt} = 6t - \frac{1}{2\sqrt{t}}$.
(*Note that the rules are independent of the names given to the variables.*)

Example:
Find the equation of the tangent line to the curve $y = f(x) = 2x^3 + 6x^2 - 4x + 3$ at $x = -2$.

The slope of the line is $m = f'(-2)$.
$f'(x) = 6x^2 + 12x - 4$, so $m = f'(-2) = 6(-2)^2 + 12(-2) - 4 = -4$.
The point on the curve has y coordinate $f(-2) = 2(-2)^3 + 6(-2)^2 - 4(-2) + 3 = 19$.

So the tangent line is $y - 19 = (-4)(x - (-2))$ or $y = -4x + 11$.



Example:
The height above the ground, in metres, of an object dropped from the top of a building of height 60 m, after t seconds, is $h(t) = 60 - 4.9t^2$.
(i) How fast is the object falling after 2 s ?
(ii) How fast does the object hit the ground ?

(i) The velocity is $v(t) = h'(t) = -9.8t$, so after 2 s, $v(2) = -19.6$ m/s (*the negative indicates downward motion*).
(ii) The object hits the ground when $h(t) = 0$ or $4.9t^2 = 60$ or $t = 3.5$ s, so $v(3.5) = 34.3$ m/s.

The Product Rule

Suppose we have a product of functions, $p(x) = f(x)g(x)$. It's tempting to think that $p'(x) = f'(x)g'(x)$, but a simple example will demonstrate that this is not true. If $f(x) = x^3$ and $g(x) = x^2$, then $p(x) = x^5$, so $p'(x) = 5x^4$. But $f'(x)g'(x) = (3x^2)(2x) = 6x^3$. So clearly, $p'(x) \neq f'(x)g'(x)$.

But what is the derivative of a product of differentiable functions ?

$$p'(x) = \lim_{h\to 0} \frac{p(x+h) - p(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \qquad (\textit{adding 0})$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0} \frac{g(x+h)[f(x+h) - f(x)]}{h} + \lim_{h\to 0} \frac{f(x)[g(x+h) - g(x)]}{h}$$

$= \left(\lim_{h\to 0} g(x+h)\right)\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right) + \left(\lim_{h\to 0} f(x)\right)\left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)$
$= g(x)f'(x) + f(x)g'(x)$

The Product Rule is $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$.
Derivative of the first times the second plus first times the derivative of the second.

Let's check our example: $f'(x)g(x) + f(x)g'(x) = (3x^2)(x^2) + (x^3)(2x) = 5x^4 = \frac{d}{dx}\left(x^5\right)$.

Examples:
(i) $\frac{d}{dx}\left((x^2+1)(x^2+3x-2)\right)$
$= \left(\frac{d}{dx}\left(x^2+1\right)\right)(x^2+3x-2) + (x^2+1)\left(\frac{d}{dx}\left(x^2+3x-2\right)\right)$
$= (2x)(x^2+3x-2) + (x^2+1)(2x+3)$
$= 2x^3 + 6x^2 - 4x + 2x^3 + 2x + 3x^2 + 3$
$= 4x^3 + 9x^2 - 2x + 3$
(*check this by expanding first and then differentiating*).
(ii) If $f(t) = (t+1)(3t^4 - t^2)$,
then $f'(t) = (1)(3t^4 - t^2) + (t+1)(12t^3 - 2t)$
$= 3t^4 - t^2 + 12t^4 + 12t^3 - 2t^2 - 2t$
$= 15t^4 + 12t^3 - 3t^2 - 2t$
(*check this, too*).
(iii) If $y = (2x^2+1)^2 = (2x^2+1)(2x^2+1)$,
then $\frac{dy}{dx} = (4x)(2x^2+1) + (2x^2+1)(4x)$
$= 2(2x^2+1)(4x)$ (*there is a pattern here*)
$= (8x)(2x^2+1)$
$= 16x^3 + 8x$.

How about $\frac{d}{dx}\left((2x^2+1)^3\right)$?
$\frac{d}{dx}\left((2x^2+1)^3\right) = \frac{d}{dx}\left((2x^2+1)^2(2x^2+1)\right)$
$= (2(2x^2+1)(4x)(2x^2+1) + (2x^2+1)^2(4x))$
$= 3(2x^2+1)^2(4x)$ (*can you see the pattern?*)

Example:
A local coffee shop sells 100 mochas per day at a price of \$2.75. They have determined that for each 25¢ price increase, they will sell 5 fewer per day.
If we let n be the number of 25¢ price increases, then the price per mocha would be $2.75 + 0.25n$ and the number sold would be $100 - 5n$.
So the revenue from the mocha sales would be $R(n) = (2.75 + 0.25n)(100 - 5n)$.
Then $\frac{dR}{dn} = (0.25)(100 - 5n) + (2.75 + 0.25n)(-5)$
$= 25 - 1.25n - 13.75 - 1.25n$

$= 11.25 - 2.5n.$

Velocity, Acceleration and Second Derivatives

Suppose an object moving in a straight line has position function $s(t)$. Then $s(t)$ gives the distance that the object is from the origin at time t. The sign of $s(t)$ tells us on which side of the origin the object is located – we usually take upward or rightward as the positive direction. The velocity is $v(t) = s'(t)$ and if $v(t) > 0$, the object is moving in the positive direction and if $v(t) < 0$, it is moving in the negative direction. The speed of the object is $|v(t)|$. If we were to differentiate again, we'd have the acceleration of the object, which is the rate of change of velocity wrt time, *ie* $a(t) = v'(t)$.

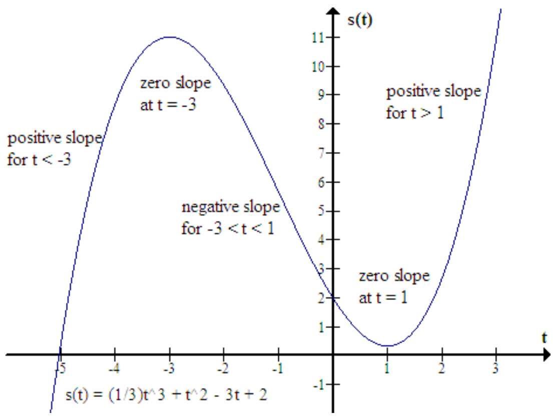
But this is $a(t) = v'(t) = \dfrac{d}{dt}(v(t)) = \dfrac{d}{dt}(s'(t)) = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d^2s}{dt^2} = s''(t)$,
the second derivative of the position function $s(t)$. And if $a(t) > 0$, the object is accelerating (the velocity is increasing). If $a(t) < 0$, the object is decelerating. Whether the object is speeding up or slowing down depends on the signs of both $v(t)$ and $a(t)$. If $v(t)$ and $a(t)$ have the same sign, the object is speeding up or going faster. If $v(t)$ and $a(t)$ have opposite signs, the object is slowing down.

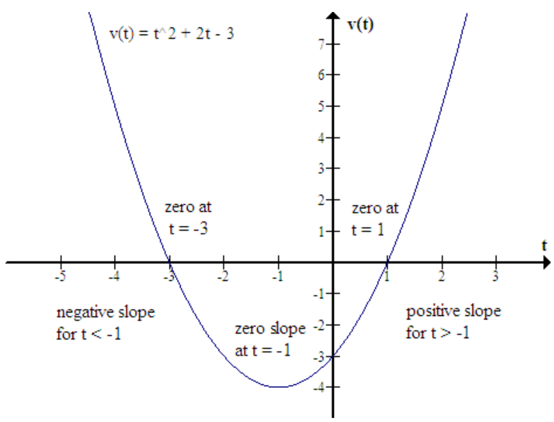
So, for example, if $v(t) < 0$ and $a(t) > 0$, the object is accelerating, which means slowing down in this case as $v(t) < 0$ means the object is moving in the negative direction whereas $a(t) > 0$ means that $v(t)$ is increasing and hence is becoming less negative and thus $|v(t)|$ is getting smaller or decreasing (*it will make more sense the second or third time you read it*).

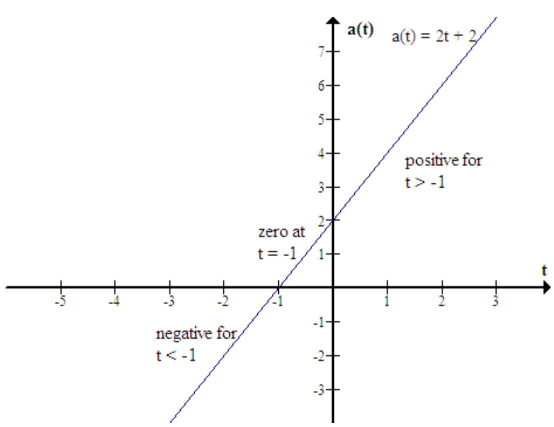
Example:
A ball is thrown straight up into the air. Its height, in metres, after t seconds, is
$h(t) = -4.9t^2 + 20t + 2$.
The velocity is $v(t) = h'(t) = -9.8t + 20$ and the acceleration is $a(t) = -9.8$ (*which is the acceleration due to gravity, usually denoted g*).
$v(t) = 0$ when $9.8t = 20$ or $t = 2.04$ s.
At this point, the ball has reached its maximum height $h(2.04) \approx 22.4$ m and will start to fall back to the ground.

Example:
An object moving in a staight line has position function $s(t) = \frac{1}{3}t^3 + t^2 - 3t + 2$ m, for t measured in seconds.
The velocity is $v(t) = s'(t) = t^2 + 2t - 3$ and the acceleration is $a(t) = v'(t) = s''(t) = 2t + 2$.

Have a look at the graphs to see the connections.





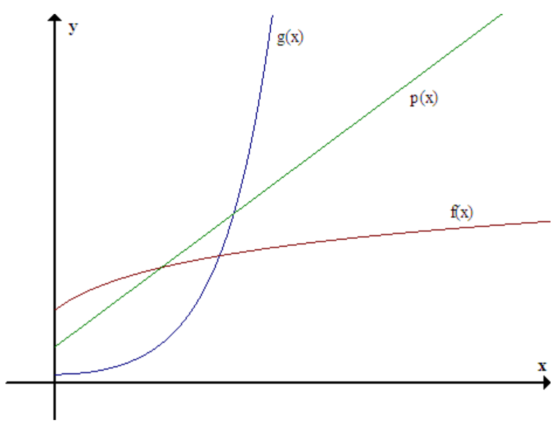


For example, $s(t)$ has zero slope at $t = -3$ and $t = 1$ and these are the roots of $v(t)$. Also,

$s(t)$ has a negative slope for $-3 < t < 1$, which is where $v(t)$ is negative. $v(t)$ has zero slope at $t = -1$, which is where $a(t) = 0$.
At time $t = 0, s(0) = 2, v(0) = -3$ and $a(0) = 2$, so the object is located 2 m to the right of the origin, moving with velocity -3 m/s (*ie* to the left) and with acceleration 2 m/s^2, so it is slowing down. It will come to a stop at time $t = 1$ s (where $v(t) = 0$ and $s(t) = 1/3$) and then start moving to the right, while continuing to speed up.

Examples:
(i) If $f(x) = \frac{1}{3}x^4 + 7x^3 - \pi x^2 - x + 1$, then
$f'(x) = \frac{4}{3}x^3 + 21x^2 - 2\pi x - 1$ and $f''(x) = 4x^2 + 42x - 2\pi$.
The third derivative would be $f'''(x) = 8x + 42$ and the fourth $f^{(4)}(x) = 8$.
(ii) If $y = 3t^2 - \sqrt{t} + 5t$, then
$\dfrac{dy}{dt} = 6t - \dfrac{1}{2\sqrt{t}} + 5$ and $\dfrac{d^2y}{dt^2} = 6 + \dfrac{1}{4t^{3/2}}$.

Example:
Consider the following graph.



What can we say about the signs of the first and second derivatives of the functions? Since all of the functions are increasing (their values increase as x increases), all have positive slope and hence $f' > 0$, $g' > 0$ and $p' > 0$. But since $p(x)$ is a straight line, $p''(x) = 0$. What about the second derivatives of f and g? The slopes of g would be increasing as x increases, so $g'' > 0$ whereas the slopes of f would be decreasing, so $f'' < 0$.
Draw tangents to the curves for different values of x to see this.

The Chain Rule

If we have two functions, $f(x)$ and $g(x)$, we can combine them to form new functions, like $f(x)g(x), f(x) + 2g(x), 3f(x) - \frac{1}{2}g(x)$, etc. Another way to combine them is to compose them, *ie* form the composite functions $f \circ g = f(g(x))$ and $g \circ f = g(f(x))$.

Example:
If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 2$, then $f \circ g = f(g(x)) = f(x^2 + 2) = \sqrt{x^2 + 2}$,
whereas $g \circ f = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 2 = x + 2$.
Notice that, in general, $f \circ g \neq g \circ f$.

In the composition $f(g(x))$, we call $g(x)$ the <u>inner function</u> and $f(x)$ the <u>outer function</u>. So if we had $q(x) = (2x^2 + x - 1)^4$, we'd have outer function $f(x) = x^4$ and inner function $g(x) = 2x^2 + x - 1$.
If you have trouble identifying the inner part of a composite function, just ask yourself what you would have to calculate first to evaluate the composite function at a specific value of x.

How do we differentiate composite functions?

Let's go back to an example we had earlier: $\dfrac{d}{dx}\left((2x^2+1)^3\right) = 3(2x^2+1)^2(4x)$.
Let $f(x) = x^3$ and $g(x) = 2x^2 + 1$, so we have $f(g(x)) = (2x^2+1)^3$.
Then $f'(x) = 3x^2$, so $f'(g(x)) = 3(2x^2+1)^2$ and $g'(x) = 4x$, so it looks like
$\dfrac{d}{dx}\left(f(g(x))\right) = \dfrac{d}{dx}\left((2x^2+1)^3\right) = 3(2x^2+1)^2(4x) = f'(g(x))g'(x)$ and that is exactly it.

<u>The Chain Rule</u> says that if f and g are differentiable, then $\dfrac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$.
Derivative of outer function wrt inner times derivative of inner.

Example:
$\dfrac{d}{dx}\left(\sqrt{2x^4+5x^2}\right)$
$= \dfrac{d}{dx}\left((2x^4+5x^2)^{1/2}\right)$
$= \dfrac{1}{2}(2x^4+5x^2)^{-1/2}\dfrac{d}{dx}\left(2x^4+5x^2\right)$
$= \dfrac{1}{2}(2x^4+5x^2)^{-1/2}(8x^3+10x)$
$= \dfrac{8x^3+10x}{2\sqrt{2x^4+5x^2}}$
$= \dfrac{4x^3+5x}{\sqrt{2x^4+5x^2}}$.

In the other notation, if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

To see that this is saying the same thing as the rule above:
if $y = f(u)$ and $u = g(x)$, then $y = f(g(x))$, so
$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(f(g(x))\right)$
$= f'(g(x))g'(x)$
$= f'(u)g'(x)$
$= \dfrac{df}{du}u'(x) \qquad$ (*since* $u = g(x)$)

$$= \frac{df}{du}\frac{du}{dx}$$
$$= \frac{dy}{du}\frac{du}{dx} \quad (\textit{since } y = f(u)).$$

When the outer function is just a power, $f(x) = x^n$, we have a special case of the Chain Rule called the Power of a Function Rule: $\frac{d}{dx}\left(f(g(x))\right) = \frac{d}{dx}\left((g(x))^n\right) = n(g(x))^{n-1}g'(x)$ (*which is all we have really dealt with so far*).

Examples:

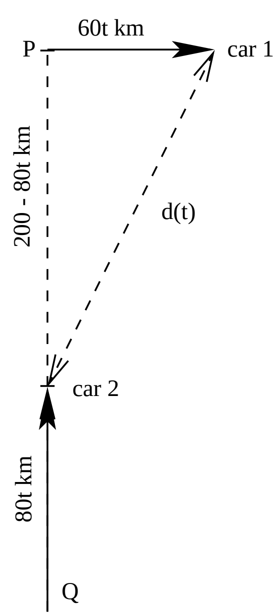
(i) $\frac{d}{dx}\left((5x^2 - x)^7\right) = 7(5x^2 - x)^6(10x - 1)$

(ii) $\frac{d}{dx}\left((2x+1)^4(6x^2-2)^3\right)$ (*product rule and chain rule*)
$= 4(2x+1)^3(2)(6x^2-2)^3 + (2x+1)^4(3)(6x^2-2)^2(12x)$
$= 4(2x+1)^3(6x^2-2)^2\left[2(6x^2-2) + 9(2x+1)\right]$
$= 4(2x+1)^3(6x^2-2)^2(12x^2+18x+5)$

(iii) If $p(t) = \sqrt[3]{t^2+3t+2} = (t^2+3t+2)^{1/3}$, then $p'(t) = \frac{1}{3}(t^2+3t+2)^{-2/3}(2t+3)$.

Example:

At time $t = 0$, a car starts driving eastward from point P at 60 km/h. A second car, initially at point Q 200 km south of the first starts driving northward at 80 km/h. After t hours, what is the distance between the cars and at what rate is that distance changing?



Car 1 has travelled $60t$ km east of P and car 2 has travelled $80t$ km north of Q, which is $200 - 80t$ km south of P.

So the distance between the cars (*by Pythagoras*) is
$d(t) = \sqrt{(60t)^2 + (200-80t)^2}$
$= \sqrt{3600t^2 + 40000 - 32000t + 6400t^2}$

$= \sqrt{10000t^2 - 32000t + 40000}$
$= 100\sqrt{t^2 - 3.2t + 4}$ km.
Rate of change is
$d'(t) = 100\left(\frac{1}{2}\right)(t^2 - 3.2t + 4)^{-1/2}(2t - 3.2)$
$= \dfrac{50(2t - 3.2)}{\sqrt{t^2 - 3.2t + 4}}$ km/h.
So, after 1 hour, for example, $d(1) \approx 134$ km and $d'(1) \approx -44.7$ km/h
($d'(1) < 0$ *means the distance between the cars is decreasing*).

Derivatives of Quotients

If $f(x)$ and $g(x)$ are differentiable functions (and $g(x) \neq 0$), what is the derivative of their quotient ?

$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{d}{dx}\left(f(x)(g(x))^{-1}\right)$
$= f'(x)(g(x))^{-1} + f(x)(-1)(g(x))^{-2}g'(x)$
$= \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ called the Quotient Rule.
Derivative of the top times the bottom minus top times derivative of the bottom all over the bottom squared.

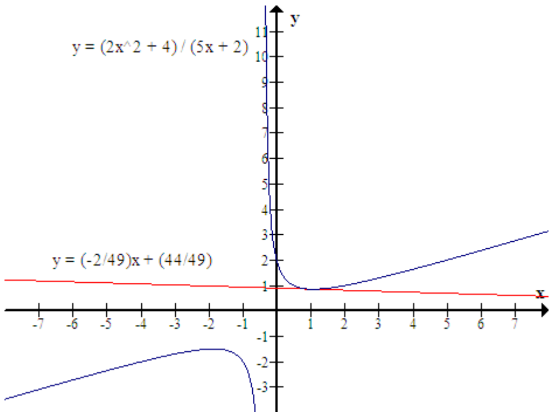
Examples:
(i) $\dfrac{d}{dx}\left(\dfrac{2x+1}{3x+2}\right)$
$= \dfrac{\left(\frac{d}{dx}(2x+1)\right)(3x+2) - (2x+1)\left(\frac{d}{dx}(3x+2)\right)}{(3x+2)^2}$
$= \dfrac{(2)(3x+2) - (2x+1)(3)}{(3x+2)^2}$
$= \dfrac{1}{(3x+2)^2}$
(ii) If $f(x) = \dfrac{\sqrt{x+1}}{2x^2+x}$, then
$f'(x) = \dfrac{\frac{1}{2}(x+1)^{-1/2}(1)(2x^2+x) - (x+1)^{1/2}(4x+1)}{(2x^2+x)^2}$
$= \dfrac{\frac{1}{2}(x+1)^{-1/2}\left[(2x^2+x) - 2(x+1)(4x+1)\right]}{(2x^2+x)^2}$
$= \dfrac{-(6x^2+9x+2)}{2(2x^2+x)^2\sqrt{x+1}}$.

Example:
Find the equation of the tangent line to the curve $y = \dfrac{2x^2+4}{5x+2}$ at $x = 1$.

$$\frac{dy}{dx} = \frac{4x(5x+2)-(2x^2+4)(5)}{(5x+2)^2} = \frac{10x^2+8x-20}{(5x+2)^2}$$

so the slope is $m = \frac{dy}{dx}|_{x=1} = \frac{10+8-20}{(7)^2} = \frac{-2}{49}$

and the line is $y - \frac{6}{7} = \frac{-2}{49}(x-1)$ or $y = \frac{-2}{49}x + \frac{44}{49}$.



Example:

The value of a car, in dollars, t years after purchase is given by $V(t) = \frac{20000+2t}{1+t}$.

Then $V'(t) = \frac{2(1+t)-(20000+2t)}{(1+t)^2} = \frac{-19998}{(1+t)^2}$.

t	$V(t)$	$V'(t)$
0	20000	-19998
1	10001	-4999.50
2	6668	-2222
3	5001.50	-1249.88
4	4001.60	-799.92
5	3335	-555.50

So, after $t = 2$ years, for example, the car is worth \$6668 and is depreciating at \$2222 per year (*this is why many people buy used cars*).

(Khan Academy video Calculus: "Quotient Rule" here)

Rate of Change Problems

We'll start with some ideas from Business and Economics. The demand or price function, $p(x)$, is the price per unit when x units of a product are sold. The revenue function is $R(x) = xp(x)$, the amount of money received for selling x units at price $p(x)$ each. The cost function, $C(x)$, is the total cost of producing x units of the product. The profit function is $P(x) = R(x) - C(x)$, the profit from the sale of x units.

The marginal cost function is $C'(x)$, the rate of change of cost.
The marginal revenue function is $R'(x)$, the rate of change of revenue.
The marginal profit function is $P'(x)$, the rate of change of profit.
All are wrt x, the number of units produced or sold.

The marginal cost at $x = 100$, $C'(100)$, for example, is an estimate of the true cost of producing the 101st unit of the product, whereas, $P'(100)$ is an estimate of the change in profit for selling the 101st unit.

Example:
The coffee shop sells 100 mochas per day at \$2.75 and we saw that they have determined that they will sell 5 fewer per day for each 25¢ price increase.
Let n be the number of 25¢ price increases, then the price is $p = 2.75 + 0.25n$ and the number sold will be $x = 100 - 5n$. Then $n = \dfrac{100 - x}{5} = 20 - 0.2x$ and so the price is $p(x) = 2.75 + 0.25(20 - 0.2x) = 7.75 - 0.05x$ and the revenue is $R(x) = xp(x) = 7.75x - 0.05x^2$. So the marginal revenue is $R'(x) = 7.75 - 0.1x$. So if they have raised the price to \$3.25, they will be selling $x = 90$ mochas per day and their revenue would be $R(90) = \$292.50$ and the marignal revenue would be $R'(90) = -\$1.25$ and thus revenue is decreasing at this point.

Example:
The cost of producing x units of a product is $C(x) = -0.002x^2 + 10x + 4000$. Compare the marginal cost at 500 units with the true cost of producing the 501st unit.

$C'(x) = -0.004x + 10$, so marginal cost is $C'(500) = 8$.
The true cost is $\Delta C = C(501) - C(500)$
$= (-0.002(501)^2 + 10(501) + 4000) - (-0.002(500)^2 + 10(500) + 4000) = 7.998$ (*very close*).

Now, let's look at an idea from Physics. The kinetic energy, in Joules J, of an object with mass m kg and velocity v m/s is $K = \frac{1}{2}mv^2$.

Example:
A baseball with mass 150 g is thrown upward with an initial velocity of 30 m/s, so its velocity function is $v(t) = 30 - 9.8t$.
So the kinetic energy is $K(t) = \frac{1}{2}mv^2 = \frac{1}{2}(0.150)(30 - 9.8t)^2 = 0.075(30 - 9.8t)^2$,
then $K'(t) = 2(0.075)(30 - 9.8t)(-9.8) = -1.47(30 - 9.8t)$.
So $K'(2) = -15.288$ J/s and the ball is slowing down at this time as its kinetic energy is decreasing.

Practice Problems

1. Find the derivatives of the following functions.
(a) $f(x) = 2x^2 - 4x + 1$
(b) $g(t) = 3t^4 - \sqrt{2}t^2 + \dfrac{7}{t} - \dfrac{3}{t^4}$
(c) $y = (2x^2 + 5x)(6x + 3)$
(d) $p(r) = (3r + 1)^5(4r^3 + r)^2$
(e) $g(x) = \sqrt{x^2 + 4x + 6}$
(f) $q(x) = \dfrac{4x + 7}{x^2 + x}$
(g) $f(t) = \dfrac{\sqrt{t^2 + 1}}{5t^3 + 6t}$
(h) $y = \sqrt{\dfrac{x^2 + x}{2x^2 + 4}}$

2. Find the second derivatives of the following functions.
(a) $f(x) = 5x^4 + 6x^3 - 4x^2 + 7x + 1$
(b) $g(t) = 4t^{3/2} + 6\sqrt{t}$
(c) $y = \dfrac{2x + 1}{4x + 3}$
(d) $p(r) = (3r^2 + 2r + 7)^4$

3. Find the equation of the tangent line to the curve $y = \sqrt{\dfrac{x + 1}{x^2 + 4}}$ at $x = 2$.

4. Find the equation of the tangent line to the curve $y = \left(\dfrac{x^2 + 1}{3x + 2}\right)^2$ at $x = -1$.

5. A model rocket is shot into the air. Its height, in metres, after t seconds, is given by $h(t) = -4.9t^2 + 20t + 1$.
(a) What is the maximum height the rocket reaches?
(b) How long does it take for the rocket to hit the ground?
(c) What is its speed when it hits the ground?

6. Are there any points on the curve $y = \dfrac{8x^2}{(x^2 + 2)^2}$ where the tangent line would be horizontal? If so, at what point(s)?

7. The population of chipmunks in a city park is given by $P(t) = \sqrt{100t + 20t^2}$ for t measured in years.
(a) What is the growth rate of the population after 3 years ?
(b) When does the population reach 30 chipmunks ?

8. Find the derivative of $f(x) = \dfrac{(3x^3 + 4x^2 - 7x)^2}{(x^2 + 6x + 2)^3}$.

9. The value, in dollars, of a new car, t years after purchase, is given by $V(t) = \dfrac{22000 + 4t}{1 + 0.5t}$.
(a) What is the initial value of the car ?
(b) What is the rate of depreciation of the car after 1, 2 and 3 years ?

10. The cost, in dollars, of producing x widgets is $C(x) = 0.1x^2 + 30x + 100$ and the demand or price function is $p(x) = 125 - 0.2x$.
(a) Find the revenue and profit functions.
(b) Determine the marginal cost at 100 widgets.
(c) Determine the actual cost of producing the 101st widget.
(d) What are the marginal revenue and profit for the sale of 100 widgets ?

Practice Problems Solutions

1. (a) $f'(x) = 4x - 4$
(b) $g'(t) = 12t^3 - 2\sqrt{2}t - \dfrac{7}{t^2} + \dfrac{12}{t^5}$
(c) $\dfrac{dy}{dx} = (4x+5)(6x+3) + (2x^2+5x)(6) = 36x^2 + 72x + 15$
(d) $p'(r) = 5(3r+1)^4(3)(4r^3+r)^2 + (3r+1)^5(2)(4r^3+r)(12r^2+1)$
$= (3r+1)^4(4r^3+r)\left[15(4r^3+r) + 2(3r+1)(12r^2+1)\right]$
$= (3r+1)^4(4r^3+r)(132r^3 + 24r^2 + 21r + 2)$
(e) $g'(x) = \dfrac{1}{2}(x^2+4x+6)^{-1/2}(2x+4) = \dfrac{x+2}{\sqrt{x^2+4x+6}}$
(f) $q'(x) = \dfrac{4(x^2+x) - (4x+7)(2x+1)}{(x^2+x)^2} = \dfrac{-(4x^2+14x+7)}{(x^2+x)^2}$
(g) $f'(t) = \dfrac{\frac{1}{2}(t^2+1)^{-1/2}(2t)(5t^3+6t) - (t^2+1)^{1/2}(15t^2+6)}{(5t^3+6t)^2}$
$= \dfrac{t(5t^3+6t) - (t^2+1)(15t^2+6)}{\sqrt{t^2+1}(5t^3+6t)^2} = \dfrac{-(10t^4+15t^2+6)}{\sqrt{t^2+1}(5t^3+6t)^2}$
(h) $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{x^2+x}{2x^2+4}\right)^{-1/2} \dfrac{d}{dx}\left(\dfrac{x^2+x}{2x^2+4}\right)$
$= \dfrac{1}{2}\left(\dfrac{x^2+x}{2x^2+4}\right)^{-1/2}\left(\dfrac{(2x+1)(2x^2+4) - (x^2+x)(4x)}{(2x^2+4)^2}\right)$
$\dfrac{1}{2}\left(\dfrac{2x^2+4}{x^2+x}\right)^{1/2}\left(\dfrac{-2x^2+8x+4}{(2x^2+4)^2}\right)$
$= \dfrac{-x^2+4x+2}{\sqrt{x^2+x}(2x^2+4)^{3/2}}$

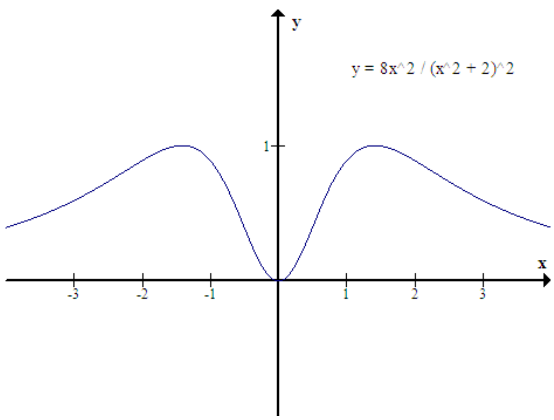
2. (a) $f'(x) = 20x^3 + 18x^2 - 8x + 7$, so $f''(x) = 60x^2 + 36x - 8$
(b) $g'(t) = 6t^{1/2} + 3t^{-1/2}$, so $g''(t) = 3t^{-1/2} - \frac{3}{2}t^{-3/2}$
(c) $\dfrac{dy}{dx} = \dfrac{2(4x+3) - (2x+1)(4)}{(4x+3)^2} = \dfrac{2}{(4x+3)^2} = 2(4x+3)^{-2}$
so $\dfrac{d^2y}{dx^2} = 2(-2)(4x+3)^{-3}(4) = \dfrac{-16}{(4x+3)^3}$
(d) $p'(r) = 4(3r^2+2r+7)^3(6r+2) = (3r^2+2r+7)^3(24r+8)$,
so $p''(r) = 3(3r^2+2r+7)^2(6r+2)(24r+8) + (3r^2+2r+7)^3(24)$
$= 3(3r^2+2r+7)^2\left[(6r+2)(24r+8) + 8(3r^2+2r+7)\right]$
$= 3(3r^2+2r+7)^2(168r^2 + 112r + 72)$

3. $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{x+1}{x^2+4}\right)^{-1/2}\left(\dfrac{(1)(x^2+4)-(x+1)(2x)}{(x^2+4)^2}\right)$
$= \dfrac{1}{2}\left(\dfrac{x^2+4}{x+1}\right)^{1/2}\left(\dfrac{4-x^2-2x}{(x^2+4)^2}\right)$
$= \dfrac{4-x^2-2x}{2\sqrt{x+1}(x^2+4)^{3/2}}$
so the slope is $m = \dfrac{dy}{dx}|_{x=2} = \dfrac{4-2(2)-(2)^2}{2\sqrt{2+1}((2)^2+4)^{3/2}} = \dfrac{-4}{2\sqrt{3}(8)^{3/2}} = \dfrac{-2}{\sqrt{3}(16\sqrt{2})} = \dfrac{-1}{8\sqrt{6}}$
and the line is $y - \sqrt{\frac{3}{8}} = \frac{-1}{8\sqrt{6}}(x-2)$ or $y = \frac{-1}{8\sqrt{6}}x + \frac{2}{8\sqrt{6}} + \sqrt{\frac{3}{8}}$ or $y \approx -0.051x + 0.714$

4. $\dfrac{dy}{dx} = 2\left(\dfrac{x^2+1}{3x+2}\right)\left(\dfrac{2x(3x+2)-(x^2+1)(3)}{(3x+2)^2}\right) = \dfrac{2(x^2+1)(3x^2+4x-3)}{(3x+2)^3}$
so the slope is $m = \dfrac{dy}{dx}|_{x=-1} = \dfrac{2((-1)^2+1)(3(-1)^2+4(-1)-3}{(3(-1)+2)^3} = \dfrac{2(2)(-4)}{(-1)^3} = 16$
and the line is $y - 4 = 16(x-(-1))$ or $y = 16x + 20$

5. (a) the rocket is at its highest point when $v(t) = 0$, so $v(t) = -9.8t + 20 \implies t \approx 2.041$ s
thus the maximum height is $h(2.041) \approx 21.41$ m
(b) $h(t) = 0$ if $t = \dfrac{-20 \pm \sqrt{(20)^2 - 4(-4.9)(1)}}{2(-4.9)} = \dfrac{-20 \pm 20.484}{-9.8} = \cancel{-0.049}, 4.131$
so it takes 4.131 s for the rocket to hit the ground
(c) its speed when it does is $|v(4.131)| = |-9.8(4.131)+20| = 20.484$ m/s

6. $\dfrac{dy}{dx} = \dfrac{16x(x^2+2)^2 - 8x^2(2)(x^2+2)(2x)}{((x^2+2)^2)^2}$
$= \dfrac{16x(x^2+2) - 32x^3}{(x^2+2)^3} = \dfrac{32x - 16x^3}{(x^2+2)^3} = \dfrac{16x(2-x^2)}{(x^2+2)^3}$
so $\dfrac{dy}{dx} = 0$ if $x = 0, \pm\sqrt{2}$
so horizontal tangents at $(0,0)$ and $(\pm\sqrt{2}, 1)$



7. (a) $P'(t) = \dfrac{1}{2}(100t + 20t^2)^{-1/2}(100 + 40t) = \dfrac{50 + 20t}{\sqrt{100t + 20t^2}}$
so $P'(3) = \dfrac{50 + 20(3)}{\sqrt{100(3) + 20(3)^2}} = \dfrac{50 + 60}{\sqrt{300 + 180}} \approx 5$ chipmunks per year
(b) $P(t) = \sqrt{100t + 20t^2} = 30 \implies 100t + 20t^2 = 900$
$\implies 20t^2 + 100t - 900 = 0 \implies t^2 + 5t - 45 = 0$
so $t = \dfrac{-5 \pm \sqrt{(5)^2 - 4(1)(-45)}}{2(1)} = \dfrac{-5 \pm 14.318}{2} = \cancel{-9.66}, 4.66$
so the chipmunk population reaches 30 at approximately 4 years and 8 months

8. $f'(x) = \dfrac{2(3x^3 + 4x^2 - 7x)(9x^2 + 8x - 7)(x^2 + 6x + 2)^3 - (3x^3 + 4x^2 - 7x)^2(3)(x^2 + 6x + 2)^2(2x + 6)}{((x^2 + 6x + 2)^3)^2}$
$= \dfrac{2(3x^3 + 4x^2 - 7x)(x^2 + 6x + 2)^2\,[(9x^2 + 8x - 7)(x^2 + 6x + 2) - 3(3x^3 + 4x^2 - 7x)(x + 3)]}{(x^2 + 6x + 2)^6}$
$= \dfrac{2(3x^3 + 4x^2 - 7x)(23x^3 + 44x^2 + 37x - 14)}{(x^2 + 6x + 2)^4}$

9. (a) $V(0) = \$22000$
(b) $V'(t) = \dfrac{4(1+0.5t) - (22000+4t)(0.5)}{(1+0.5t)^2} = \dfrac{-10996}{(1+0.5t)^2}$
$V'(1) = -\$4887.11$ per year,
$V'(2) = -\$2749$ per year and
$V'(3) = -\$1759.36$ per year

10. (a) $R(x) = xp(x) = 125x - 0.2x^2$
$P(x) = R(x) - C(x) = 125x - 0.2x^2 - (0.1x^2 + 30x + 100) = 95x - 0.3x^2 - 100$
(b) $C'(x) = 0.2x + 30$, so $C'(100) = \$50$
(c) $\Delta C = C(101) - C(100) = 0.1(101)^2 + 30(101) + 100 - (0.1(100)^2 + 30(100) + 100) = \$50.10$
(d) $R'(x) = 125 - 0.4x$, so $R'(100) = \$85$
$P'(x) = 95 - 0.6x$, so $P'(100) = \$35$
and so both revenue and profit are increasing at 100 units sold

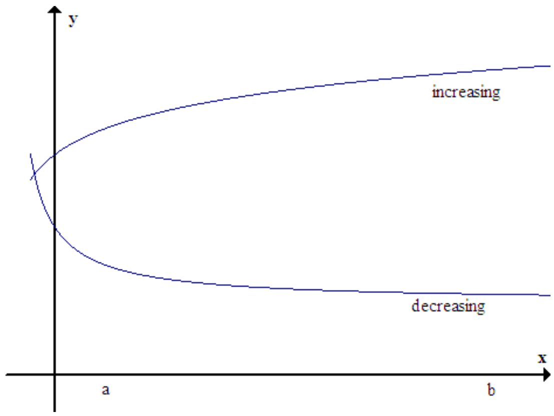
Chapter 3

Goals

- to understand what it means for a function to be increasing or decreasing and the connection with the first derivative
- to understand what a critical number (or point) of a function is
- to understand local and absolute extrema of functions and be able to find and identify them using the First or Second Derivative Tests and the Closed Interval Method
- to understand what it means for a function to be concave up or down and the connection with the second derivative
- to be able to find the points of inflection of a function
- to be to recognize when the graph of a rational function would have vertical asymptotes and be able to identify them
- to be able to sketch the graph of a function using the first and second derivatives
- to be able to use derivatives to solve optimization problems

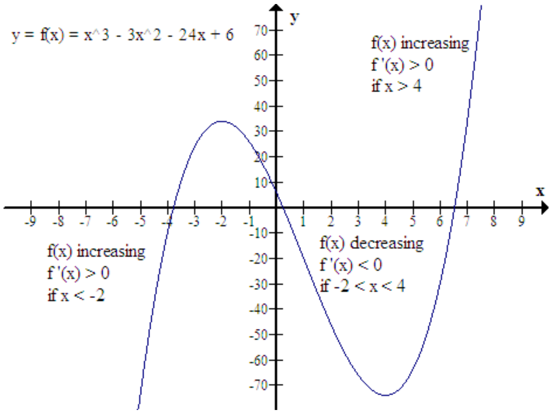
Increasing and Decreasing Functions

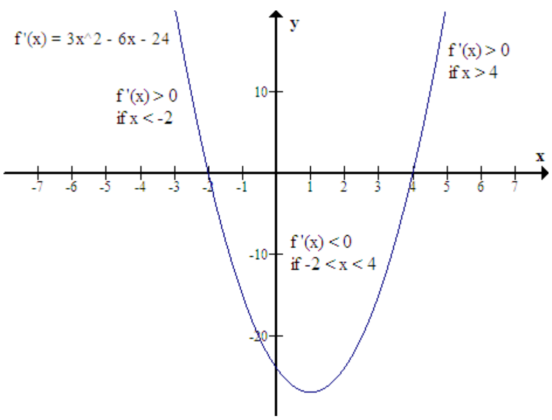
A function $f(x)$ is said to be increasing on an interval $a \leq x \leq b$ if its values get larger as x gets larger on that interval, *ie* $x_2 > x_1 \implies f(x_2) > f(x_1)$. It is decreasing if its values get smaller, *ie* $x_2 > x_1 \implies f(x_2) < f(x_1)$.



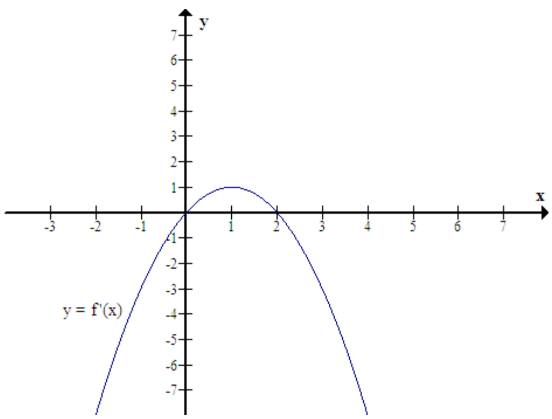
Notice the connection with the first derivative – if the function is increasing, then slopes of tangents will be positive and hence $f'(x) > 0$. If it is decreasing, then the slopes of tangents will be negative and so $f'(x) < 0$.

Example:
Consider $f(x) = x^3 - 3x^2 - 24x + 6$.
Then $f'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x + 2)(x - 4)$,
so $f'(x) = 0$ if $x = -2$ or $x = 4$.
This divides the domain into three intervals $x < -2$, $-2 < x < 4$ and $x > 4$.
If $x < -2$, $f'(x) > 0$, so $f(x)$ is increasing.
If $-2 < x < 4$, $f'(x) < 0$, so $f(x)$ is decreasing.
If $x > 4$, $f'(x) > 0$, so $f(x)$ is increasing.

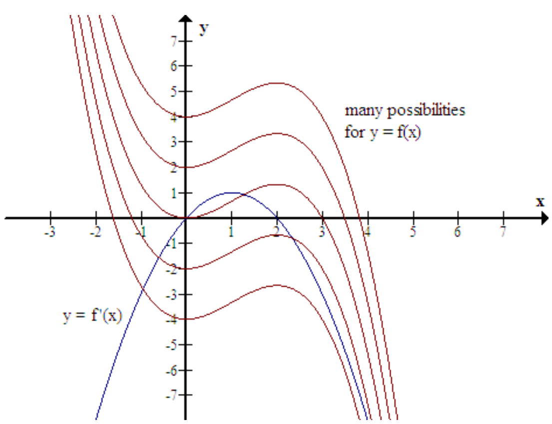

y = f(x) = x^3 - 3x^2 - 24x + 6
f(x) increasing
f '(x) > 0
if x > 4
f(x) increasing
f '(x) > 0
if x < -2
f(x) decreasing
f '(x) < 0
if -2 < x < 4


f '(x) = 3x^2 - 6x - 24
f '(x) > 0
if x < -2
f '(x) > 0
if x > 4
f '(x) < 0
if -2 < x < 4

Example:
Suppose the graph of $f'(x)$ looks like

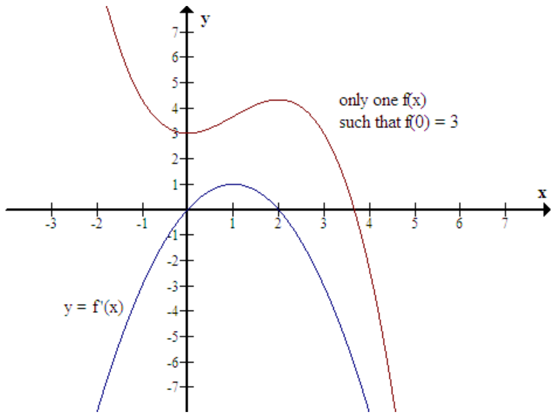


then $f'(x) < 0$ for $x < 0$ and $x > 2$ tells us that $f(x)$ is decreasing on these intervals. Whereas $f'(x) > 0$ on $0 < x < 2$ means that $f(x)$ is increasing there. Notice that $f'(x) = 0$ at $x = 0$ and $x = 2$ and so $f(x)$ would have a horizontal tangent at these points (*ie* the curve is flat there). So we can make a sketch of the graph of $f(x)$.



But notice that there are several (actually infinitely many) such $f(x)$ as we can add any constant to $f(x)$ and still have the same $f'(x)$.

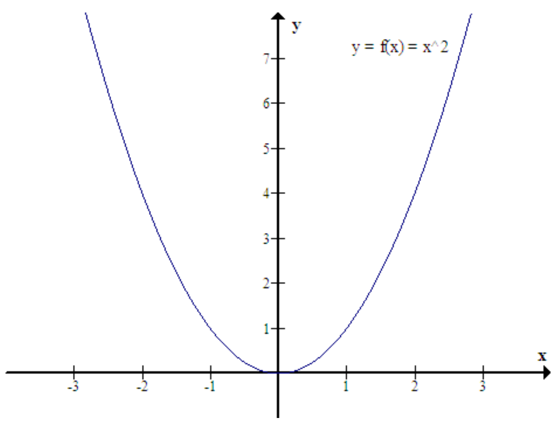
We would have to know another piece of information to choose a specific $f(x)$. For example, suppose we know that we were supposed to have that $f(0) = 3$, then we would have only one choice for $f(x)$.



Maxima and Minima

A function $f(x)$ has an absolute or global maximum at $x = a$ if $f(a) \geq f(x)$ for all x in the domain (*ie* $f(a)$ is the largest value the function achieves). It has an absolute or global minimum at $x = a$ if $f(a) \leq f(x)$ for all x in the domain (*ie* $f(a)$ is the smallest value the function achieves). *Any given function may have one, both or neither.*

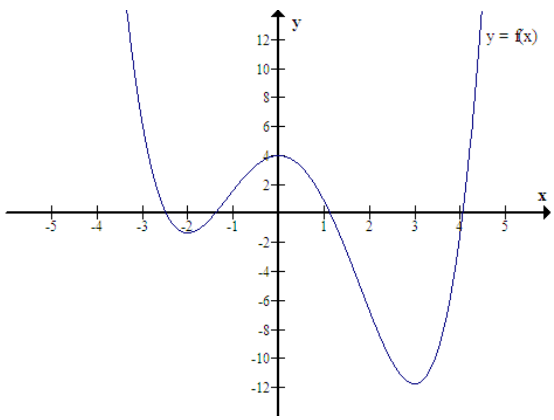
Example:
Consider $f(x) = x^2$.



This function has an absolute min of 0 at $x = 0$, but no absolute max as $f(x) \to \infty$ as $x \to \pm\infty$.

In addition to global or absolute extrema, a function can also have local extrema, the largest or smallest values in a neighbourhood or vicinity. The function $f(x)$ has a local maximum at $x = a$ if $f(a) \geq f(x)$ for all x near a (*ie* for all x in an open interval around a). It has a local minimum at $x = a$ if $f(a) \leq f(x)$ for all x near a.

Example:
Consider the function $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 + 4$ as shown in the graph below.

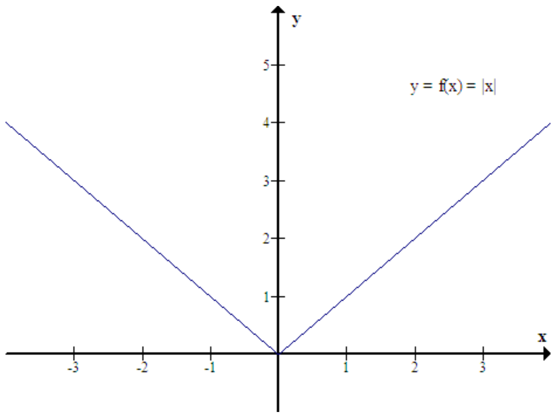


This function has a local max at $x = 0$, a local min at $x = -2$, a local (and absolute) min at $x = 3$ and no absolute max.

Notice that $f'(x) = x^3 - x^2 - 6x = x(x^2 - x - 6) = x(x+2)(x-3)$ and so $f'(x) = 0$ at $x = 0, -2$ and 3 – *ie* the local extrema, also called turning points, occur at places where $f'(x) = 0$ for this function.

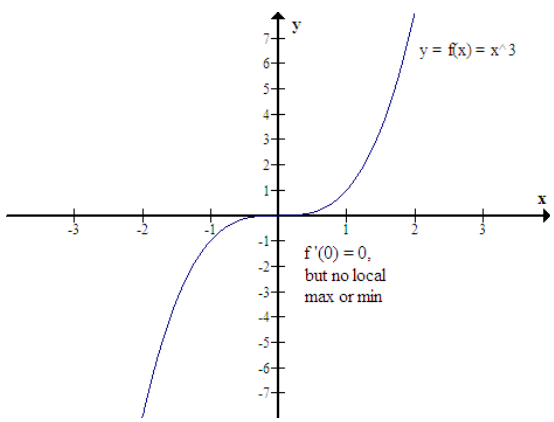
A critical number or point for a function $f(x)$ is a value a in the domain such that $f'(a) = 0$ or $f'(a)$ is undefined. Local extrema will occur at critical numbers – *ie* if $f(x)$ has a local extremum at $x = a$, then $f'(a) = 0$ or $f'(a)$ is undefined.

To see that $f'(a)$ being undefined can lead to a local extremum, consider $f(x) = |x|$.



$f'(0)$ is undefined, there is a local (and absolute) min there.

But it is also very important to recognize that just because $x = a$ is a critical number of $f(x)$, $f(x)$ need not have a local extremum at $x = a$. For example, if $f(x) = x^3$, then $f'(x) = 3x^2 = 0$ if $x = 0$, but there is no local max or min there.

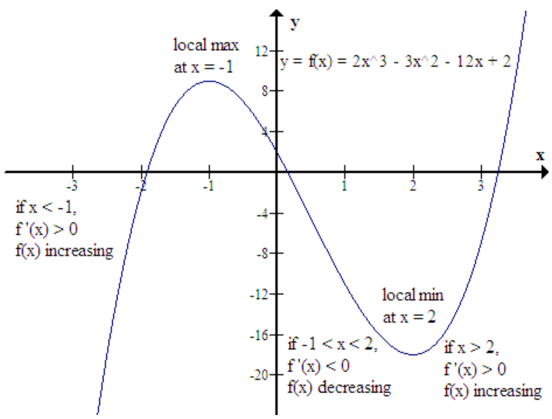


We can distinguish a local max from a local min by noticing that the way the sign of the derivative changes is different (*look back at our quartic example above*).

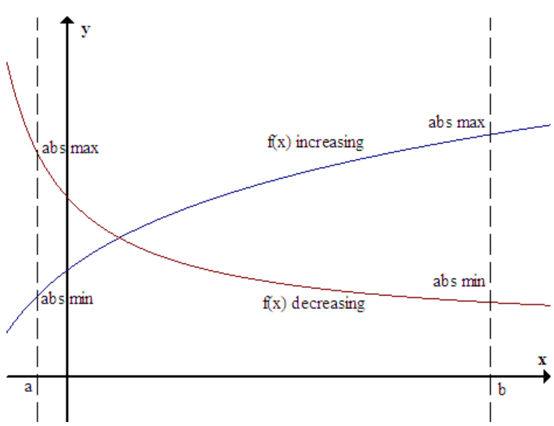
<u>The First Derivative Test</u>
Suppose that $x = c$ is a critical number of $f(x)$.
(i) If $f'(x)$ changes sign from $+$ to $-$ at $x = c$, then $f(x)$ has a local max at $x = c$.
(ii) If $f'(x)$ changes sign from $-$ to $+$ at $x = c$, then $f(x)$ has a local min at $x = c$.
(iii) If $f'(x)$ does not change sign at $x = c$, then there is no local extremum there.

To see that (iii) is so, see our example with $f(x) = x^3$ above.

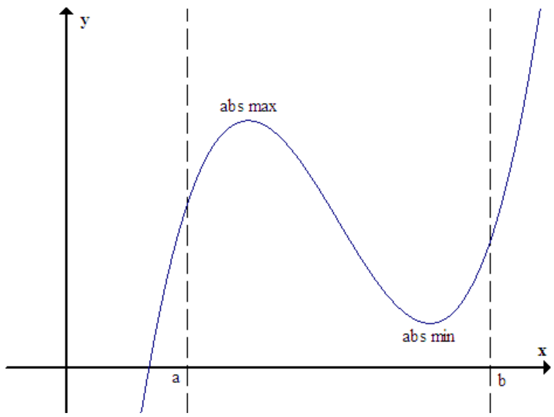
Example:
Consider $f(x) = 2x^3 - 3x^2 - 12x + 2$.
Then $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x + 1)(x - 2)$,
so the critical numbers are $x = -1$ and $x = 2$.
If $x < -1$, $f'(x) > 0$ and $f(x)$ is increasing.
If $-1 < x < 2$, $f'(x) < 0$ and $f(x)$ is decreasing. So there is a local max at $x = -1$.
If $x > 2$, $f'(x) > 0$ and $f(x)$ is increasing. So there is a local min at $x = 2$.



If $f(x)$ is a continuous function defined on a closed interval $a \le x \le b$, then $f(x)$ attains both an absolute max and an absolute min on the interval. But where would they occur? If $f(x)$ were monotonic (always increasing or always decreasing), they would occur at the endpoints of the interval (see the graph below).

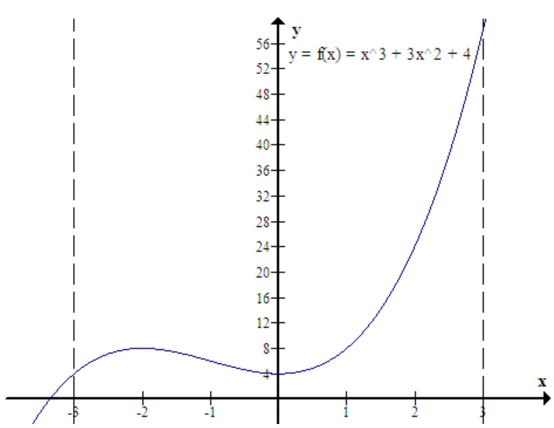


What would have to happen for the extrema to occur inside the interval? The function would have to increase and decrease inside the interval – meaning that the derivative would have to change sign – meaning that there would be local extrema inside the interval.



So we can put it all together to get the Closed Interval Method to find the absolute max and/or min of a continuous function $f(x)$ on a closed interval $a \leq x \leq b$.
(i) Find the critical numbers of $f(x)$ that are inside the interval and evaluate the function at them.
(ii) Evaluate the function at the endpoints of the interval $x = a$ and $x = b$.
(iii) Compare the values and identify the max and/or min.

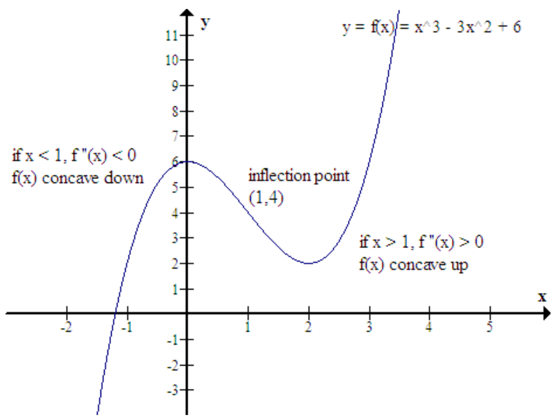
Example:
Find the absolute extrema of $f(x) = x^3 + 3x^2 + 4$ on the interval $-3 \leq x \leq 3$.
$f'(x) = 3x^2 + 6x = 3x(x+2)$, so the critical numbers are $x = -2$ and $x = 0$.
$f(-2) = 8, f(0) = 4$
$f(-3) = 4, f(3) = 58$
So the absolute max is 58 (at $x = 3$) and the absolute min is 4 (at $x = -3$ and $x = 0$).



Concavity and the Second Derivative Test

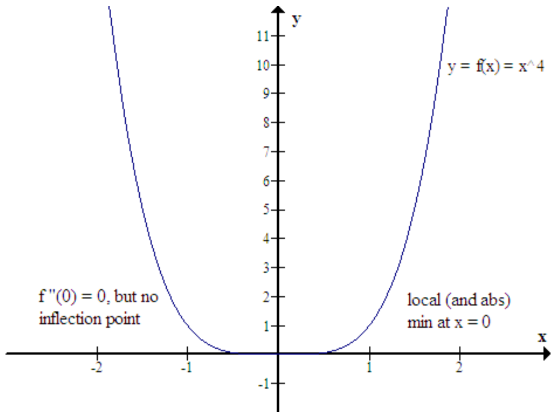
If $f''(x) > 0$ on the interval $a < x < b$, we say that $f(x)$ is concave up on the interval and the curve would be bending upwards as slopes of tangents would be increasing. If $f''(x) < 0$, $f(x)$ is concave down and the curve bends downwards. A point where the concavity changes (from up to down or down to up) is called an inflection point. The concavity changing requires $f''(x)$ to change sign and hence inflection points occur where $f''(x) = 0$ or is undefined. *But $f''(c) = 0$ or being undefined does not mean that there must be an inflection point at $x = c$ – see the x^4 example below.*

Example:
If $f(x) = x^3 - 3x^2 + 6$, then $f'(x) = 3x^2 - 6x$ and $f''(x) = 6x - 6$,
so $f''(x) = 0$ if $x = 1$.
If $x < 1$, $f''(x) < 0$ and so $f(x)$ is concave down.
If $x > 1$, $f''(x) > 0$ and so $f(x)$ is concave up and there is an inflection point at $(1, 4)$.

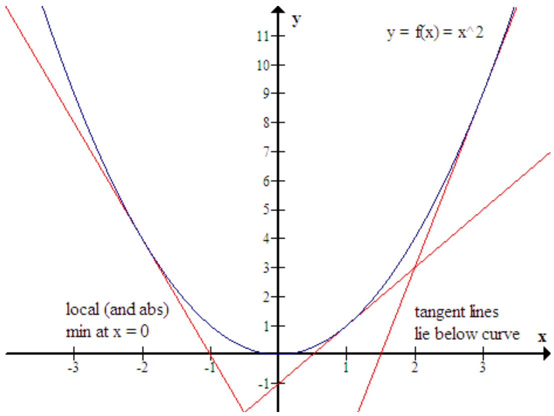


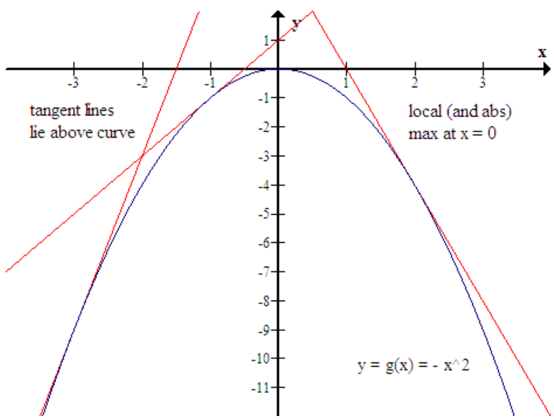
Example:
If $f(x) = x^4$, then $f'(x) = 4x^3$ and $f''(x) = 12x^2$.
Then $f''(0) = 0$, but $f''(x) \geq 0$ for all x and the function is always concave up.
There is no point of inflection as the concavity has not changed.

But there is a local (and absolute) minimum at $x = 0$.



Consider the functions $f(x) = x^2$ and $g(x) = -x^2$. Then $f'(x) = 2x, f''(x) = 2$ and so the function is always concave up and tangent lines lie below the curve and there is a local min, whereas $g'(x) = -2x, g''(x) = -2$ and that function is always concave down and tangent lines always lie above the curve and there is a local max. (*This tells us that the tangent line* *would have to cross the curve at a point of inflection.*)





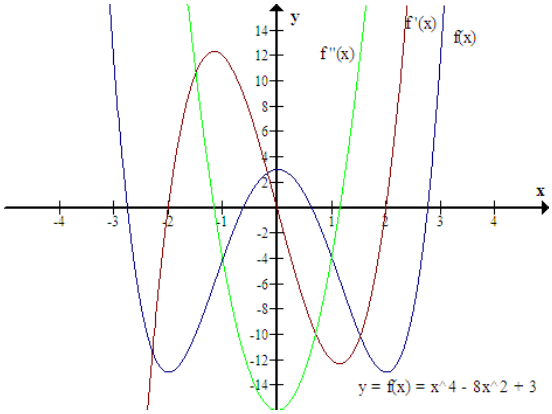
These graphs also show us the connection between concavity and local extrema.

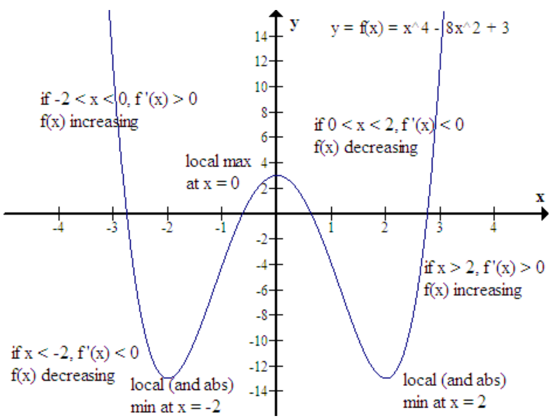
<u>The Second Derivative Test</u>
Suppose that $f'(c) = 0$ and $f''(c) \neq 0$.
(i) If $f''(c) > 0$, $f(x)$ has a local min at $x = c$.
(ii) If $f''(c) < 0$, $f(x)$ has a local max at $x = c$.

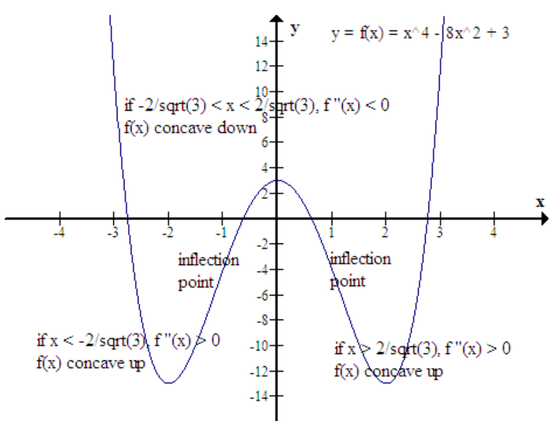
Example:
Consider our example above, $f(x) = x^3 - 3x^2 + 6$.
Then $f'(x) = 3x^2 - 6x = 3x(x - 2)$ and $f''(x) = 6x - 6$.
So $f'(x) = 0$ if $x = 0$ or 2.
$f''(0) = -6 < 0 \implies$ local max at $x = 0$.
$f''(2) = 6 > 0 \implies$ local min at $x = 2$.

Example:
Consider the function $f(x) = x^4 - 8x^2 + 3$
(*this function has* <u>*even symmetry*</u> *as* $f(-x) = f(x)$ *for all* x).
Then $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2)$
and so $f'(x) = 0$ if $x = -2, 0$ or 2.
If $x < -2$, $f'(x) < 0 \implies f(x)$ decreasing.
If $-2 < x < 0$, $f'(x) > 0 \implies f(x)$ increasing, so $(-2, -13)$ is a local min (FDT).
If $0 < x < 2$, $f'(x) < 0 \implies f(x)$ decreasing, so $(0, 3)$ is a local max (FDT).
If $x > 2$, $f'(x) > 0 \implies f(x)$ increasing, so $(2, -13)$ is a local min (FDT).
And $f''(x) = 12x^2 - 16 = 4(3x^2 - 4)$,
so $f''(x) = 0$ if $x = \pm\sqrt{4/3} = \pm 2/\sqrt{3}$.
If $x < -2/\sqrt{3}$, $f''(x) > 0$ and $f(x)$ is concave up (*agrees with min at* $x = -2$).
If $-2/\sqrt{3} < x < 2/\sqrt{3}$, $f''(x) < 0$ and $f(x)$ is concave down (*agrees with max at* $x = 0$) and there is an inflection point at $(-2/\sqrt{3}, -53/9)$.
If $x > 2/\sqrt{3}$, $f''(x) > 0$ and $f(x)$ is concave up (*agrees with min at* $x = 2$) and there is an inflection point at $(2/\sqrt{3}, -53/9)$.
Or, we could have used the SDT to classify the extrema:
$f''(-2) > 0 \implies$ local min at $x = -2$,
$f''(0) < 0 \implies$ local max at $x = 0$,
and $f''(2) > 0 \implies$ local min at $x = 2$.

See the graphs below.







(Khan Academy video Calculus: “Inflection Points and Concavity Intuition” here)

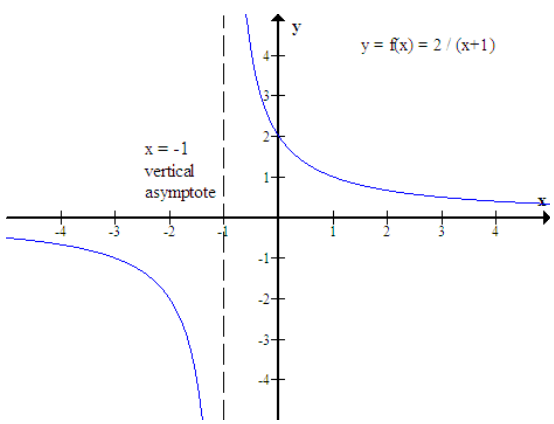
Simple Rational Functions

A rational function has the form $f(x) = \dfrac{P(x)}{Q(x)}$, where both $P(x)$ and $Q(x)$ are polynomials. A rational function will be undefined wherever $Q(x) = 0$ (*as we cannot divide by 0*) and it will be continuous everywhere on its domain, $\{x \mid Q(x) \neq 0\}$.

If $x = a$ is a value where the denominator is zero and the numerator is not, *ie* $Q(a) = 0$ and $P(a) \neq 0$, the rational function will have a vertical asymptote at $x = a$. More exactly, $f(x)$ has a vertical asymptote $x = a$ if $f(a)$ is undefined and $\lim_{x\to a^-} f(x) = \pm\infty$ and/or $\lim_{x\to a^+} f(x) = \pm\infty$. *If we graph $f(x)$, the curve cannot cross a vertical asymptote.*

Example:

Consider $f(x) = \dfrac{2}{x+1}$.
$f(x)$ is undefined at $x = -1$ (denominator is 0)
and $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \dfrac{2}{x+1} = -\infty$
and $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \dfrac{2}{x+1} = \infty$
and so $x = -1$ is a vertical asymptote.



The signs of $f'(x)$ and $f''(x)$ can change from one side of a vertical asymptote $x = a$ to the other, so we must check for that when we look for intervals of increase/decrease and/or intervals of concavity. *But there will be no local extremum or inflection point at $x = a$ even if there is a sign change in the appropriate derivative as there is no point on the curve when $x = a$.*

Example:

Consider $f(x) = \dfrac{4}{x^2+x-6} = \dfrac{4}{(x+3)(x-2)}$.

$f(x)$ is undefined at $x=-3$ and $x=2$.

$\displaystyle\lim_{x\to-3^-} f(x) = \lim_{x\to-3^-} \frac{4}{(x+3)(x-2)} = \infty$

and $\displaystyle\lim_{x\to-3^+} f(x) = \lim_{x\to-3^+} \frac{4}{(x+3)(x-2)} = -\infty$,

so there is a vertical asymptote at $x=-3$.

$\displaystyle\lim_{x\to2^-} f(x) = \lim_{x\to2^-} \frac{4}{(x+3)(x-2)} = -\infty$

and $\displaystyle\lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{4}{(x+3)(x-2)} = \infty$,

so there is a vertical asymptote at $x=2$.

$f(x) = \dfrac{4}{x^2+x-6} = 4(x^2+x-6)^{-1}$,

so $f'(x) = -4(x^2+x-6)^{-2}(2x+1) = \dfrac{-4(2x+1)}{(x^2+x-6)^2}$.

$f'(x)=0$ if $2x+1=0$ or $x=-1/2$ (*where the numerator is 0*).

If $x<-3$, $f'(x)>0$, so $f(x)$ is increasing.

If $-3<x<-1/2$, $f'(x)>0$, so $f(x)$ is increasing.

If $-1/2<x<2$, $f'(x)<0$, so $f(x)$ is decreasing and there is a local max at $(-1/2,-16/25)$.

If $x>2$, $f'(x)<0$, so $f(x)$ is decreasing.

$f''(x) = -4\left[\dfrac{2(x^2+x-6)^2-(2x+1)(2)(x^2+x-6)(2x+1)}{(x^2+x-6)^4}\right]$

$= -4(2)\left[\dfrac{(x^2+x-6)-(2x+1)^2}{(x^2+x-6)^3}\right]$

$= -8\left(\dfrac{x^2+x-6-(4x^2+4x+1)}{(x^2+x-6)^3}\right)$

$= \dfrac{-8(-3x^2-3x-7)}{(x^2+x-6)^3}$

$= \dfrac{8(3x^2+3x+7)}{(x^2+x-6)^3}$,

so $f''(x)=0$ if $3x^3+3x+7=0$,

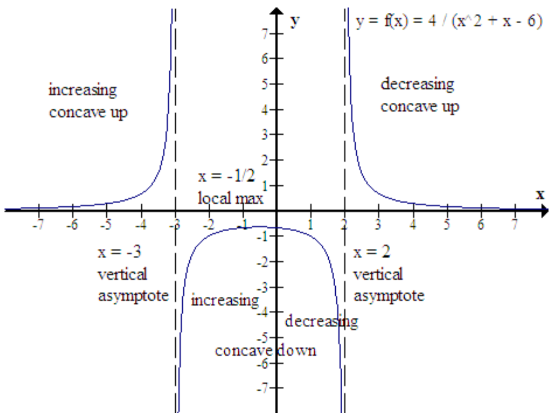
or if $x = \dfrac{-3\pm\sqrt{(3)^2-4(3)(7)}}{2(3)} = \dfrac{-3\pm\sqrt{9-84}}{6}$, so there are no real roots.

If $x<-3$, $f''(x)>0$ and $f(x)$ is concave up.

If $-3<x<2$, $f''(x)<0$ and $f(x)$ is concave down.

If $x>2$, $f''(x)>0$ and $f(x)$ is concave up.

But there are no inflection points. *Why not?*

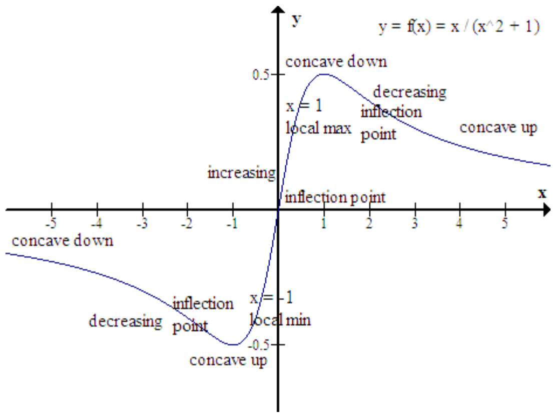


Example:
Consider $f(x) = \dfrac{x}{x^2+1}$
(*this function has* <u>*odd symmetry*</u> *as* $f(-x) = -f(x)$ *for all* x).
The denominator is never zero, so no vertical asymptotes (*ie a rational function does not have to have them*).
$f'(x) = \dfrac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$,
so $f'(x) = 0$ if $x = \pm 1$.
If $x < -1$, $f'(x) < 0 \implies f(x)$ is decreasing.
If $-1 < x < 1$, $f'(x) > 0 \implies f(x)$ is increasing and there is a local min at $(-1, -1/2)$.
If $x > 1$, $f'(x) < 0 \implies f(x)$ is decreasing and there is a local max at $(1, 1/2)$.

$$f''(x) = \frac{-2x(x^2+1)^2 - (1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$$
$$= \frac{-2x(x^2+1) - 2(2x)(1-x^2)}{(x^2+1)^3}$$
$$= \frac{-2x[x^2+1+2(1-x^2)]}{(x^2+1)^3}$$
$$= \frac{-2x(3-x^2)}{(x^2+1)^3}$$
$$= \frac{2x(x^2-3)}{(x^2+1)^3}$$

and so $f''(x) = 0$ if $x = 0$ or $x = \pm\sqrt{3}$.
If $x < -\sqrt{3}$, $f''(x) < 0$ and $f(x)$ is concave down.
If $-\sqrt{3} < x < 0$, $f''(x) > 0$ and $f(x)$ is concave up and thus there is an inflection point at $(-\sqrt{3}, -\sqrt{3}/4)$.
If $0 < x < \sqrt{3}$, $f''(x) < 0$ and $f(x)$ is concave down and thus there is an inflection point at $(0, 0)$.
If $x > \sqrt{3}$, $f''(x) > 0$ and $f(x)$ is concave up and thus there is an inflection point at

$(\sqrt{3}, \sqrt{3}/4)$.



Putting It All Together

If we want to sketch the graph of a function $y = f(x)$, we can obtain information about the shape and behaviour of the curve and about special points on the curve by finding the following:
(i) the domain
(ii) the intercepts (*where the curve crosses the x and y axes*)
(iii) symmetry (*if the function is* <u>*even*</u> $f(-x) = f(x)$ *or* <u>*odd*</u> $f(-x) = -f(x)$, *though most functions are neither*)
(iv) vertical asymptotes
(v) horizontal asymptotes (*check if either* $\lim_{x\to-\infty} f(x)$ *or* $\lim_{x\to\infty} f(x)$ *is a constant*)
(vi) intervals of increase and decrease and local extrema
(vii) intervals of concavity and points of inflection.

Example:
$y = f(x) = x + \dfrac{1}{x}$
This function is defined for all x except $x = 0$, so $x = 0$ could be a vertical asymptote.
$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} x + \dfrac{1}{x} = 0 - \infty = -\infty$
and $\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x + \dfrac{1}{x} = 0 + \infty = \infty$,
so, yes, $x = 0$ is a vertical asymptote.
$f(0)$ is undefined, so there is no y-intercept.
$f(x) = 0$ only if $x + \dfrac{1}{x} = 0$ or $x^2 + 1 = 0$, which is impossible (x *is real*), so no x-intercepts either.
$f(-x) = (-x) + \dfrac{1}{(-x)} = -x - \dfrac{1}{x} = -\left(x + \dfrac{1}{x}\right) = -f(x)$, so this function is odd (*and the*

symmetry is about the origin as we'll see in the graph).

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x + \dfrac{1}{x} = -\infty - 0 = -\infty$

and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x + \dfrac{1}{x} = \infty + 0 = \infty$,

so no horizontal asymptotes.

$f'(x) = 1 - \dfrac{1}{x^2}$,

so $f'(x) = 0$ if $x = \pm 1$.

If $x < -1$, $f'(x) > 0$ and so $f(x)$ is increasing.

If $-1 < x < 0$, $f'(x) < 0$ and so $f(x)$ is decreasing and there is a local max at $(-1, -2)$.

If $0 < x < 1$, $f'(x) < 0$ and so $f(x)$ is decreasing.

If $x > 1$, $f'(x) > 0$ and so $f(x)$ is increasing and there is a local min at $(1, 2)$.

Why did we use $x = 0$ as one of the divisions for the intervals?

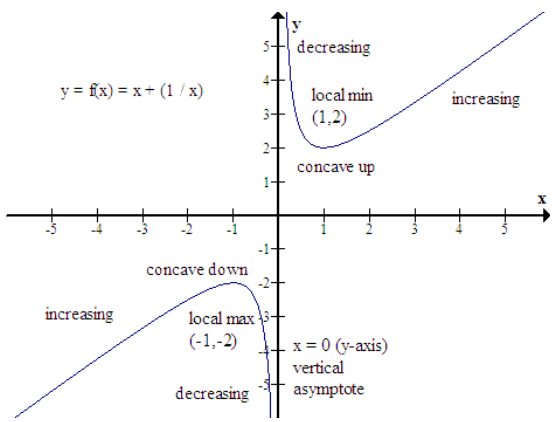
$f''(x) = \dfrac{2}{x^3} \neq 0$ for any x.

If $x < 0$, $f''(x) < 0$ and $f(x)$ is concave down.

If $x > 0$, $f''(x) > 0$ and $f(x)$ is concave up.

But there is no inflection point. *Why not?*

Put it all together to get the graph.



Example:

$y = f(x) = x^3 - 6x^2 - 36x$

$f(x)$ is a polynomial, so it is defined for all x and there are no vertical asymptotes.

$f(0) = 0 \implies (0, 0)$ is the y-intercept.

$f(x) = 0$ if $x^3 - 6x^2 - 36x = x(x^2 - 6x - 36) = 0$ or if $x = 0$

or if $x = \dfrac{6 \pm \sqrt{(-6)^2 - 4(-36)}}{2(1)} = \dfrac{6 \pm \sqrt{5(36)}}{2} = \dfrac{6 \pm 6\sqrt{5}}{2} = 3 \pm 3\sqrt{5}$,

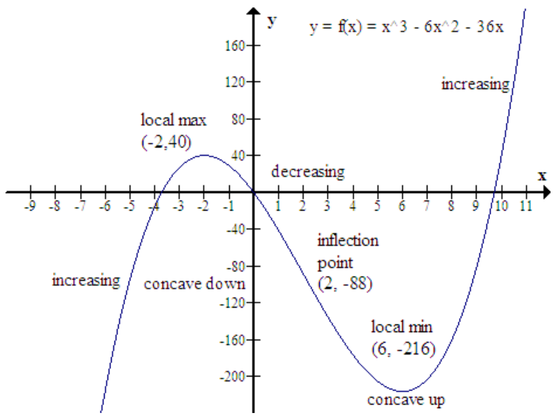
and thus the x-intercepts are $0, 3 - 3\sqrt{5} \approx -3.71$ and $3 + 3\sqrt{5} \approx 9.71$.

This function has no symmetry (*it has both even and odd powered terms*).

$\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$, so no horizontal asymptotes.

$f'(x) = 3x^2 - 12x - 36 = 3(x^2 - 4x - 12) = 3(x + 2)(x - 6)$,

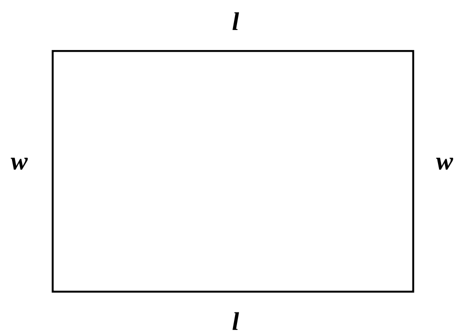
so $f'(x) = 0$ if $x = -2$ or $x = 6$.
If $x < -2$, $f'(x) > 0$, so $f(x)$ is increasing.
If $-2 < x < 6$, $f'(x) < 0$, so $f(x)$ is decreasing and there is a local max at $(-2, 40)$.
If $x > 6$, $f'(x) > 0$, so $f(x)$ is increasing and there is a local min at $(6, -216)$.
$f''(x) = 6x - 12 = 6(x - 2)$,
so $f''(x) = 0$ if $x = 2$.
If $x < 2$, $f''(x) < 0$ and $f(x)$ is concave down.
If $x > 2$, $f''(x) > 0$ and $f(x)$ is concave up and there is an inflection point at $(2, -88)$.



Optimization Problems

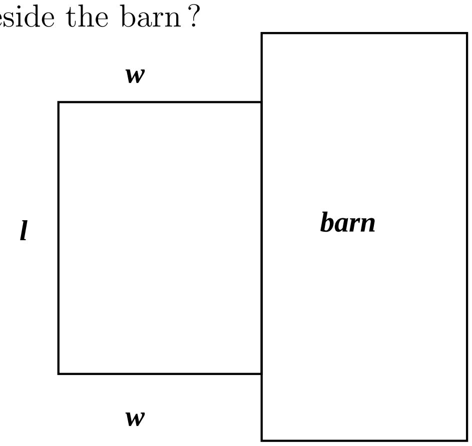
We can use our knowledge of how to find the absolute extrema of a function on a closed interval and the Derivative Tests to solve real-world optimization problems like minimizing costs, maximizing area, etc. When solving these problems, there are simple things to keep in mind:
(i) we need to be sure what the question is asking for and what the given information is
(ii) we need to be able to identify what function needs to be optimized and what the interval is (if appropriate), which usually requires reduction to a single variable and determination of constraints
(iii) a diagram often helps.

Example:
A farmer has 800 m of fence and wishes to enclose a rectangular field. What dimensions will maximize the area enclosed ?



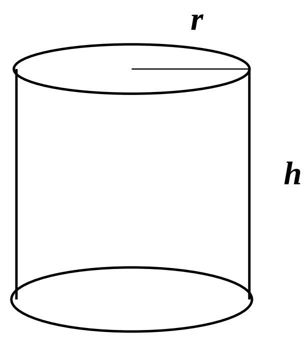
Let l and w be the length and width of the field.
We are told that he has 800 m of fence, so the perimeter of the rectangle must be 800 m.
So we have that $2l + 2w = 800$ or $l + w = 400$.
The area enclosed is $A = lw$, but $w = 400 - l$, so $A(l) = l(400 - l) = 400l - l^2$.
Clearly, $l \geq 0$ and since $w \geq 0$, we must have $l \leq 400$ (*because* $l + w = 400$), so the interval is $0 \leq l \leq 400$.
$A'(l) = 400 - 2l$, so $A'(l) = 0$ if $l = 200$.
$A(0) = 0$, $A(200) = 40\,000$ and $A(400) = 0$,
so the maximum area enclosed is $40\,000$ m^2 if the field is a 200 m by 200 m square.

Example:
What if the farmer only needed to enclose three sides with 100 m of fence to create a pen beside the barn ?

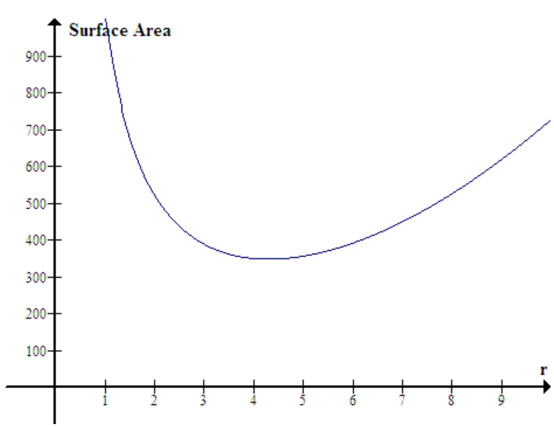


Now $l + 2w = 100$, so $w = (100 - l)/2$
and the area is $A(l) = l(100 - 1)/2 = 50l - l^2/2$ and the interval is $0 \leq l \leq 100$. *Why ?*
Then $A'(l) = 50 - l$, so $A'(l) = 0$ if $l = 50$.
$A(0) = A(100) = 0$ and $A(50) = 1250$,
so the maximum area enclosed is 1250 m^2 with dimensions 50 m by 25 m, with the 50 m side parallel to the barn.

Example:
A soup can is to have a volume of 500 mL. What dimensions will minimize the amount of tin used ?



Let the radius and height of the can be r and h, respectively.
The amount of tin used corresponds to the surface area of the can,
which is $A = 2\pi r^2 + 2\pi rh$ (*can you see that ?*).
But the volume is supposed to be $V = \pi r^2 h = 500\text{ cm}^3$, so $h = \dfrac{500}{\pi r^2}$.
And so $A(r) = 2\pi r^2 + 2\pi r\left(\dfrac{500}{\pi r^2}\right) = 2\pi r^2 + \dfrac{1000}{r}$.
Clearly, the dimensions must be positive values.
$A'(r) = 4\pi r - \dfrac{1000}{r^2}$,
so $A'(r) = 0$ if $4\pi r^3 = 1000$ or $r^3 = \dfrac{1000}{4\pi} = \dfrac{250}{\pi}$ or $r = \sqrt[3]{\dfrac{250}{\pi}} \approx 4.30$ cm.
$A''(r) = 4\pi + \dfrac{2000}{r^3} > 0$ for all $r > 0$, so this value must be a local (and absolute) min for the function (*see the graph for confirmation*).
The height is then $h = \dfrac{500}{\pi(4.3)^2} \approx 8.61$ cm
(*notice that the height and the diameter are equal here*).



Example:
Suppose that the material for the top and bottom of the can costs twice as much as that for the side. What dimensions would minimize the cost of producing the can ?

The (relative) cost would be $C = 2(2\pi r^2) + 2\pi rh = 4\pi r^2 + 2\pi rh$,
so $C(r) = 4\pi r^2 + \dfrac{1000}{r}$.
Then $C'(r) = 8\pi r - \dfrac{1000}{r^2}$, so $C'(r) = 0$ if $r = \sqrt[3]{\dfrac{125}{\pi}} \approx 3.41$ cm
and then the height would be $h \approx 13.69$ cm
(*notice that the height is now twice the diameter*).

Example:
If the cost of producing x widgets is $C(x) = 0.1x^2 + 30x + 100$ and the price function is $p(x) = 125 - 0.2x$, how many widgets should be produced and sold to maximize profits?
The profit function is
$P(x) = R(x) - C(x)$
$= xp(x) - C(x)$
$= x(125 - 0.2x) - (0.1x^2 + 30x + 100)$
$= -0.3x^2 + 95x - 100.$
So $P'(x) = -0.6x + 95$, and then $P'(x) = 0$ if $x = 95/0.6 \approx 158$ (*round to nearest integer*).
$P''(x) = -0.6 < 0 \implies$ this is a local (and absolute) max for the function (*which is a parabola opening downwards*).

Practice Problems

1. Find the intervals of increase and decrease of the following functions. Find and identify any local extrema.
(a) $f(x) = 2x^2 - 5x + 6$
(b) $g(t) = 2t^3 + 15t^2 - 84t + 13$
(c) $y = \dfrac{5}{x+3}$
(d) $f(x) = \dfrac{x^2}{x^2+1}$

2. Find the intervals of concavity and any inflection points for the following functions.
(a) $f(x) = 2x^2 - 5x + 6$
(b) $g(t) = 2t^3 + 15t^2 - 84t + 13$
(c) $y = \dfrac{5}{x+3}$
(d) $f(x) = \dfrac{x^2}{x^2+1}$

3. Find the absolute max and min of the given function on the given interval.
(a) $f(x) = \dfrac{3}{x+2}$ on $[0, 5]$
(b) $y = \dfrac{t^2}{2} + \dfrac{8}{t}$ on $[1, 4]$
(c) $g(x) = 2x^3 - 15x^2 + 24x + 7$ on $[-1, 6]$
(d) $f(t) = t^4 - 8t^3 + 22t^2 - 24t$ on $[0, 4]$

4. Identify any vertical asymptotes of the following functions.
(a) $f(x) = \dfrac{4}{x+6}$
(b) $f(x) = \dfrac{x}{x^2+3}$
(c) $y = \dfrac{2t}{t^2-4}$
(d) $g(r) = \dfrac{3r+1}{r^2-4r-12}$

5. Analyze the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x$ and sketch its graph.

6. Analyze the function $f(x) = \dfrac{x^2}{2} + \dfrac{8}{x}$ and sketch its graph.

7. Analyze the function $f(x) = \dfrac{x^2}{x^2+1}$ and sketch its graph.

8. Analyze the function $f(x) = \dfrac{2}{x^2 - x - 2}$ and sketch its graph.

9. An open-top box is to be constructed so that its base is twice as long as it is wide and its volume is to be 2800 cm^3. Find the dimensions that will minimize the amount of cardboard required.

10. The cost of fuel per kilometre for a truck travelling v km/h is $C(v) = \dfrac{v}{450} + \dfrac{13}{v}$.
(a) What speed will result in the lowest fuel cost per km ?
(b) If the driver is paid \$25 per hour, what speed would give the lowest total cost for a 1500 km trip ?

Practice Problems Solutions

1. (a) $f'(x) = 4x - 5$, so $f'(x) = 0$ if $x = 5/4$
if $x < 5/4, f'(x) < 0$ so $f(x)$ decreasing
if $x > 5/4, f'(x) > 0$ so $f(x)$ increasing and local min at $x = 5/4$
(b) $g'(t) = 6t^2 + 30t - 84 = 6(t^2 + 5t - 14) = 6(t + 7)(t - 2)$, so $g'(t) = 0$ if $t = 2$ or -7
if $t < -7, g'(t) > 0$ so $g(t)$ increasing
if $-7 < t < 2, g'(t) < 0$ so $g(t)$ decreasing and local max at $t = -7$
if $t > 2, g'(t) > 0$ so $g(t)$ increasing and local min at $t = 2$
(c) $\dfrac{dy}{dx} = \dfrac{-5}{(x+3)^2} < 0$ for all $x \neq -3$, so always decreasing
(d) $f'(x) = \dfrac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \dfrac{2x}{(x^2+1)^2}$, so $f'(x) = 0$ if $x = 0$
if $x < 0, f'(x) < 0$ so $f(x)$ decreasing
if $x > 0, f'(x) > 0$ so $f(x)$ increasing and local min at $x = 0$

2. (a) $f'(x) = 4x - 5, f''(x) = 4 > 0 \implies$ always concave up
(b) $g'(t) = 6t^2 + 30t - 84, g''(t) = 12t + 30$, so $g''(t) = 0$ if $t = -30/12 = -5/2$
if $t < -5/2, g''(t) < 0$ so $g(t)$ concave down
if $t > -5/2, g''(t) > 0$ so $g(t)$ concave up and there is an inflection point at $t = -5/2$
(c) $\dfrac{dy}{dx} = \dfrac{-5}{(x+3)^2}, \dfrac{d^2y}{dx^2} = \dfrac{10}{(x+3)^3} \neq 0$ for any x
if $x < -3, y'' < 0$ so concave down
if $x > -3, y'' > 0$ so concave up (but no inflection point)
(d) $f'(x) = \dfrac{2x}{(x^2+1)^2}$,

$$f''(x) = \frac{2(x^2+1)^2 - 2x(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2[(x^2+1) - 4x^2]}{(x^2+1)^3} = \frac{2(1-3x^2)}{(x^2+1)^3}$$

so $f''(x) = 0$ if $x = \pm 1/\sqrt{3} \approx \pm 0.577$
if $x < -1/\sqrt{3}, f''(x) < 0$, so $f(x)$ concave down
if $-1/\sqrt{3} < x < 1/\sqrt{3}, f''(x) > 0$, so $f(x)$ concave up and inflection point at $x = -1/\sqrt{3}$
if $x > 1/\sqrt{3}, f''(x) < 0$, so $f(x)$ concave down and inflection point at $x = 1/\sqrt{3}$

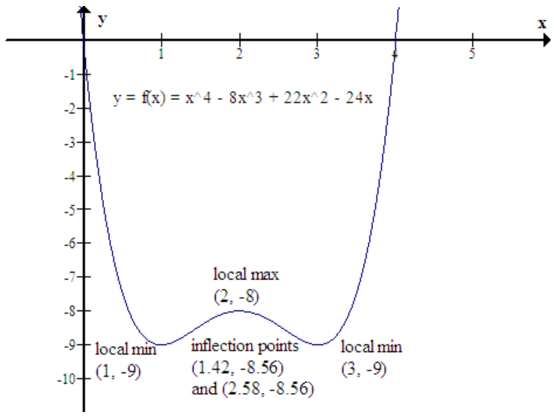
3. (a) $f'(x) = \dfrac{-3}{(x+2)^2} < 0$ for all $x \neq -2$, so $f(x)$ always decreasing
so $f(0) = 3/2$ is the max and $f(5) = 3/7$ is the min
(b) $\dfrac{dy}{dt} = t - \dfrac{8}{t^2}, \dfrac{dy}{dt} = 0$ if $t^3 = 8 \implies t = 2$
critical number $t = 2$ is in the interval $[1, 4]$
$f(1) = 17/2$, $f(2) = 6$ is min, $f(4) = 10$ is max
(c) $g'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x - 1)(x - 4)$
critical numbers $x = 1$ and $x = 4$ are in interval
$g(-1) = -34$ is min, $g(1) = 18$, $g(4) = -9$, $g(6) = 43$ is max
(d) $f'(t) = 4t^3 - 24t^2 + 44t - 24 = 4(t^3 - 6t^2 + 11t - 6) = 4(t-1)(t^2 - 5t + 6) = 4(t-1)(t-2)(t-3)$

all three critical points are in the interval
$f(0) = f(4) = 0$ is max, $f(1) = f(3) = -9$ is min, $f(2) = -8$

4. (a) $x = -6$
(b) none
(c) $t = \pm 2$
(d) $r = -2$ and $r = 6$

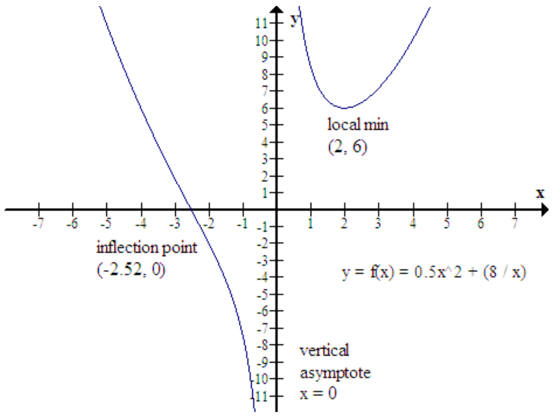
5. $f(x)$ is a polynomial, so defined for all x and no vertical asymptotes
$\lim\limits_{x\to\pm\infty} f(x) = \infty$ so no horizontal asymptotes
no symmetry
$f(0) = 0 \implies (0,0)$ is y-intercept
$f(x) = 0 \implies x^4 - 8x^3 + 22x^2 - 24x = 0$ or $x(x^3 - 8x^2 + 22x - 24) = 0$
or $x(x-4)(x^2 - 4x + 6) = 0 \implies x$-intercepts are $x = 0$ and $x = 4$
$f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x^3 - 6x^2 + 11x - 6) = 4(x-1)(x-2)(x-3)$
$\implies f'(x) = 0$ if $x = 1, 2, 3$
if $x < 1, f'(x) < 0, f(x)$ decreasing
if $1 < x < 2, f'(x) > 0, f(x)$ increasing and local min at $(1, -9)$
if $2 < x < 3, f'(x) < 0, f(x)$ decreasing and local max at $(2, -8)$
if $x > 3, f'(x) < 0, f(x)$ increasing and local min at $(3, -9)$
$f''(x) = 12x^2 - 48x + 44 = 4(3x^2 - 12x + 11)$,
so $f''(x) = 0$ if $x = \dfrac{12 \pm \sqrt{(-12)^2 - 4(3)(11)}}{2(3)} = \dfrac{12 \pm \sqrt{12}}{6} = \dfrac{12 \pm 2\sqrt{3}}{6}$
$= 2 \pm 1/\sqrt{3} \approx 1.42, 2.58$
if $x < 2 - 1/\sqrt{3}, f''(x) > 0, f(x)$ concave up
if $2 - 1/\sqrt{3} < x < 2 + 1/\sqrt{3}, f''(x) < 0, f(x)$ concave down and inflection point at $(1.42, -8.56)$

if $x > 2 + 1/\sqrt{3}, f''(x) > 0, f(x)$ concave up and inflection point at $(2.58, -8.56)$



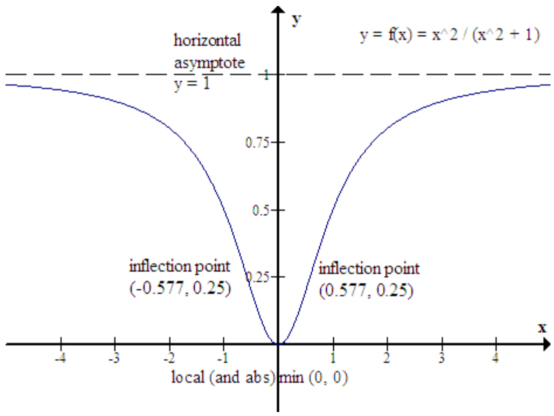
6. $f(x)$ defined for all $x \neq 0$, so no y-intercept
$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(\frac{x^2}{2} + \frac{8}{x}\right) = 0 - \infty = -\infty$
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(\frac{x^2}{2} + \frac{8}{x}\right) = 0 + \infty = \infty$, so $x = 0$ is vertical asymptote
no symmetry
$f(x) = 0$ if $\frac{x^2}{2} + \frac{8}{x} = 0$ or $x^3 = -16$ or $x = \sqrt[3]{-16} \approx -2.52$
$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \left(\frac{x^2}{2} + \frac{8}{x}\right) = \infty - 0 = \infty$
$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \left(\frac{x^2}{2} + \frac{8}{x}\right) = \infty + 0 = \infty$, so no horizontal asymptotes
$f'(x) = x - \frac{8}{x^2}$, so $f'(x) = 0$ if $x = 2$
if $x < 0, f'(x) < 0, f(x)$ is decreasing
if $0 < x < 2, f'(x) < 0, f(x)$ is decreasing
if $x > 2, f'(x) > 0, f(x)$ is increasing and there is a local min at $(2, 6)$
$f''(x) = 1 + \frac{16}{x^3}$, so $f''(x) = 0$ if $x^3 = -16$ or $x = \sqrt[3]{-16} \approx -2.52$
if $x < \sqrt[3]{-16}, f''(x) > 0, f(x)$ is concave up
if $\sqrt[3]{-16} < x < 0, f''(x) < 0, f(x)$ is concave down and there is an inflection point at $(\sqrt[3]{-16}, 0)$

if $x > 0, f''(x) > 0, f(x)$ is concave up



7. $f(x)$ defined for all x, so no vertical asymptotes
$f(0) = 0$ and $f(x) = 0$ only if $x = 0$, so $(0, 0)$ is the only intercept
$f(-x) = f(x)$ and so the function is even
$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \dfrac{x^2}{x^2+1} = \lim_{x\to\pm\infty} \dfrac{1}{1+\frac{1}{x^2}} = 1$, so $y = 1$ is a horizontal asymptote
$f'(x) = \dfrac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \dfrac{2x}{(x^2+1)^2}$, so $f'(x) = 0$ if $x = 0$
if $x < 0, f'(x) < 0, f(x)$ decreasing
if $x > 0, f'(x) > 0, f(x)$ increasing and there is a local (and abs) min at $(0, 0)$
$f''(x) = \dfrac{2(x^2+1)^2 - 2x(2)(x^2+1)(2x)}{(x^2+1)^4} = \dfrac{2[(x^2+1) - 4x^2]}{(x^2+1)^3} = \dfrac{2(1-3x^2)}{(x^2+1)^3}$
so $f''(x) = 0$ if $x = \pm 1/\sqrt{3} \approx \pm 0.577$
if $x < -1/\sqrt{3}, f''(x) < 0$, so $f(x)$ concave down
if $-1/\sqrt{3} < x < 1/\sqrt{3}, f''(x) > 0$, so $f(x)$ concave up and inflection point at $(-1/\sqrt{3}, 1/4)$

if $x > 1/\sqrt{3}, f''(x) < 0$, so $f(x)$ concave down and inflection point at $(1/\sqrt{3}, 1/4)$



8. $f(x) = \dfrac{2}{x^2 - x - 2} = \dfrac{2}{(x+1)(x-2)}$ defined for all $x \neq -1, 2$

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \dfrac{2}{(x+1)(x-2)} = \infty$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \dfrac{2}{(x+1)(x-2)} = -\infty$, and so $x = -1$ is a vertical asymptote

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \dfrac{2}{(x+1)(x-2)} = -\infty$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \dfrac{2}{(x+1)(x-2)} = \infty$, and so $x = 2$ is a vertical asymptote

$f(0) = -1$

$f(x) \neq 0$ for any x, so no x-intercepts

no symmetry

$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \dfrac{2}{x^2 - x - 2} = \lim_{x \to \pm\infty} \dfrac{\frac{2}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}} = \dfrac{0}{1} = 0$ and so $y = 0$ is a horizontal asymptote

$f'(x) = \dfrac{-2(2x-1)}{(x^2 - x - 2)^2} = \dfrac{2(1-2x)}{(x^2 - x - 2)^2}$, so $f'(x) = 0$ if $x = 1/2$

if $x < -1, f'(x) > 0, f(x)$ increasing

if $-1 < x < 1/2, f'(x) > 0, f(x)$ increasing

if $1/2 < x < 2, f'(x) < 0, f(x)$ decreasing and local max at $(1/2, -8/9)$

if $x > 2, f'(x) < 0, f(x)$ decreasing

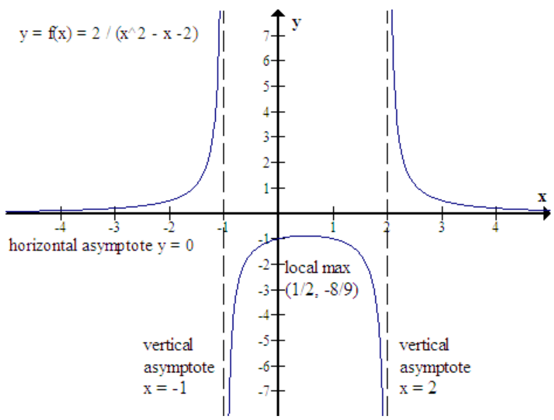
$f''(x) = 2\left[\dfrac{-2(x^2 - x - 2)^2 - (1-2x)(2)(x^2 - x - 2)(2x-1)}{(x^2 - x - 2)^4}\right]$

$= -4\left[\dfrac{(x^2 - x - 2) - (2x-1)^2}{(x^2 - x - 2)^3}\right]$

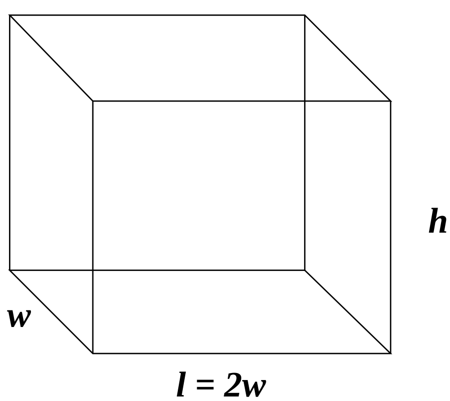
$= \dfrac{-4(x^2 - x - 2 - (4x^2 - 4x + 1)}{(x^2 - x - 2)^3}$

$$= \frac{-4(-3x^2 + 3x - 3)}{(x^2 - x - 2)^3}$$
$$= \frac{12(x^2 - x + 1)}{(x^2 - x - 2)^3} \implies f''(x) \neq 0 \text{ for any } x$$

if $x < -1, f''(x) > 0, f(x)$ is concave up
if $-1 < x < 2, f''(x) < 0, f(x)$ is concave down
if $x > 2, f''(x) > 0, f(x)$ is concave up
but there are no inflection points



9.



let w be the width of the base, then the length of the base is $l = 2w$
and let h be the height
the surface area is
$A = lw + 2wh + 2lh = (2w)w + 2wh + 2(2w)h = 2w^2 + 2wh + 4wh = 2w^2 + 6wh$
but the volume is $V = lwh = (2w)wh = 2w^2h = 2800$
so $h = \frac{2800}{2w^2} = \frac{1400}{w^2}$
and so $A(w) = 2w^2 + 6w\left(\frac{1400}{w^2}\right) = 2w^2 + \frac{8400}{w}$

and then $A'(w) = 4w - \dfrac{8400}{w^2}$
so $A'(w) = 0$ if $4w^3 = 8400$ or $w^3 = 2100$ or $w \approx 12.81$ cm
then $l = 2w \approx 25.61$ cm and $h = \dfrac{1400}{w^2} \approx 8.54$ cm
$A''(w) = 4 + \dfrac{16\,800}{w^3} > 0$ for all $w > 0$
so this a local (and abs) min for the area function

10. (a) clearly, $v > 0$
$C'(v) = \dfrac{1}{450} - \dfrac{13}{v^2}$, so $C'(v) = 0$ if $v^2 = (13)(450) = 5850$ or $v \approx 76.5$ km/h
since $C''(v) = \dfrac{26}{v^3} > 0$ for all $v > 0$, the function is always concave up and hence we have found the local (and absolute) minimum of it
(b) total cost = wages + fuel
or $T = 1500\left[\dfrac{25}{v} + \left(\dfrac{v}{450} + \dfrac{13}{v}\right)\right] = 1500\left(\dfrac{v}{450} + \dfrac{38}{v}\right)$
so $T'(v) = 1500\left(\dfrac{1}{450} - \dfrac{38}{v^2}\right)$
and thus, $T'(v) = 0$ if $v^2 = (38)(450) = 17\,100$ or $v \approx 130.8$ km/h

Chapter 4

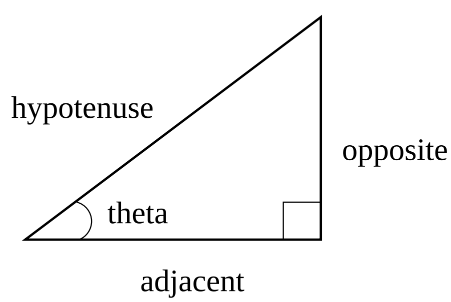
Goals

- to know the derivatives of the trigonometric functions and be able to use them in applications
- to know the derivatives of exponential functions
- to understand the properties of the exponential function $f(x) = e^x$
- to understand the properties of the natural logarithm function $f(x) = \ln x$
- to be able solve application problems involving exponentials

Derivatives of Trigonometric Functions

In Calculus, we always use radian measure for angles – the formulas we will find below for the derivatives of the trigonometric functions require radian measure to be true.

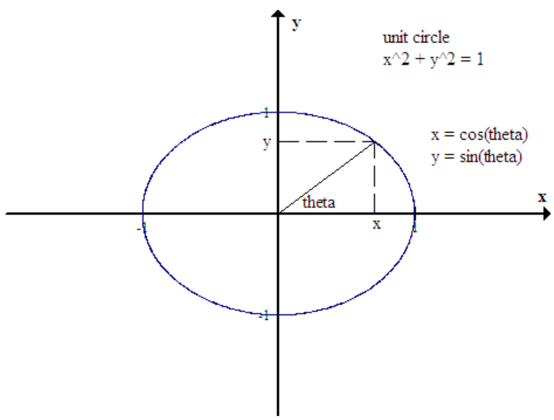
Recall the definitions of the trigonometric functions from a right-angled triangle.



$$\cos\theta = \frac{adjacent}{hypotenuse}$$
$$\sin\theta = \frac{opposite}{hypotenuse}$$
$$\tan\theta = \frac{opposite}{adjacent}$$

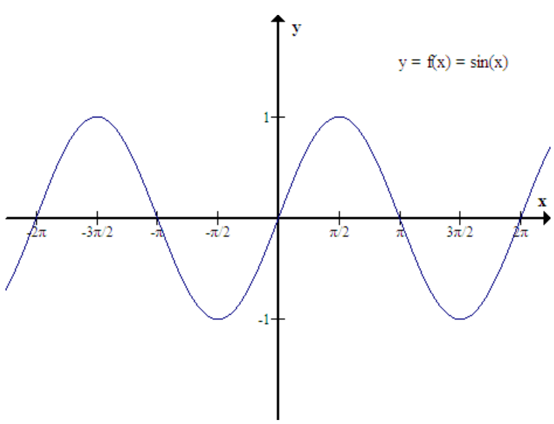
$$\sec\theta = \frac{1}{\cos\theta} = \frac{hypotenuse}{adjacent}$$
$$\csc\theta = \frac{1}{\sin\theta} = \frac{hypotenuse}{opposite}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{adjacent}{opposite}$$

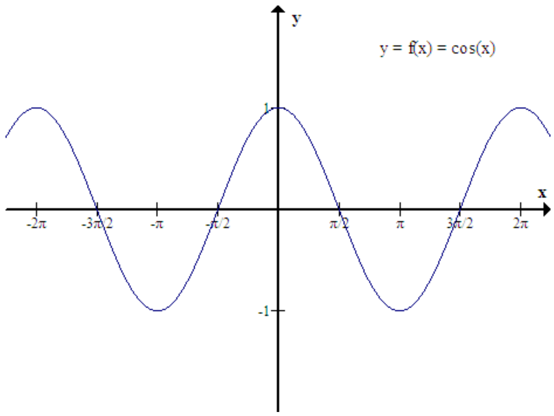
Or, from the unit circle, $x^2 + y^2 = 1$.



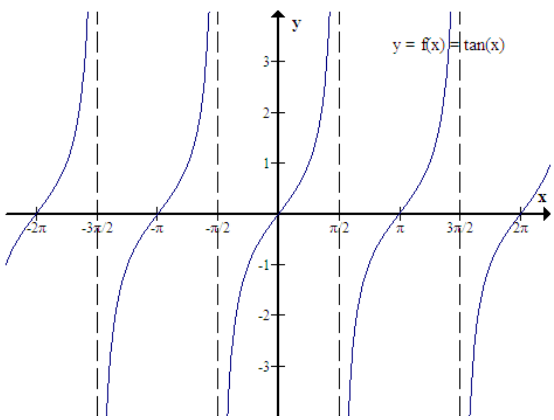
$x = \cos\theta$ and $y = \sin\theta$.

The graphs of the functions follow below.

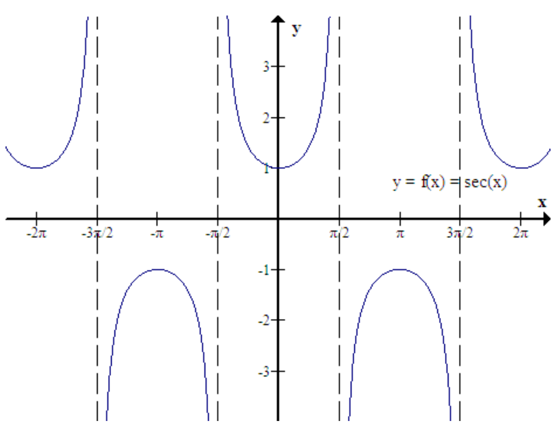




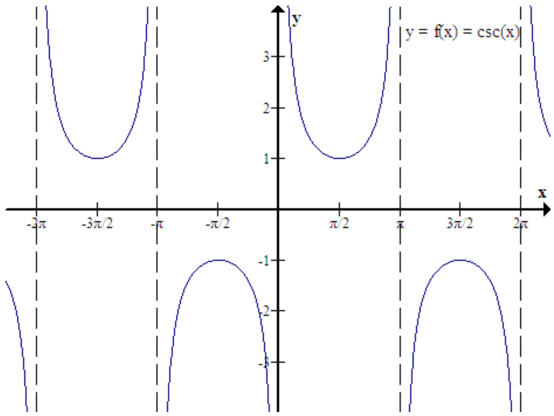
y
y = f(x) = cos(x)
1
-1
x
-2π
-3π/2
-π
-π/2
π/2
π
3π/2
2π

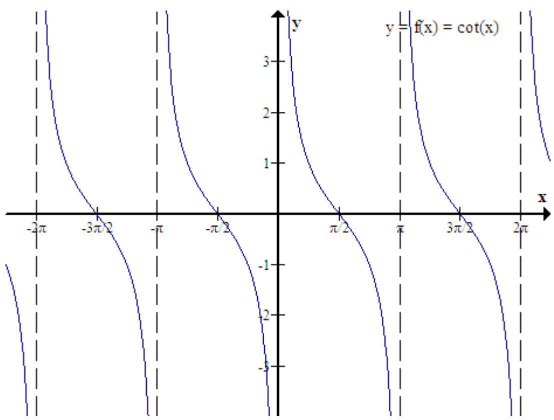


y
y = f(x) = tan(x)
3
2
1
-1
-2
-3
x
-2π
-3π/2
-π
-π/2
π/2
π
3π/2
2π



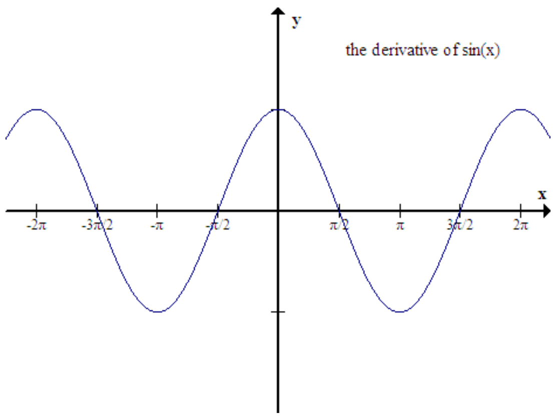
y
y = f(x) = sec(x)
3
2
1
-1
-2
-3
x
-2π
-3π/2
-π
-π/2
π/2
π
3π/2
2π

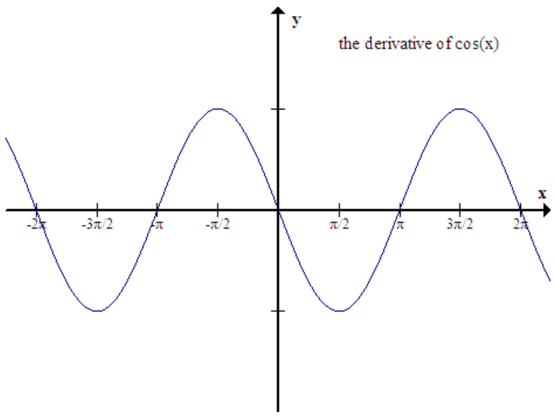




All of the functions are periodic, *ie* they repeat themselves over and over again. *Some have period* 2π *while the others have period* π *– can you tell which are which ?*

What would the derivatives of $\sin x$ and $\cos x$ be like ? We can make graphical approximations and see that they will also be periodic and look like trigonometric functions themselves.



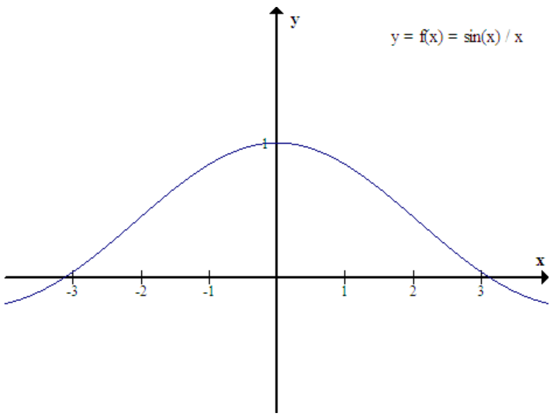


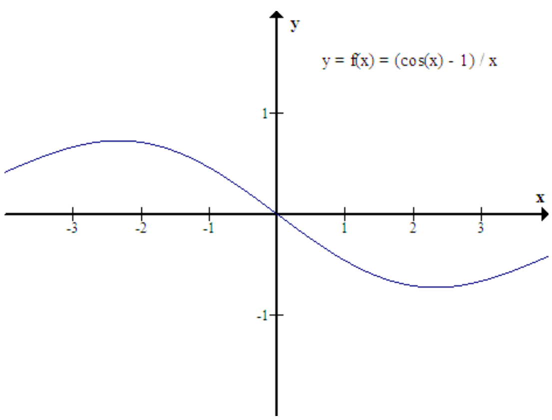
Notice that $\dfrac{d}{dx}(\sin x)$ has the shape of $\cos x$ and that $\dfrac{d}{dx}(\cos x)$ has the shape of $-\sin x$. However, because we cannot be sure of the y values of the derivatives from our graphical approximations, we cannot say at this point that the derivatives are exactly these functions. But we will see that they are.

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$
$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$
$$(\textit{using the trig identity}\ \sin(\theta\pm\phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi)$$
$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}$$
$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\cos x \sin h}{h}$$

$$= (\sin x) \lim_{h \to 0} \frac{\cos h - 1}{h} + (\cos x) \lim_{h \to 0} \frac{\sin h}{h}$$

To see what these limits are, look at the graphs. *The functions are not defined at $x = 0$, but the graphs show us that the limits exist.*





We can see that $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

And thus $\frac{d}{dx}(\sin x) = \cos x$ (*for x in radians*).

We can use the trig identity $\cos x = \sin(\frac{\pi}{2} - x)$ and the chain rule to get
$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(\sin\left(\frac{\pi}{2} - x\right)\right) = \cos\left(\frac{\pi}{2} - x\right)\frac{d}{dx}\left(\frac{\pi}{2} - x\right) = (\sin x)(-1) = -\sin x.$

The derivatives of the other trigonometric functions are as follows.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2}$$
$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \qquad (\textit{using notation convention } (\sin x)^n = \sin^n x)$$
$$= \frac{1}{\cos^2 x} \qquad (\textit{using trig identity } \cos^2 x + \sin^2 x = 1)$$
$$= \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left((\cos x)^{-1}\right)$$
$$= -(\cos x)^{-2}\frac{d}{dx}(\cos x)$$
$$= -(\cos x)^{-2}(-\sin x)$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)$$
$$= \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left((\sin x)^{-1}\right)$$
$$= -(\sin x)^{-2}\frac{d}{dx}(\sin x)$$
$$= -(\sin x)^{-2}(\cos x)$$
$$= \frac{-\cos x}{\sin^2 x}$$
$$= \left(\frac{-1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$
$$= -\csc x \cot x$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$
$$= \frac{\frac{d}{dx}(\cos x)(\sin x) - (\cos x)\frac{d}{dx}(\sin x)}{(\sin x)^2}$$
$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$
$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$
$$= \frac{-1}{\sin^2 x}$$
$$= -\csc^2 x$$

Examples:

(i) $\frac{d}{dx}(\cos(3x)) = -\sin(3x)(3) = -3\sin(3x)$

(ii) $\frac{d}{dx}\left(\sin(x^2)\right) = \cos(x^2)(2x) = 2x\cos(x^2)$

(iii) $\dfrac{d}{d\theta}\,(\theta\tan\theta) = \tan\theta + \theta\sec^2\theta$

(iv) $\dfrac{d}{dt}\,\big(\sec(t^2+4t)\big) = \sec(t^2+4t)\tan(t^2+4t)(2t+4) = (2t+4)\sec(t^2+4t)\tan(t^2+4t)$

(v) $\dfrac{d}{dx}\,(\sin(\tan(x+1))) = \cos(\tan(x+1))\sec^2(x+1)$

(vi) If $y = \cos^2(4x)$,

$\dfrac{dy}{dx} = 2\cos(4x)(-\sin(4x))(4) = -8\sin(4x)\cos(4x) = -4\sin(8x)$

(*using the double angle trig identity* $\sin(2\theta) = 2\sin\theta\cos\theta$).

(vii) If $g(x) = 2\cos^2 x - 3x^4\sin(x^2)$,

then $g'(x) = 4\cos x(-\sin x) - 12x^3\sin(x^2) - 3x^4(\cos(x^2)(2x))$

$= -4\sin x\cos x - 12x^3\sin(x^2) - 6x^5\cos(x^2)$

$= -2\sin(2x) - 12x^3\sin(x^2) - 6x^5\cos(x^2)$.

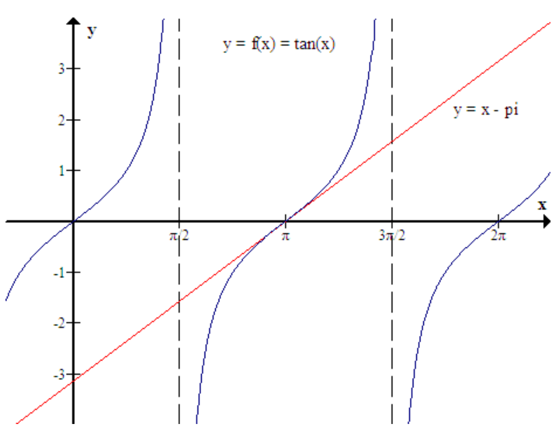
(viii) If $y = \dfrac{t\cos t}{t+\sin t}$,

$\dfrac{dy}{dt} = \dfrac{(\cos t - t\sin t)(t+\sin t) - (t\cos t)(1+\cos t)}{(t+\sin t)^2}$

$= \dfrac{t\cos t - t^2\sin t + \cos t\sin t - t\sin^2 t - t\cos t - t\cos^2 t}{(t+\sin t)^2}$

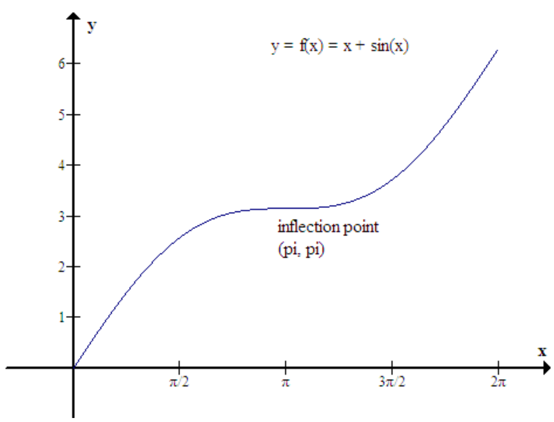
$= \dfrac{\cos t\sin t - t^2\sin t - t}{(t+\sin t)^2}$.

Example:
Find the equation of the tangent line to $y = \tan x$ at $x = \pi$.
The point on the curve is $(\pi, 0)$.
The slope is $m = \dfrac{dy}{dx}|_{x=\pi} = \sec^2(\pi) = (-1)^2 = 1$,
so the line is $y - 0 = (1)(x-\pi)$ or $y = x - \pi$.



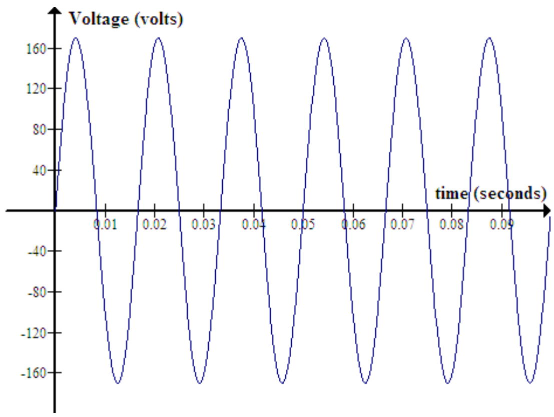
Example:
Analyze the function $y = f(x) = x + \sin x$ on $[0, 2\pi]$ and sketch its graph.
$f(x)$ is defined for all x in $[0, 2\pi]$, so no vertical asymptotes.

We're considering only a finite interval, so no horizontal asymptotes.
There is no symmetry because the interval is not symmetric around origin (*though function is odd for any interval of form* $[-a, a]$).
$f(0) = 0$ and $f(2\pi) = 2\pi$.
$f(x) = 0$ if $x + \sin x = 0$, which has no solution.
$f'(x) = 1 + \cos x$, so $f'(x) = 0$ if $\cos x = -1 \implies x = \pi$.
If $0 < x < \pi$, $f'(x) > 0$ and $f(x)$ is increasing.
If $\pi < x < 2\pi$, $f'(x) > 0$ and $f(x)$ is increasing.
So there are no local extrema, but the graph does have a horizontal tangent at $x = \pi$.
$f''(x) = -\sin x$, so $f''(x) = 0$ if $x = \pi$.
If $0 < x < \pi$, $f''(x) < 0$ and $f(x)$ is concave down.
If $\pi < x < 2\pi$, $f''(x) > 0$ and $f(x)$ is concave up and there is an inflection point at (π, π).



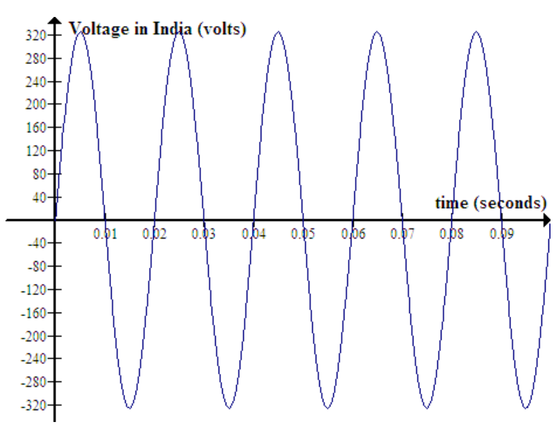
Applications of Trigonometric Functions

Example:
The voltage in a wall socket is given by $V(t) = 170\sin(120\pi t)$ volts for t measured in seconds.



$V'(t) = 170\cos(120\pi t)(120\pi) = 20\,400\pi\cos(120\pi t)$,
so $V'(t) = 0$ if $\cos(120\pi t) = 0$ or if $120\pi t = (2k+1)\frac{\pi}{2}$ for $k \in \mathbb{Z}$.
So we have the maximum voltage of 170 V if $120\pi t = (4k+1)\frac{\pi}{2}$ or if $t = \frac{4k+1}{240}$ s.
And we have the minimum voltage of -170 V if $120\pi t = (4k+3)\frac{\pi}{2}$ or if $t = \frac{4k+3}{240}$ s.
The period of oscillation is $T = \frac{2\pi}{120\pi} = \frac{1}{60}$ s and the frequency is $f = 60$ Hz.

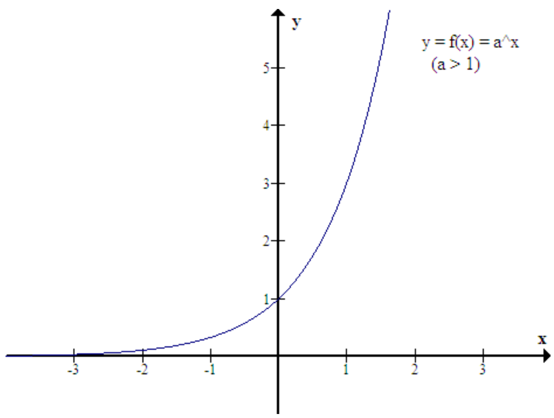
Example:
The standard household voltage in many other countries is given by a different function. The wall socket voltage in India, for example, is given by $V(t) = 325\sin(100\pi t)$.



Which delivers the maximum 325 V when $t = \frac{4k+1}{200}$ s and the minimum -325 V when $t = \frac{4k+3}{200}$ s, with a frequency of $f = 50$ Hz.

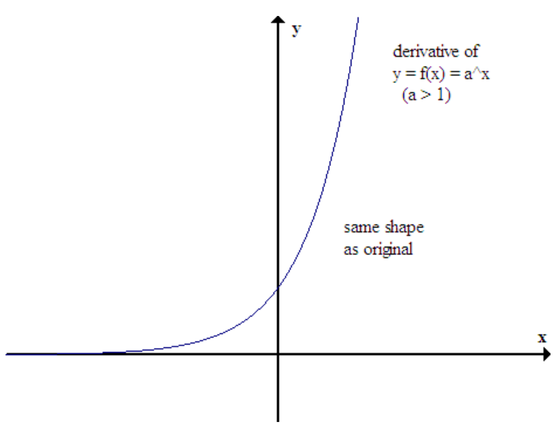
Derivatives of Exponential Functions

Recall that the graph of an exponential function $y = f(x) = a^x$ (for $a > 1$) would look like the one below.



The function is always positive, increasing, concave up, passes through $(0, 1)$ and has $\lim_{x \to -\infty} a^x = 0$ and $\lim_{x \to \infty} a^x = \infty$.

What is the derivative of this function? Graphically, we can see that $\frac{d}{dx}(a^x)$ would have the same shape as a^x itself – increasing, concave up, always positive, starts small (for negative x) and gets larger.



So it appears that the derivative of an exponential is another exponential function. Is it really?

$$\frac{d}{dx}(a^x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$
$$= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$
$$= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \quad (\textit{because } a^x \textit{ is independent of the limit})$$
$$= ka^x,$$

where $k = \lim_{h \to 0} \frac{a^h - 1}{h}$ is a constant.

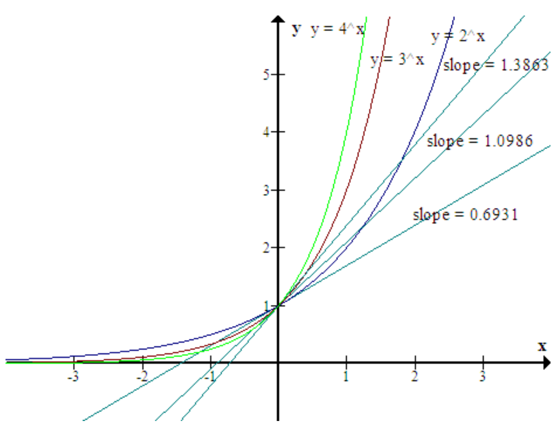
Indeed, $\frac{d}{dx}(a^x)$ is an exponential function – in fact, $\frac{d}{dx}(a^x) = ka^x$.

But, we can do a little better, since if $f(x) = a^x$,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{a^{0+h} - a^0}{h}$$
$$= \lim_{h \to 0} \frac{a^h - 1}{h}$$

and so $k = f'(0)$, the slope of the tangent line to the curve at $x = 0$.

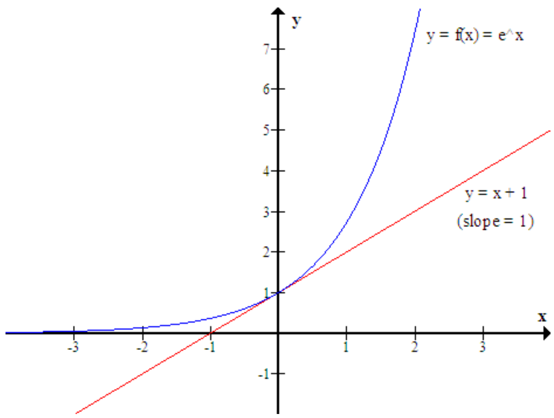
Let's see what happens with $2^x, 3^x$ and 4^x.



For 2^x, $f'(0) \approx 0.6931$, for 3^x, $f'(0) \approx 1.0986$ and for 4^x, $f'(0) \approx 1.3863$. So we can see that $\lim_{h \to 0} \frac{a^h - 1}{h}$ increases as a increases. And we can see something else – there must be a number between 2 and 3 (*closer to* 3) such that the slope of the tangent at $x = 0$, or the limit, is 1. We call that number e.

So e is the number such that $\frac{e^h - 1}{h} \approx 1$ for small h. Or $e^h - 1 \approx h$. Or $e^h \approx h + 1$. So $e \approx (1 + h)^{1/h}$.

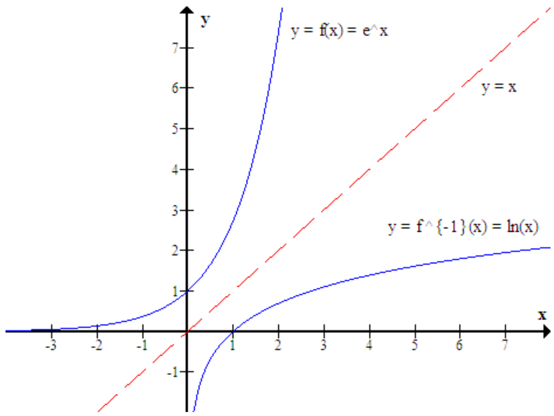
And thus $e = \lim_{h \to 0} (1+h)^{1/h}$
$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \quad (n = 1/h)$
$\approx 2.718\,281\,828\,459\ldots$.



And we have that $\frac{d}{dx}(e^x) = e^x$, *ie* this exponential function is everywhere equal to its own derivative (*and second derivative and third derivative, etc.*).

The Natural Logarithm

Since we have defined the exponential function $f(x) = e^x$, we can define its inverse function, called the natural logarithm, $f^{-1}(x) = \log_e x = \ln x$ (*pronounced as "lawn of x"*), defined for all $x > 0$.



So we have that $e^{\ln x} = x = \ln(e^x)$ (*since they are inverses*).

If $y = \ln x$, then $x = e^y$,
so $\dfrac{d}{dx}(x) = \dfrac{d}{dx}(e^y)$
or $1 = e^y \dfrac{dy}{dx}$ (*by the chain rule*).
So $\dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{x}$.
And thus, we have that $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ for $x > 0$.

Example:
Suppose the population of a bacterial culture after t days is given by $P(t) = 1500e^{0.08t}$.
Then the initial population is $P(0) = 1500e^{0.08(0)} = 1500e^0 = 1500$.
The population after two days is $P(2) = 1500e^{0.08(2)} = 1500e^{0.16} = 1500(1.1735) \approx 1760$.
How long would it take for the population to reach 2000 ?
$2000 = 1500e^{0.08t}$ means that $e^{0.08t} = \frac{2000}{1500} = \frac{4}{3}$,
so $\ln(e^{0.08t}) = \ln(4/3)$ or $0.08t = \ln(4/3)$
and so $t = \frac{\ln(4/3)}{0.08} \approx 3.6$ days.

Since we have defined $\ln x$, we can identify the constant $k = f'(0)$ in the derivative of a^x. Recall that $\dfrac{d}{dx}(a^x) = ka^x = f'(0)a^x$. Actually, $k = f'(0) = \ln a$ and so $\dfrac{d}{dx}(a^x) = a^x \ln a$.

Examples:
(i) $\dfrac{d}{dx}(3^x) = 3^x \ln 3$
(ii) $\dfrac{d}{dt}\left(t^2e^t\right) = 2te^t + t^2e^t = (2t + t^2)e^t$
(iii) $\dfrac{d}{dx}(x5^x) = 5^x + x5^x \ln 5 = (1 + x\ln 5)5^x$

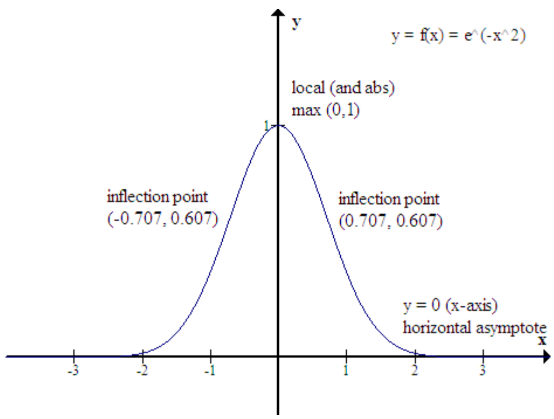
Applications of Exponential Functions

If we have $f(x) = e^{g(x)}$, then the chain rule tells us that
$f'(x) = \dfrac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}\dfrac{d}{dx}\left(g(x)\right) = e^{g(x)}g'(x)$.

Examples:
(i) $f'(x) = e^{\cos x}$, $f'(x) = -(\sin x)e^{\cos x}$
(ii) $y = 2e^{t^2+3}$, $\dfrac{dy}{dt} = 4te^{t^2+3}$
(iii) $\dfrac{d}{dx}\left(e^{4x-\sin x}\right) = (4-\cos x)e^{4x-\sin x}$
(iv) $\dfrac{d}{dx}\left(e^x \sin x\right) = e^x \sin x + e^x \cos x$
(v) $\dfrac{d}{dx}\left(x^2 e^{x^2}\right) = 2xe^{x^2} + 2x^3 e^{x^2}$
(vi) If $f(x) = \dfrac{xe^x}{\cos x}$,
then $f'(x) = \dfrac{(e^x + xe^x)\cos x - xe^x(-\sin x)}{\cos^2 x}$
$= \dfrac{e^x(\cos x + x(\cos x + \sin x))}{\cos^2 x}$.
(vii) $\dfrac{d}{dx}\left(\sin(e^{x^2+5x+1})\right)$
$= \cos(e^{x^2+5x+1})\dfrac{d}{dx}\left(e^{x^2+5x+1}\right)$
$= \cos(e^{x^2+5x+1})e^{x^2+5x+1}\dfrac{d}{dx}\left(x^2+5x+1\right)$
$= (2x+5)e^{x^2+5x+1}\cos(e^{x^2+5x+1})$

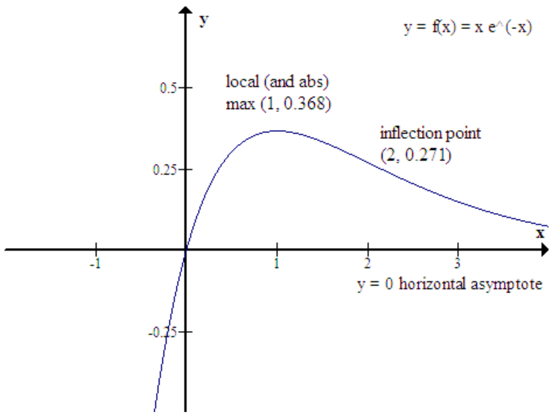
Example:
Analyze the function $y = f(x) = e^{-x^2}$ and sketch its graph.
$f(x)$ is defined for all x, so no vertical asymptotes.
$f(0) = 1 \implies (0,1)$ is the y-intercpet.
$f(x) \neq 0$ for any x, so no x-intercepts.
$f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$, so the function is even.
$\lim_{x\to\pm\infty} e^{-x^2} = 0$, so $y = 0$ is a horizontal asymptote.
$f'(x) = -2xe^{-x^2}$, so $f'(x) = 0$ if $x = 0$.
If $x < 0$, $f'(x) > 0$ and so $f(x)$ is increasing.
If $x > 0$, $f'(x) < 0$ and so $f(x)$ is decreasing and $(0,1)$ is a local (and abs) maximum.
$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 2(2x^2-1)e^{-x^2}$, so $f''(x) = 0$ if $x = \pm 1/\sqrt{2}$.
If $x < -1/\sqrt{2}$, $f''(x) > 0$ and $f(x)$ is concave up.
If $-1/\sqrt{2} < x < 1/\sqrt{2}$, $f''(x) < 0$ and $f(x)$ is concave down (*agrees with local max*).
So there is an inflection point at $(-1/\sqrt{2}, e^{-1/2}) \approx (-0.707, 0.607)$.

If $x > 1/\sqrt{2}$, $f''(x) > 0$ and $f(x)$ is concave up.
So there is another inflection point at $(1/\sqrt{2}, e^{-1/2}) \approx (0.707, 0.607)$.



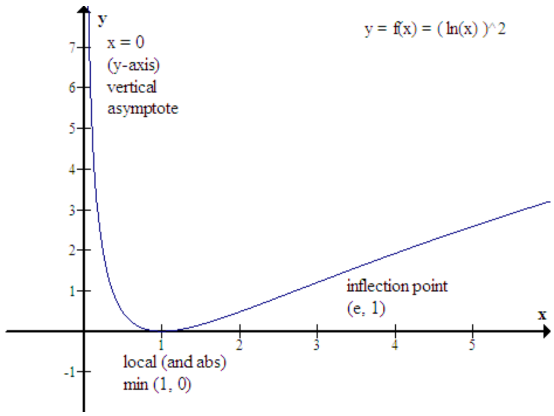
Example:
Analyze the function $y = f(x) = xe^{-x}$ and sketch its graph.
$f(x)$ is defined for all $x \implies$ no vertical asymptotes.
$f(0) = 0$ and so $(0, 0)$ is the y-intercept.
$f(x) = 0$ only if $x = 0$ and so $(0, 0)$ is the only intercept.
This function has no symmetry.
$\lim_{x\to-\infty} xe^{-x} = -\infty$
$\lim_{x\to\infty} xe^{-x} = 0$ and so $y = 0$ is a horizontal asymptote.
$f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x}$, and so $f'(x) = 0$ if $x = 1$.
If $x < 1$, $f'(x) > 0$ and $f(x)$ is increasing.
If $x > 1$, $f'(x) < 0$ and $f(x)$ is decreasing and thus $(1, e^{-1}) \approx (1, 0.368)$ is a local (and abs) max.
$f''(x) = -e^{-x} - e^{-x} + xe^{-x} = (x - 2)e^{-x}$, so $f''(x) = 0$ if $x = 2$.
If $x < 2$, $f''(x) > 0$ and $f(x)$ is concave up.
If $x > 2$, $f''(x) < 0$ and $f(x)$ is concave down and so $(2, 2e^{-2}) \approx (2, 0.271)$ is an inflection

point.



Example:
Analyze the function $y = f(x) = (\ln x)^2$ and sketch its graph.
$f(x)$ is defined for all $x > 0$.
$\lim_{x\to 0^+} (\ln x)^2 = (-\infty)^2 = \infty$ and so $x = 0$ is a vertical asymptote.
No y-intercept.
There is no symmetry.
$f(x) = 0$ if $\ln x = 0$, or if $x = 1$. So there is an x-intercept at $(1, 0)$.
$\lim_{x\to\infty} (\ln x)^2 = \infty$ so no horizontal asymptotes.
$f'(x) = 2(\ln x)\dfrac{1}{x}$, so $f'(x) = 0$ if $x = 1$.
If $0 < x < 1$, $f'(x) < 0$ and $f(x)$ is decreasing.
If $x > 1$, $f'(x) > 0$ and $f(x)$ is increasing and thus $(1, 0)$ is a local (and abs) min.
$f''(x) = \dfrac{2}{x^2} - \dfrac{2\ln x}{x^2} = \dfrac{2}{x^2}(1 - \ln x)$ and thus $f''(x) = 0$ if $\ln x = 1 \implies x = e$.
If $0 < x < e$, $f''(x) > 0$ and so $f(x)$ is concave up.

If $x > e$, $f''(x) < 0$ and $f(x)$ is concave down and there is an inflection point at $(e, 1)$.



Radioactive decay can be modelled using exponential functions. The mass of the radioactive element at time t is given by $m(t) = m_0 e^{-kt} = m_0 e^{-t(\ln 2)/t_{1/2}}$ where m_0 is the initial mass of the sample and $t_{1/2}$ is the half-life (time for half of the sample to decay) of the element.

Example:
Radon 222, ^{222}Rn, has a half-life of 3.8 days. How much of a 100 mg sample will remain in 5 days? How long does it take for the sample to be reduced to 10 mg?
$m(t) = m_0 e^{-t(\ln 2)/t_{1/2}} = 100e^{-t(\ln 2)/3.8}$,
so $m(5) = 100e^{-5(\ln 2)/3.8} \approx 40.2$ mg.
We want t such that $m(t) = 10$ mg.
So $100e^{-t(\ln 2)/3.8} = 10$,
or $e^{-t(\ln 2)/3.8} = 0.1$,
or $\frac{-t(\ln 2)}{3.8} = \ln(0.1)$,
and so $t = \frac{-(\ln(0.1))(3.8)}{\ln 2} \approx 12.6$ days.

Practice Problems

1. Find the derivatives of the following functions.
(a) $f(x) = x^2 \cos x + 7e^x$
(b) $g(t) = 5e^{\sin(t^2)}$
(c) $y = \cos(\sin(e^x))$
(d) $r(\theta) = 2\tan^2(5\theta + e^\theta)$

2. Find the absolute maximum and minimum of $f(x) = x + e^{-x}$ on the interval $[-1, 3]$.

3. Show that $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$ (for $x > 0$).

4. Analyze the function $y = f(x) = x - \cos x$ on the interval $[0, 2\pi]$ and sketch its graph.

5. Analyze the function $y = f(x) = xe^x$ and sketch its graph.

6. Suppose that voltage is given by $V(t) = 125\sin(4\pi t)$ volts, for t measured in seconds.
(a) Find the max and min voltages and the times at which they occur.
(b) Determine the period and the frequency.

7. The radioactive element Unstablium has a half-life of 2.3 hours.
(a) How much of a 10 mg sample is left after 1 day?
(b) How long does it take for the sample to be reduced to 2 mg?

8. Find the equation of the tangent line the curve $y = f(x) = e^x \cos x$ at $x = 0$.

9. On what interval(s) is the function $f(x) = (x^2 + x)e^x$ concave down?

10. Consider the function $g(x) = e^x \sin x$. Find the first four derivatives. What do you suspect the 8th derivative is?

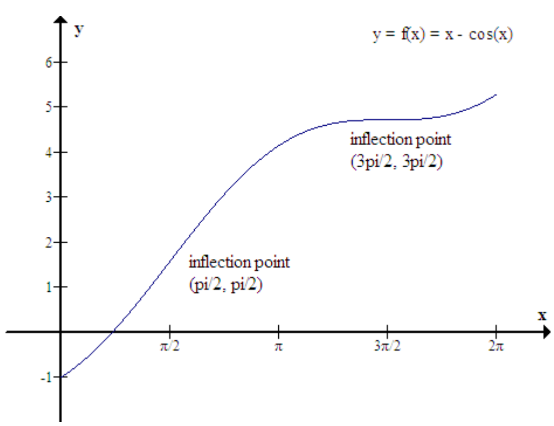
Practice Problems Solutions

1. (a) $f'(x) = 2x\cos x - x^2\sin x + 7e^x$
(b) $g'(t) = 10t\cos(t^2)e^{\sin(t^2)}$
(c) $\dfrac{dy}{dx} = -e^x\sin(\sin(e^x))\cos(e^x)$
(d) $r'(\theta) = 4(5+e^\theta)\tan(5\theta+e^\theta)\sec^2(5\theta+e^\theta)$

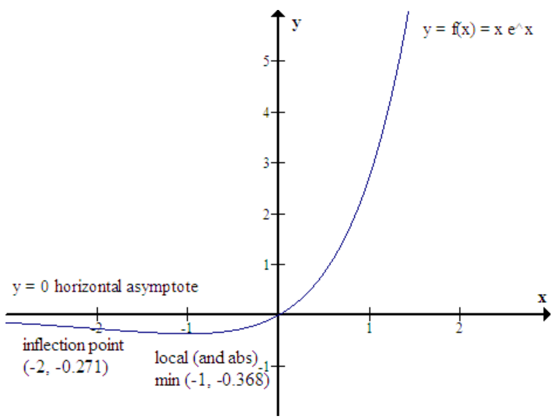
2. $f'(x) = 1 - e^{-x} = 0$ if $x = 0$
$f(-1) = -1 + e \approx 1.72$, $f(0) = 1$ is the min, $f(3) = 3 + e^{-3} \approx 3.05$ is the max

3. $\dfrac{d}{dx}(\log_a x) = \dfrac{d}{dx}\left(\dfrac{\ln x}{\ln a}\right) = \dfrac{1}{\ln a}\dfrac{d}{dx}(\ln x) = \dfrac{1}{x\ln a}$

4. $f(0) = -1, f(2\pi) = 2\pi - 1$
no symmetry, no asymptotes
$f(x) = 0$ if $x = \cos x$, so $x \approx 0.74$
$f'(x) = 1 + \sin x$, $f'(x) = 0$ if $x = 3\pi/2$
if $0 < x < 3\pi/2$, $f'(x) > 0$, $f(x)$ increasing
if $3\pi/2 < x < 2\pi$, $f'(x) > 0$, $f(x)$ increasing so no local extrema
but there is a horizontal tangent at $x = 3\pi/2$
$f''(x) = \cos x$, so $f''(x) = 0$ if $x = \pi/2, 3\pi/2$
if $0 < x < \pi/2$, $f''(x) > 0$ so $f(x)$ is concave up
if $\pi/2 < x < 3\pi/2$, $f''(x) < 0$ so $f(x)$ is concave down
and there is an inflection point at $(\pi/2, \pi/2)$
if $3\pi/2 < x < 2\pi$, $f''(x) > 0$ so $f(x)$ is concave up
and there is also an inflection point at $(3\pi/2, 3\pi/2)$



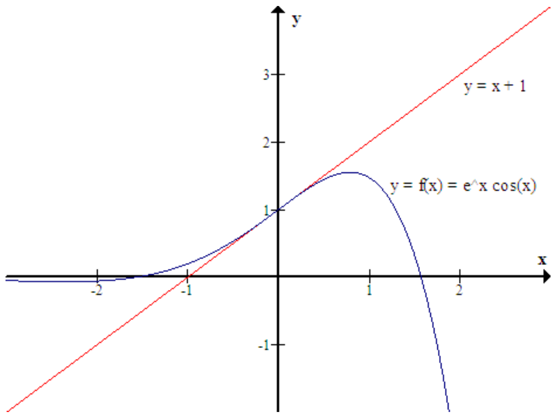
5. $f(x)$ is defined for all x, so no vertical asymptotes.
$f(0) = 0$ and $f(x) = 0$ only if $x = 0$, so $(0, 0)$ is only intercept
no symmetry
$\lim_{x\to-\infty} xe^x = 0$ so $y = 0$ is horizontal asymptote
$\lim_{x\to\infty} xe^x = \infty$
$f'(x) = e^x + xe^x = (1 + x)e^x$, so $f'(x) = 0$ if $x = -1$
if $x < -1$, $f'(x) < 0$ and $f(x)$ is decreasing
if $x > -1$, $f'(x) > 0$ and $f(x)$ is increasing
and thus there is a local (and abs) min at $(-1, -e^{-1}) \approx (-1, -0.368)$
$f''(x) = e^x + e^x + xe^x = (2 + x)e^x$ and so $f''(x) = 0$ if $x = -2$
if $x < -2$, $f''(x) < 0$ and $f(x)$ is concave down
if $x > -2$, $f''(x) > 0$ and $f(x)$ is concave up
and there is an inflection point at $(-2, -2e^{-2}) \approx (-2, -0.271)$



6. (a) $V'(t) = 125(4\pi)\cos(4\pi t) = 0$ if $4\pi t = (2n + 1)\pi/2$
have max of 125 V if $4\pi t = (4n + 1)\pi/2$ or $t = \dfrac{4n+1}{8}$ s
and have min of -125 V if $4\pi t = (4n + 3)\pi/2$ or $t = \dfrac{4n+3}{8}$ s
(b) the period is $T = \dfrac{2\pi}{4\pi} = \dfrac{1}{2}$ s and the frequency is $f = 2$ Hz

7. (a) $m(t) = m_0 e^{-t(\ln 2)/t_{1/2}} = 10e^{-t(\ln 2)/2.3}$
so after 1 day, $m(24) = 10e^{-24(\ln 2)/2.3} \approx 7.22 \times 10^{-3}$ mg
(b) $m(t) = 2 \implies 10e^{-t(\ln 2)/2.3} = 2$
or $e^{-t(\ln 2)/2.3} = 0.2$
or $\frac{-t(\ln 2)}{2.3} = \ln(0.2)$
so $t = \frac{-(2.3)\ln(0.2)}{\ln 2} \approx 5.3$ hours

8. $f(0) = 1, f'(x) = e^x \cos x - e^x \sin x$, so slope is $m = f'(0) = 1$
so the tangent line is $y - 1 = (1)(x - 0) \implies y = x + 1$



9. $f'(x) = (2x+1)e^x + (x^2+x)e^x = (x^2+3x+1)e^x$
$f''(x) = (2x+3)e^x + (x^2+3x+1)e^x = (x^2+5x+4)e^x = (x+4)(x+1)e^x$
$f''(x) = 0$ if $x = -4, -1$ and $f''(x) < 0$ if $-4 < x < -1$

10. $g(x) = e^x \sin x$
$g'(x) = e^x \sin x + e^x \cos x$
$g''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x$
$g'''(x) = 2e^x \cos x - 2e^x \sin x$
$g^{(4)}(x) = 2e^x \cos x - 2e^x \sin x - 2e^x \sin x - 2e^x \cos x = -4e^x \sin x$
so it looks like we should have $g^{(8)}(x) = 16e^x \sin x$ (*verfiy directly*)