When zombies attack!

Mathematical modelling of an outbreak of zombie infection

Robert Smith?

Department of Mathematics and Faculty of ~~Medicine~~ BLOOD
The University of Ottawa

Outline... of DOOM!

- A short history of zombie outbreaks
- The basic SZR model
- Including a latent class
- Intervention 1: Quarantine
- Intervention 2: Treatment
- Intervention 3: Impulsive attacks
- Implications.

Definition

- Zombie: a reanimated corpse that feeds on living flesh
- Origin: African-Carribean belief systems of voodoo
- Main organs and all bodily functions operate at minimal levels.

How to identify a zombie from far away

- Mindless monsters who do not feel pain
- They have an immense appetite for human flesh
- Their aim is to kill, eat or infect people
- The ‘undead’ move in small, irregular steps, and show signs of physical decomposition, eg
  - rotting flesh
  - discoloured eyes
  - open wounds.

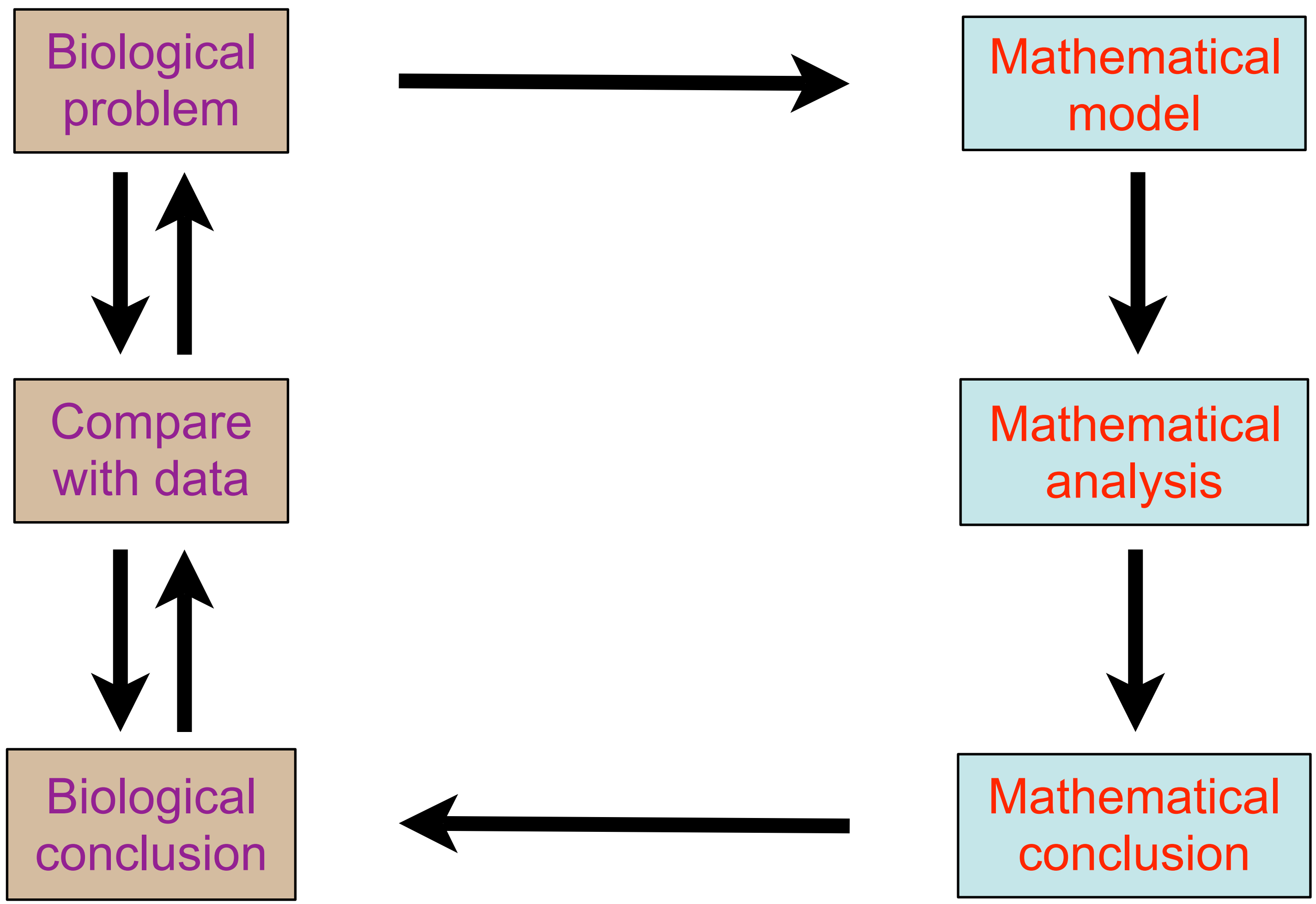
Historical outbreaks

- Major outbreaks of zombies have been recorded since 1968
- Primarily in the US and the UK
- These largely involve zombies overwhelming isolated farmhouses, shopping malls, or British pubs.

Dawn of the Night of the Living Dead

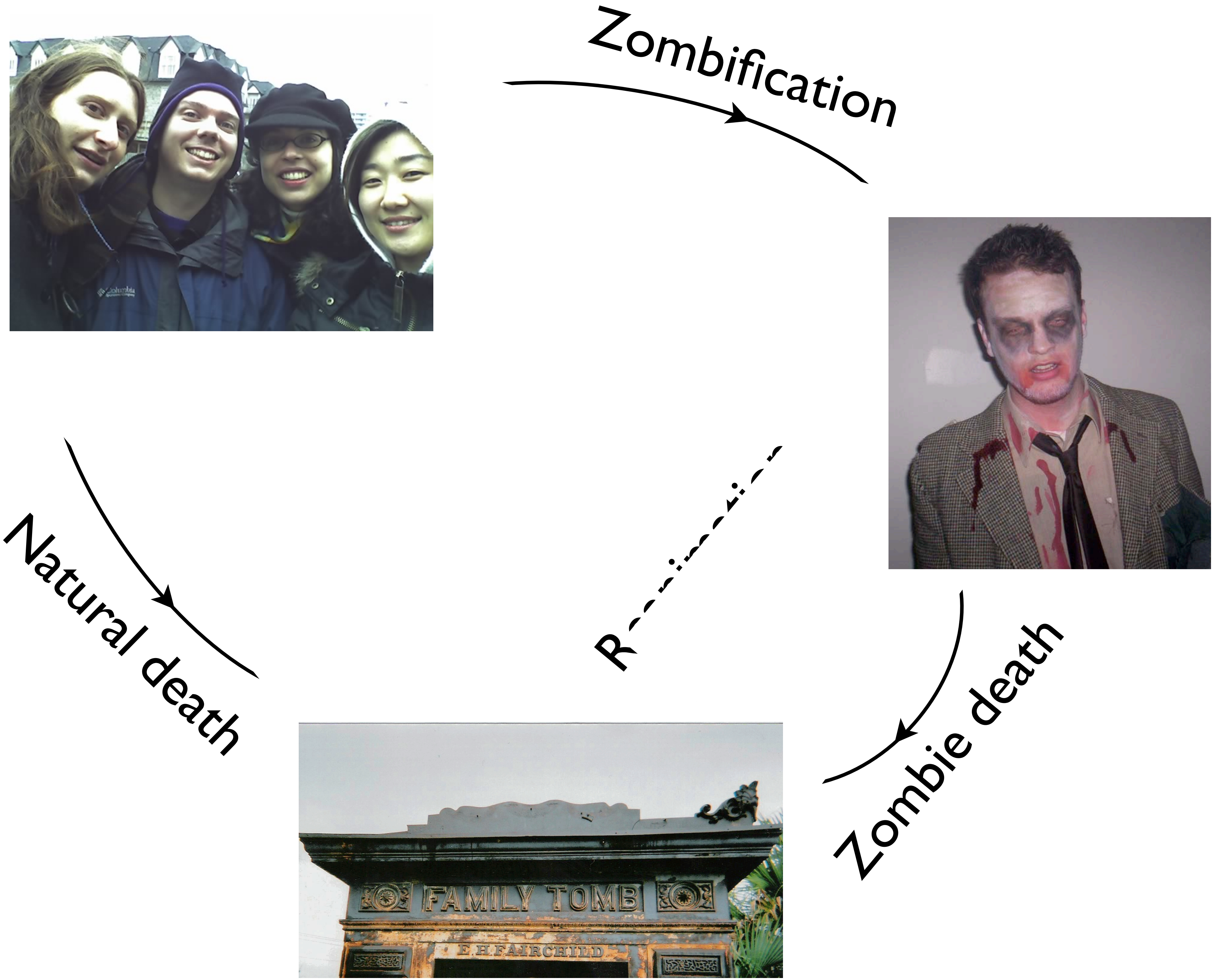
- Possible causes include:
  - radiation emanating from a Venus space probe
  - a virus in chimpanzees
- Zombies defeated by:
  - guns
  - the army
  - eventual starvation
  - Dire Straits records.

Using math to solve real problems

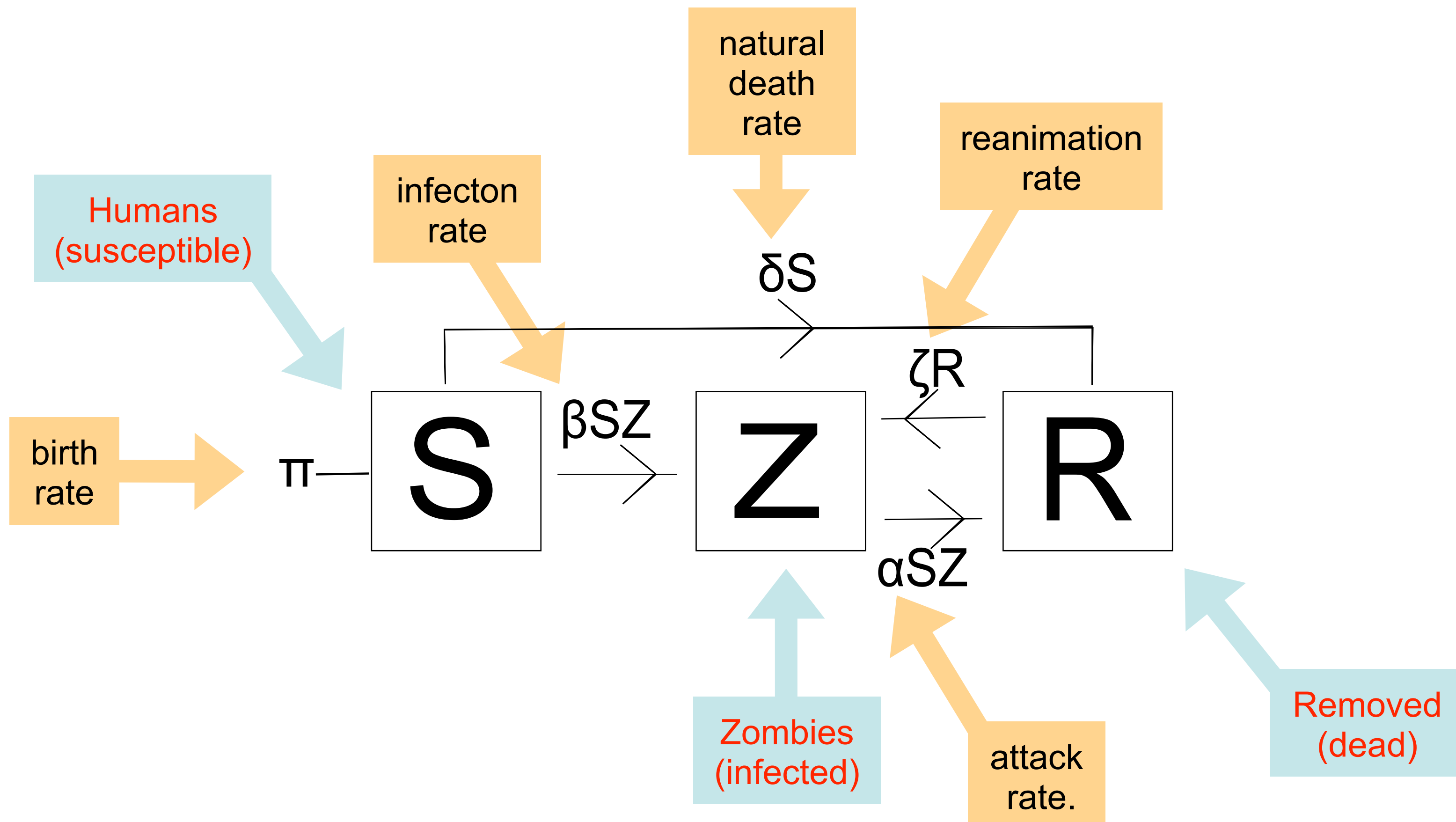


Modelling a zombie outbreak

- Humans are infection by contact with a zombie
- Zombies are created either through converting a human, or by reanimating the dead
- Susceptibles can die of natural causes
- Zombies can be killed in an encounter with humans.


Zombification
Zombie death
Natural death
FAMILY TOMB
E.H.FAIRCHILD

The SZR model



The SZR model

- The basic model is thus

$$
\begin{aligned}
S' &= \Pi - \beta S Z - \delta S \\
Z' &= \beta S Z + \zeta R - \alpha S Z \\
R' &= \delta S + \alpha S Z - \zeta R
\end{aligned}
$$

- Key factor: two mass-action terms
  - one for infection
  - one for attack
- We also keep track of the Removed (dead) class.

Demographics... of DESTINY!

- The ODEs satisfy

$$S' + Z' + R' = \Pi$$

and hence

$$S + Z + R \to \infty$$

as $t \to \infty$, if $\Pi \neq 0$

- Thus, we assume the outbreak happens over a short timescale and set $\Pi = \delta = 0$.

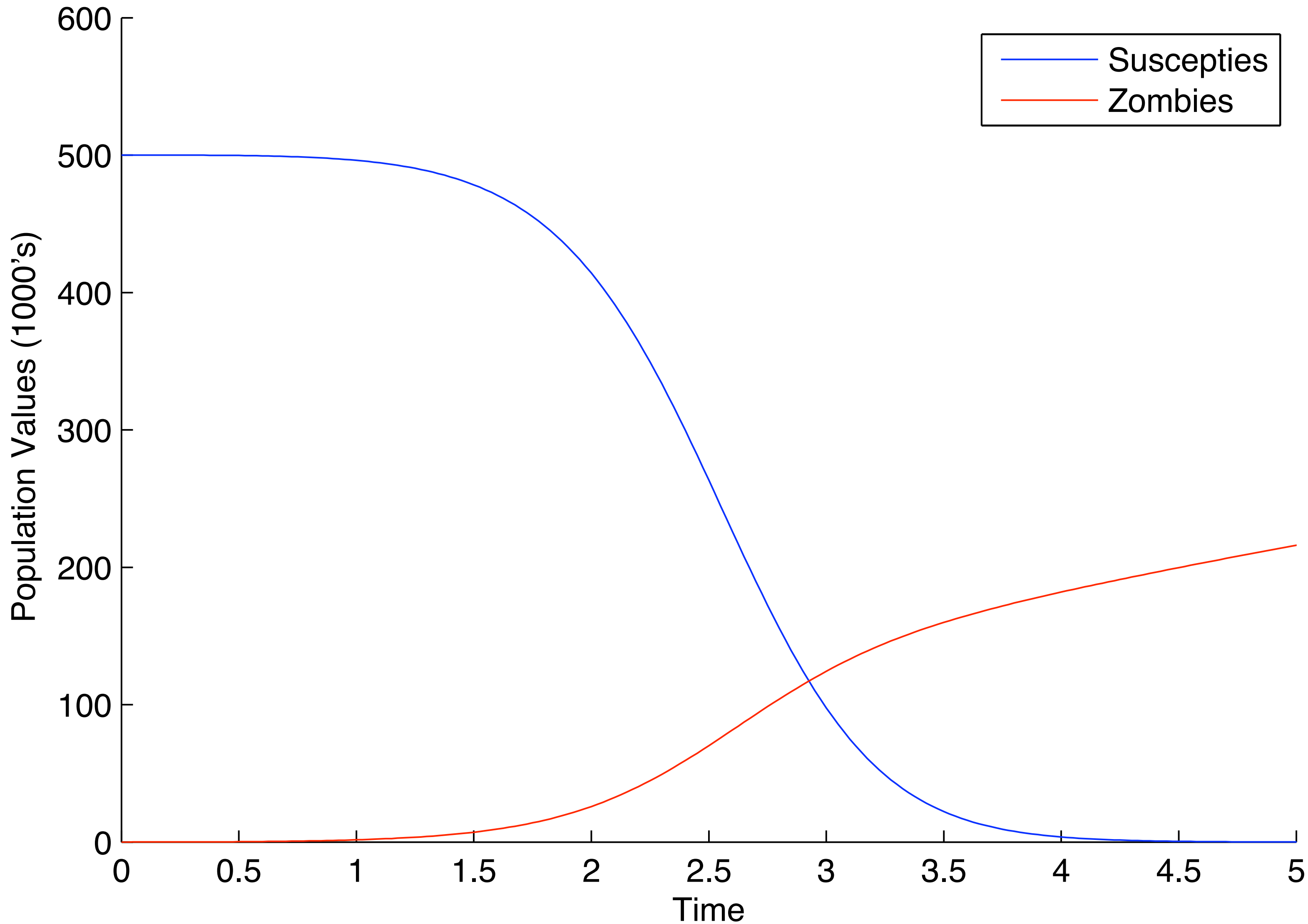
S=humans Z=zombies
R=dead Π=birth rate
δ=natural death rate

Analysis of the SZR model

- Two equilibria: the disease-free equilibrium
$$(\bar{S}, \bar{Z}, \bar{R}) = (N, 0, 0)$$
(no zombies)
and the doomsday equilibrium
$$(\bar{S}, \bar{Z}, \bar{R}) = (0, \bar{Z}, 0)$$
(everyone is a zombie)
- We can prove: the disease-free equilibrium is unstable and the doomsday equilibrium is stable
- This is not good.

S=humans Z=zombies
R=dead N=total population

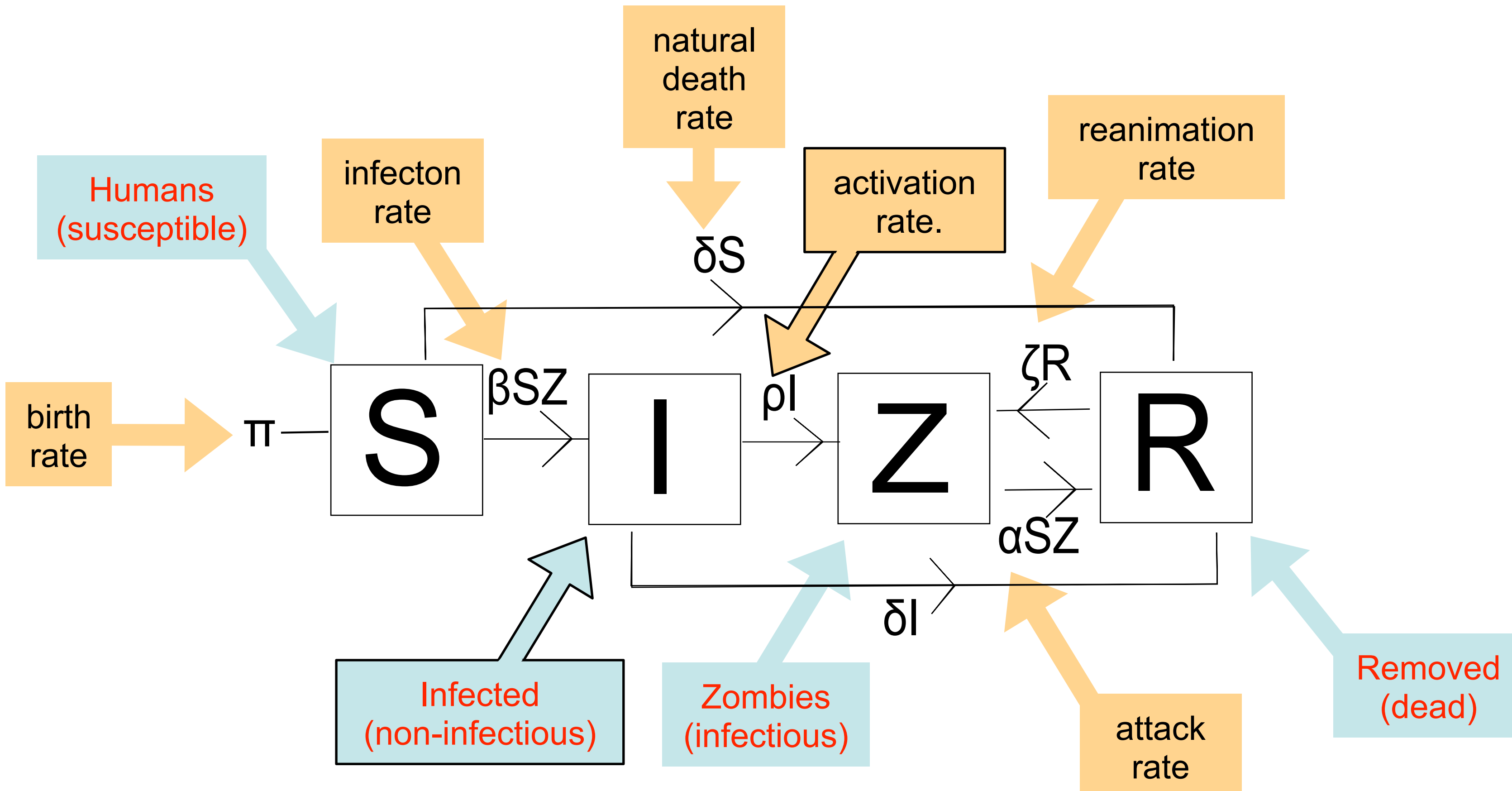
Zombies take over, infecting everyone



Model revision: adding a latent class

- There is a period of time between (approximately 24 hours) after the human susceptible gets bitten before they succumb to their wounds and become a zombie
- We thus extend the basic model to include the possibility that a susceptible individual becomes infected before succumbing to zombification
- This is much more realistic.

The SIZR model



The SIZR model

- The model with latent infection is thus

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S \\ I' &= \beta SZ - \rho I - \delta I \\ Z' &= \rho I + \zeta R - \alpha SZ \\ R' &= \delta S + \delta I + \alpha SZ - \zeta R \end{aligned}$$

- I=infected, not yet infectious
- As before, we model a short outbreak and set $\Pi = \delta = 0$

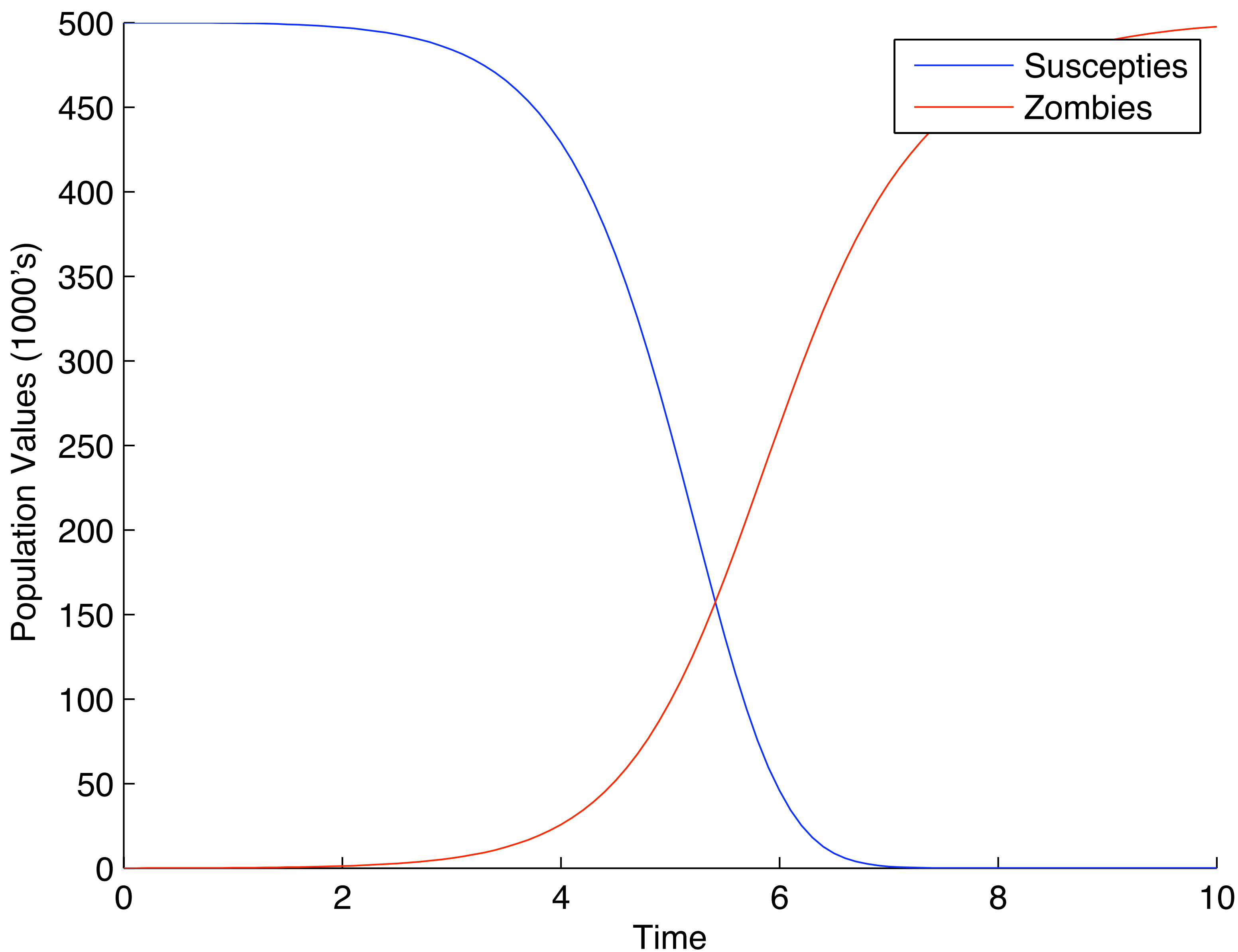
  (or else $S+I+Z+R \to \infty$).

Analysis of the SIZR model

- Two equilibria: the disease-free equilibrium
$$(\bar{S}, \bar{I}, \bar{Z}, \bar{R}) = (N, 0, 0, 0)$$
(no zombies)
and the doomsday equilibrium
$$(\bar{S}, \bar{I}, \bar{Z}, \bar{R}) = (0, 0, \bar{Z}, 0)$$
(everyone is a zombie)
- The disease-free equilibrium is unstable and the doomsday equilibrium is stable
- Thus, even with a latent class, zombies take over the population.

S=humans I=infected Z=zombies
R=dead N=total population

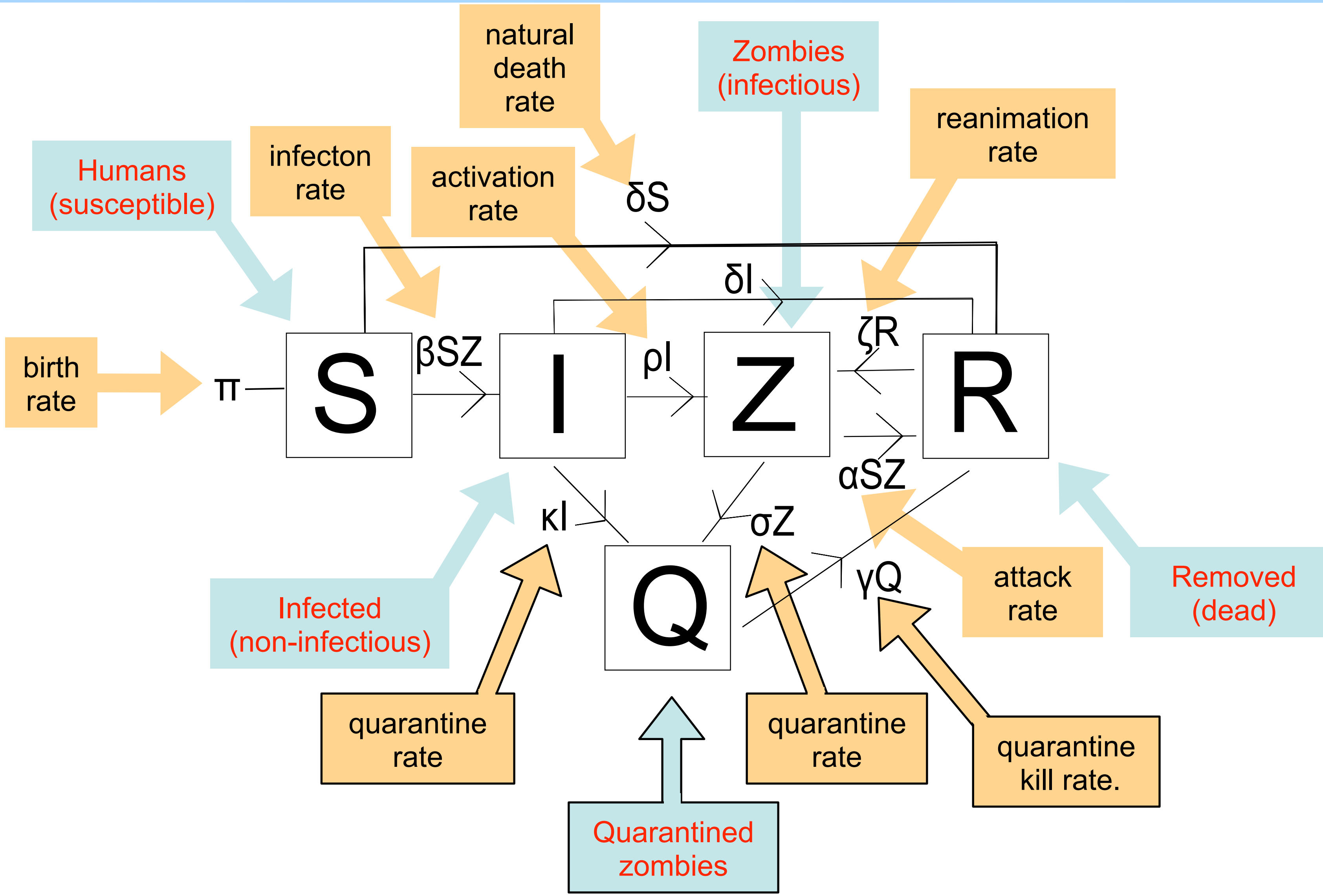
Zombies again take over, destroying humanity



Intervention 1: Quarantine

- To contain the outbreak, we modelled the effects of partial quarantine of zombies
- Quarantined individuals are removed from the population and cannot infect new humans while they remain quarantined
- There is a chance some will try to escape, but any that tried to would be killed before finding their “freedom”.

The SIZRQ model



The SIZRQ model

- The model with quarantine is thus

$$\begin{aligned} S' &= \Pi - \beta SZ - \delta S \\ I' &= \beta SZ - \rho I - \delta I - \kappa I \\ Z' &= \rho I + \zeta R - \alpha SZ - \sigma Z \\ R' &= \delta S + \delta I + \alpha SZ - \zeta R + \gamma Q \\ Q' &= \kappa I + \sigma Z - \gamma Q \end{aligned}$$

- We assume individuals who attempt to escape from quarantine are killed...

  ...whereupon they enter the removed class (and can of course later become zombified).

Equilibria... of ANNIHILATION!

- For a short outbreak ($\Pi = \delta = 0$), we have two equilibria: disease-free

$$(\bar{S}, \bar{I}, \bar{Z}, \bar{R}, \bar{Q}) = (N, 0, 0, 0, 0)$$

and coexistence

$$(\bar{S}, \bar{I}, \bar{Z}, \bar{R}, \bar{Q}) = (0, 0, \bar{Z}, \bar{R}, \bar{Q})$$

(but no humans).

S=humans I=infected Z=zombies
R=dead Q=quarantined
Π=birth rate δ=natural death rate

Basic reproductive ratio

- Using the next-generation method, we determined

$$R_0 = \frac{\beta N \rho}{(\rho + \kappa)(\alpha N + \sigma)}$$

- If the population is large, then

$$R_0 \approx \frac{\beta \rho}{(\rho + \kappa)\alpha}$$

- The disease-free equilibrium is stable if $R_0 < 1$.

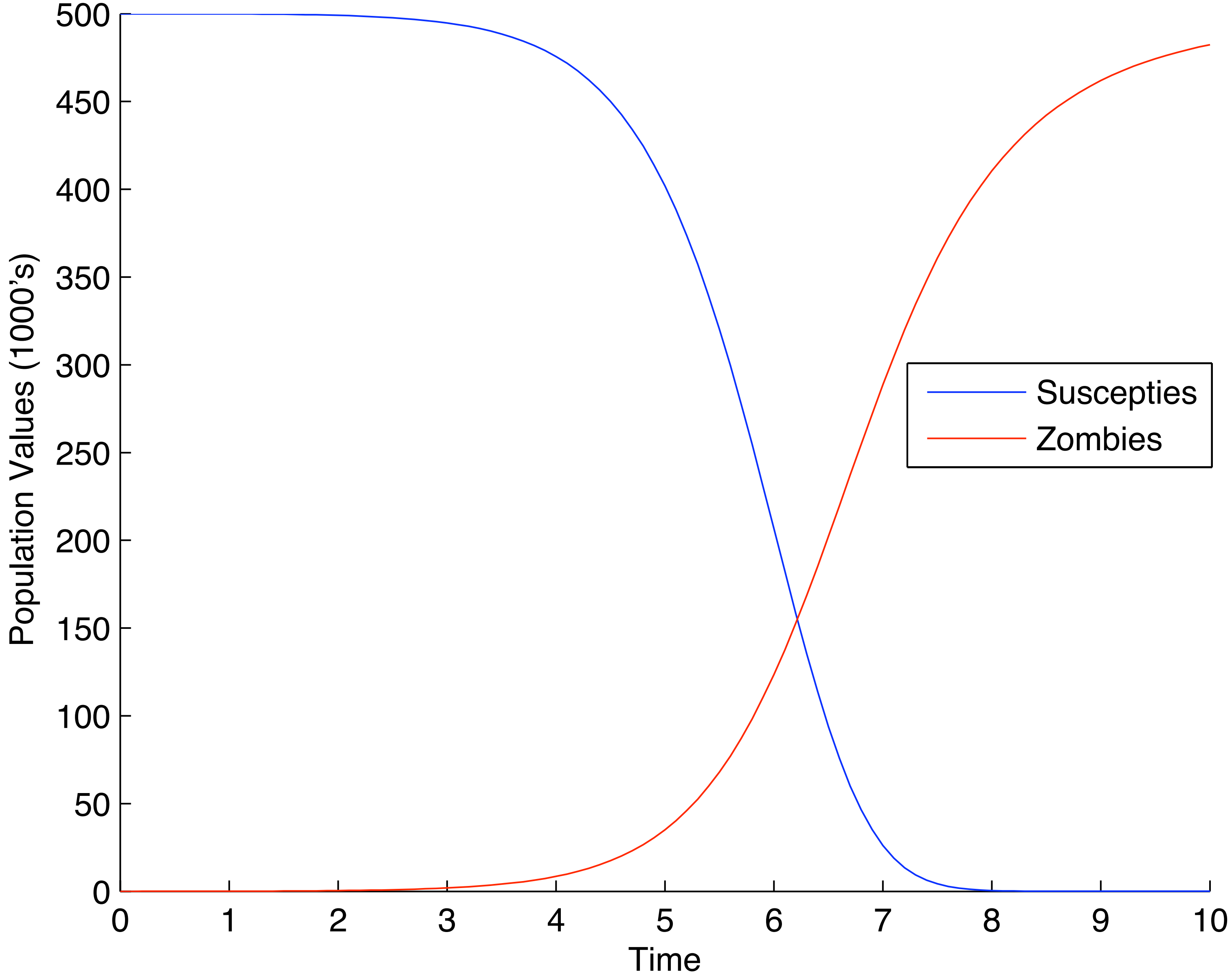
β=*infection rate* N=*total population*
ρ=*activation rate* α=*attack rate*
κ=*quarantine rate (infected)*
σ=*quarantine rate (zombies)*

Invasion of the living dead

- We can reduce $R_0 < 1$ by increasing the quarantine rates κ or σ
- However, we expect that quarantining a large percentage of infected individuals is unrealistic, due to infrastructure limitations
- Thus, we expect $R_0 > 1$
- Hence, zombies can invade.

R_0=basic reproductive ratio
κ=quarantine rate (infected)
σ=quarantine rate (zombies)

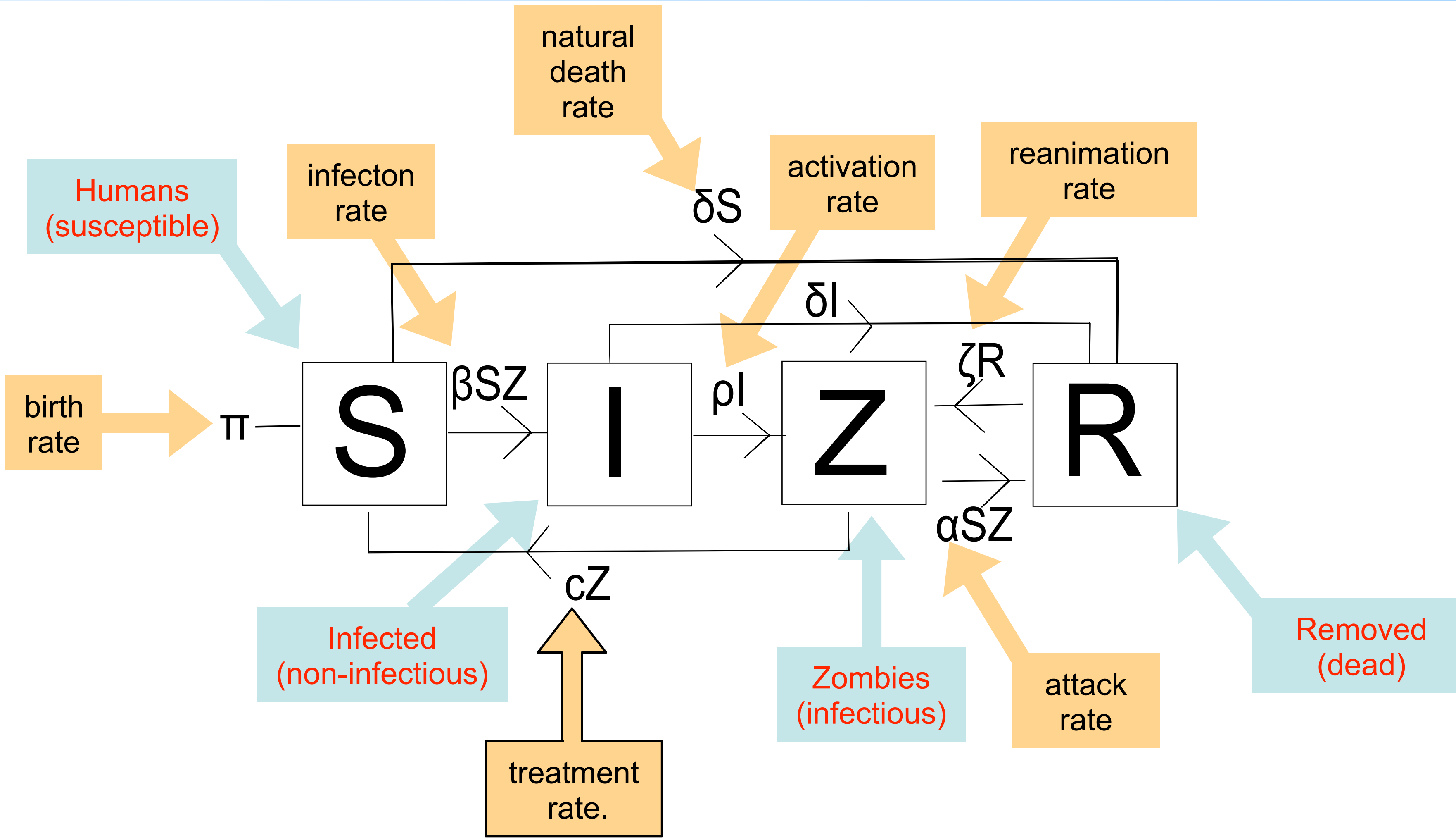
Quarantine delays the inevitable (slighty)



Intervention 2: Treatment

- Suppose we are able to quickly produce a cure for zombie-ism
- Our treatment would be able to allow the zombie individual to return to their human form again
- Since we have treatment, we no longer need the quarantine
- Treatment does not provide immunity.

The model with treatment



The model with treatment

- The model with treatment is thus

$$
\begin{aligned}
S' &= \Pi - \beta SZ - \delta S + cZ \\
I' &= \beta SZ - \rho I - \delta I \\
Z' &= \rho I + \zeta R - \alpha SZ - cZ \\
R' &= \delta S + \delta I + \alpha SZ - \zeta R
\end{aligned}
$$

- As before, we model a short outbreak and set $\Pi = \delta = 0$ (or else $S+I+Z+R \to \infty$).

Analysis of the treatment model

- We have the usual disease-free equilibrium

$$(\bar{S}, \bar{I}, \bar{Z}, \bar{R}) = (N, 0, 0, 0)$$

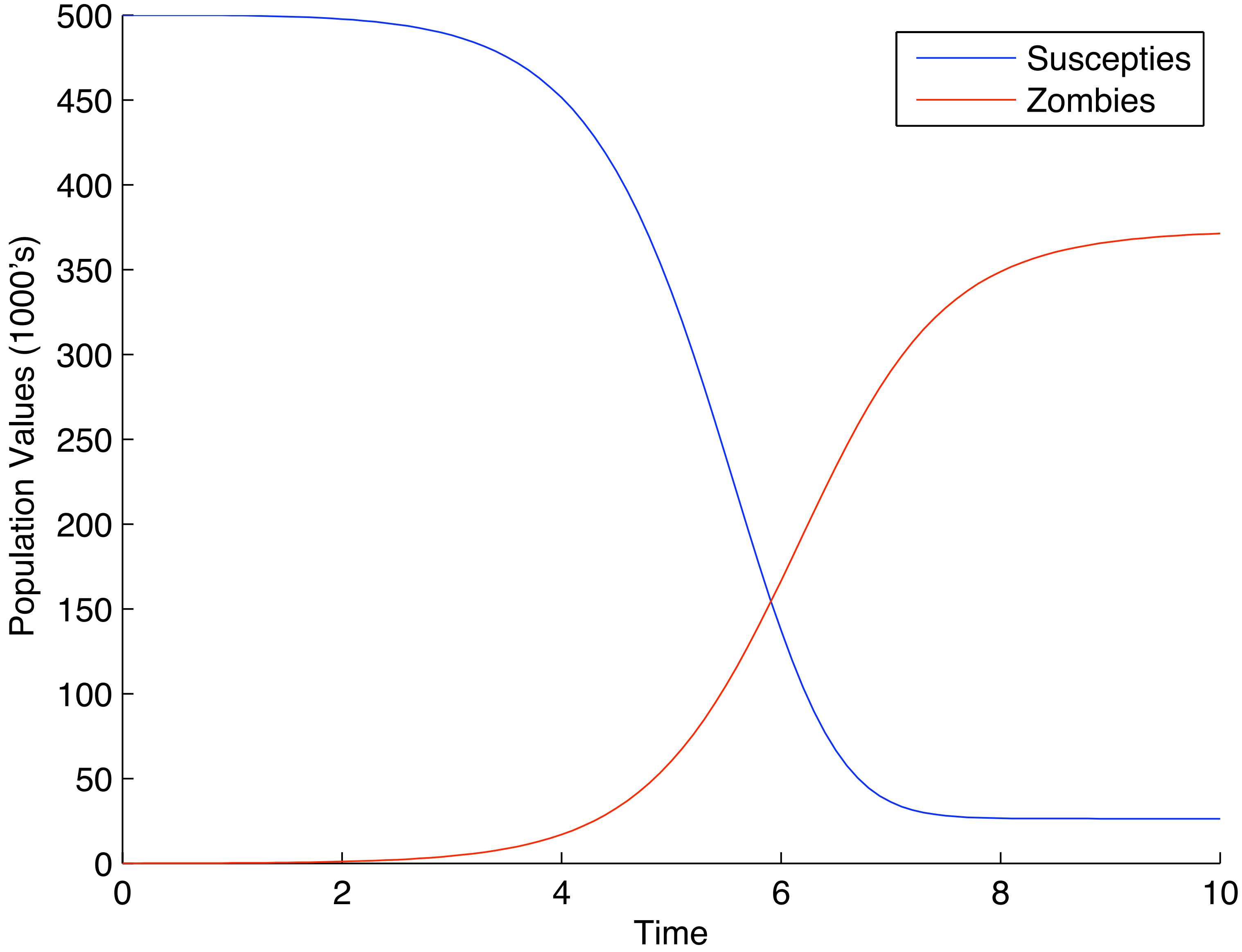
- But now we have the possibility of coexistence

$$(\bar{S}, \bar{I}, \bar{Z}, \bar{R}) = \left(\frac{c}{\beta}, \frac{c}{\rho}\bar{Z}, \bar{Z}, \frac{\alpha c}{\zeta\beta}\bar{Z}\right)$$

- We can prove that the DFE is unstable and the coexistence equilibrium is stable
- Thus, humans and zombies can live in (relative) harmony.

S=humans I=infected Z=zombies R=dead N=total population β=infection rate ρ=activation rate α=attack rate c=treatment rate ζ=reanimation rate

Humans are not eradicated, but exist only in low numbers



Intervention 3: Impulsive attack

- Finally, we attempted to control the zombie population by strategically destroying them at such times that our resources permit
- It was assumed that it would be difficult to have the resources and coordination, so we would need to attack more than once, and with each attack try and destroy more zombies
- This results in an impulsive effect.

Impulsive effect... of TERROR!

- According to impulsive theory, we can describe the nature of the impulse at time r_k via the difference equation

$$\Delta y \equiv y(r_k^+) - y(r_k^-) = f(r_k, y(r_k^-))$$

Difference equation

Depends on the time of impulse and the state immediately beforehand.

Impulsive differential equations

- Solutions are continuous for $t \neq r_k$
- Solutions undergo an instantaneous change in state when $t = r_k$
- Such approximations are reasonable when the cycle time is sufficiently large, compared to the time being approximated
- The model thus consists of a system of ODEs together with a difference equation.

r_k=*impulse time*

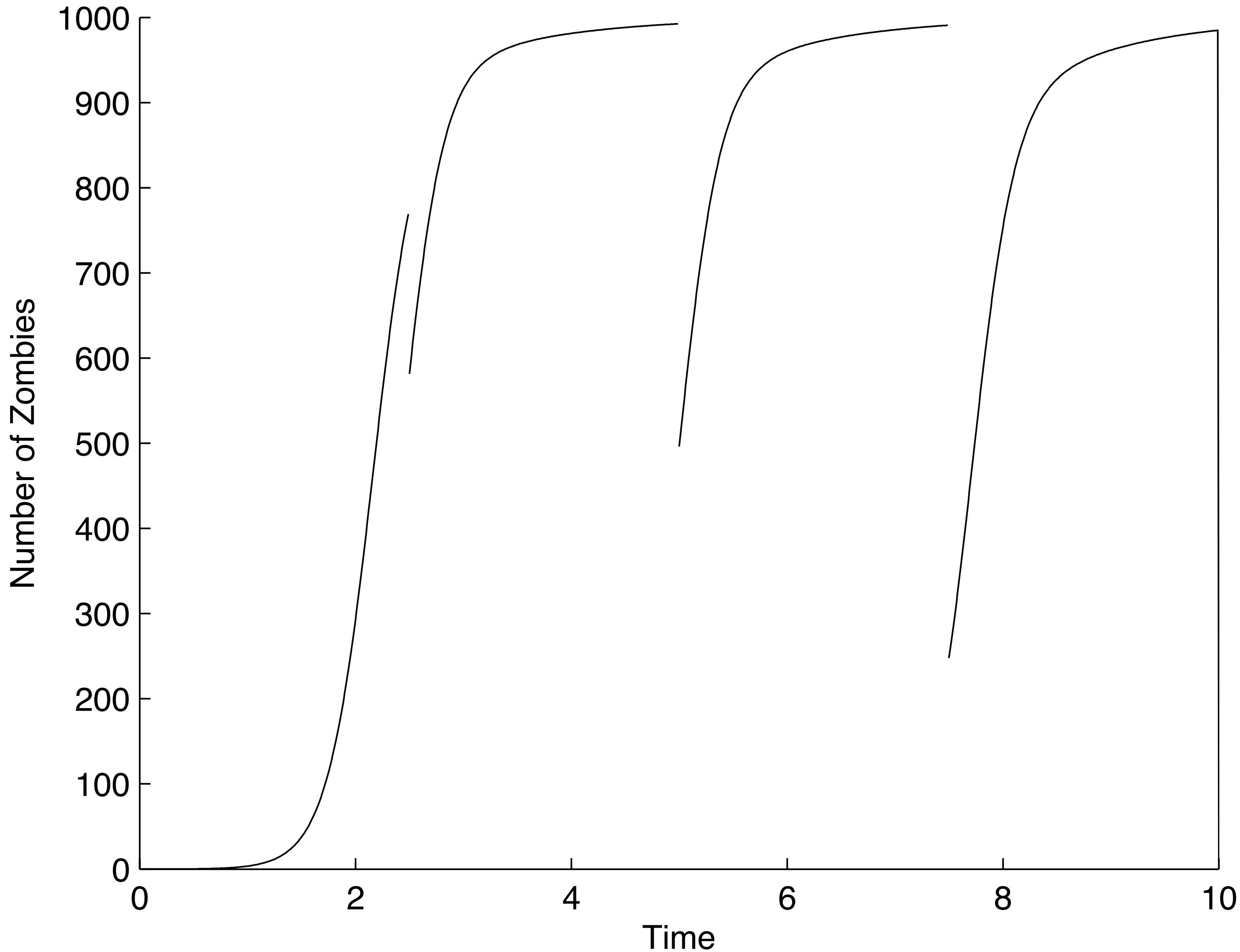
SZRΔZ model

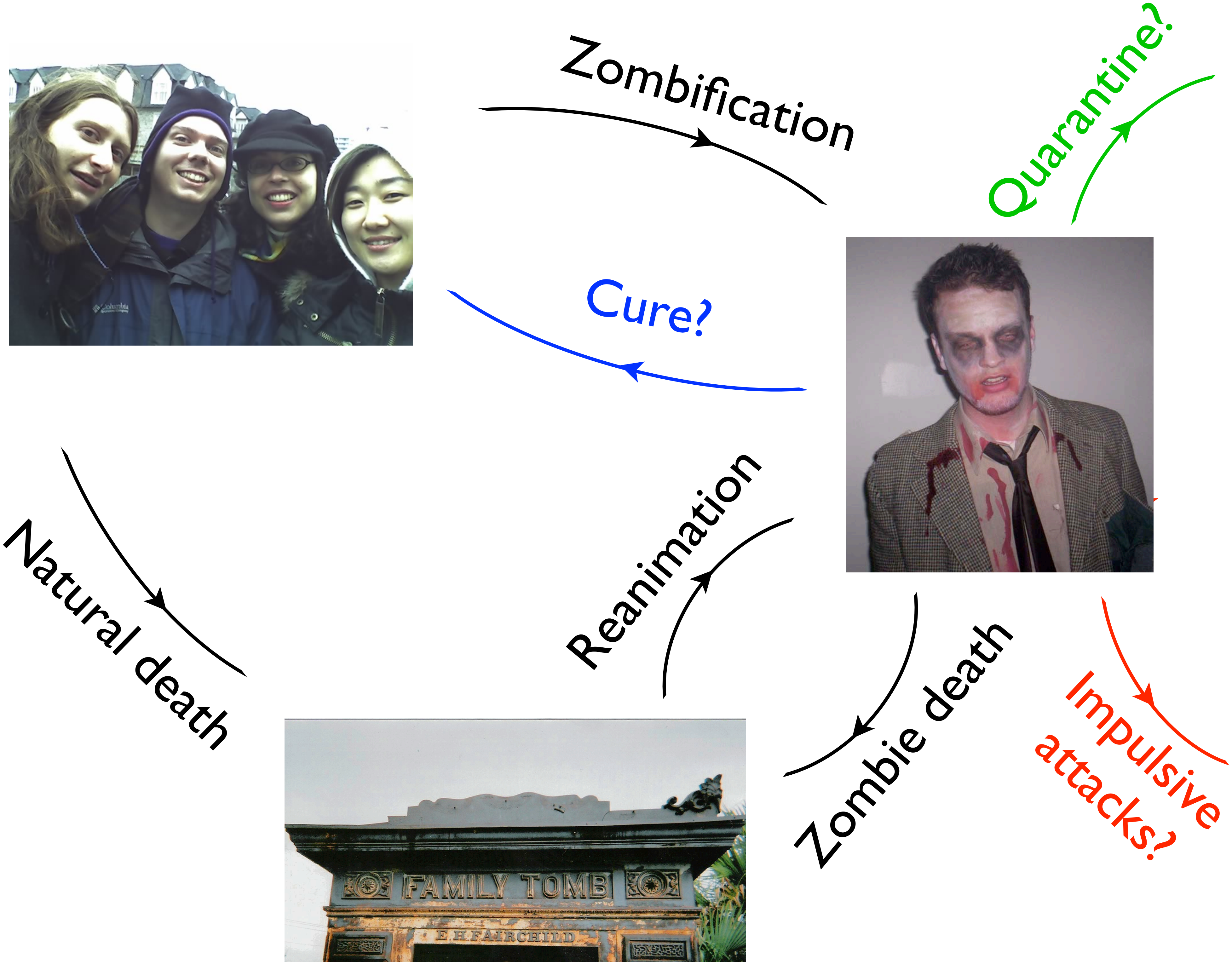
- We return to the basic model and add in impulsive effect:

$$\begin{array}{rcl@{\qquad}rcl}
S' & = & \Pi - \beta SZ - \delta S & t & \neq & t_n \\
Z' & = & \beta SZ + \zeta R - \alpha SZ & t & \neq & t_n \\
R' & = & \delta S + \alpha SZ - \zeta R & t & \neq & t_n \\
\Delta Z & = & -knZ & t & = & t_n
\end{array}$$

- $k \in (0,1)$ is the kill ratio
- n = number of attacks required until $kn > 1$
- Thus, we hit zombies with ever-increasing force.

Only ever-more powerful attacks will stop the zombies




Zombification
Quarantine?
Cure?
Reanimation
Natural death
Zombie death
Impulsive attacks?
FAMILY TOMB
E.H.FAIRCHILD

Summary... of DEATH!

- Extremely aggressive tactics are required
- Quarantine is unable to save us
- Treatment results in coexistence, but only at low levels for humans
- Only frequent attacks, with increasing force, result in eradication...

  ...assuming available resources can be mustered in time.

Limitations

- We only modelled a short timescale
- Otherwise, the result is the doomsday scenario: an outbreak of zombies will result in the collapse of civilisation, with every human infected, or dead
- Because human births and deaths will provide the undead with a limitless supply of new bodies to infect, resurrect and convert
- Thus, if zombies arrive, we must act quickly and decisively to eradicate them before they eradicate us.

Conclusions... of TERROR!

- A zombie outbreak is likely to lead to the collapse of civilisation, unless it is dealt with quickly
- While aggressive quarantine may contain the epidemic, or a cure may lead to coexistence of humans and zombies, the most effective way to contain the rise of the undead is to hit hard and hit often
- It is imperative that zombies are dealt with quickly...

  ...or else we are all in a great deal of trouble.

Key reference... of FEAR!

- P. Munz, I. Hudea, J. Imad and R.J. Smith? When zombies attack!: Mathematical modelling of an outbreak of zombie infection (in: J.M. Tchuenche and C. Chiyaka, eds, Infectious Disease Modelling Research Progress 2009, pp133-150).

http://mysite.science.uottawa.ca/rsmith43