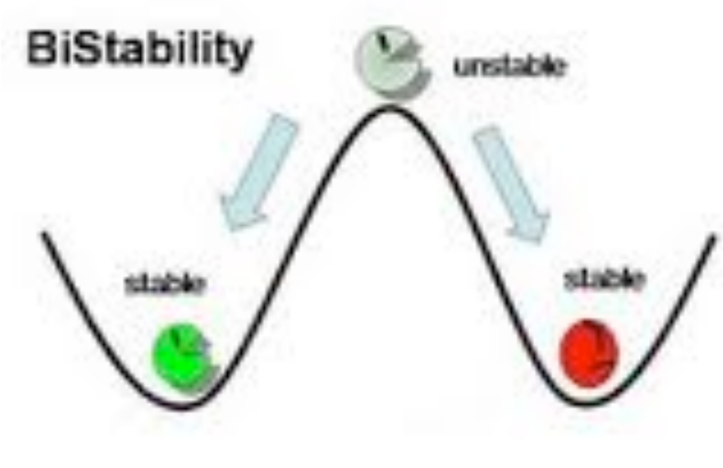
When humans strike back!

Adaptive strategies for zombie attacks



The victim becomes the aggressor

- Humans are usually portrayed as fairly passive victims of a zombie attack
- What if humans become the aggressor against the undead?
- An individual zombie is relatively vulnerable
- It has little agility or speed
- It is incapable of strategic action
- Thus, the average person should be easily able to dispatch a lone walking corpse
  - provided they are equipped with the proper knowledge and minimal armaments.

The danger of many zombies

- The true danger of the zombie is when he is in the company of his peers
- They have a method of unwittingly attracting other zombies...

  ...their despair-inducing moan
- This happens when a zombie becomes aware of his next potential meal
- A human can easily become surrounded overwhelmed.

“Kill the brain and you kill the ghoul”

- Consider the scene at the house early in *Night of the Living Dead*
- Ben and the others are able to fend off attacks by single zombies and successfully kill them
- This is because the zombies of this particular strain are slow and uncoordinated
- Thus, in one-on-one encounters with alert and prepared susceptibles, the single zombie will generally be destroyed.

Safety in numbers

- Conversely, groups of two or more zombies can successfully attack and defeat all but the most well-armed human
- A human with a chain gun could survive a concerted attack by zombies
- However, a single individual with a screwdriver, as in *Dawn of the Dead*, would probably be killed in an attack by multiple zombies
- These empirical observations will allow us to build our first model.

An empirical model

- Let S denote susceptibles, Z denote zombies and ✶ denote a state outside the system
- Then the interactions are

$$
\begin{aligned}
\star &\xrightarrow{a_1} S && \text{(human migration)} \\
S &\xrightarrow{a_2} \star && \text{(natural human death)} \\
S + Z &\xrightarrow{a_3} S && \text{(humans kill individual zombies)} \\
S + 2Z &\xrightarrow{a_4} 3Z && \text{(two zombies convert a human)} \\
Z &\xrightarrow{a_5} \star && \text{(zombie death).}
\end{aligned}
$$

A two-population model

- For simplicity, we assume killed humans are instantly transformed into zombies
- If (s,z) denotes the population of susceptible humans and zombies, respectively, then

$$\frac{ds}{dt} = a_1 s_0 - a_2 s - a_4 s z^2$$
$$\frac{dz}{dt} = -a_3 s z + a_4 s z^2 - a_5 z$$

where s_0 is the number of people outside infested areas who can enter the region where the zombie outbreak has occurred.

a_1: human migration
a_2: human death
a_3: zombie kill rate
a_4: zombie conversion
a_5: zombie death

Dimensionless parameters

$$\frac{ds}{dt} = a_1 s_0 - a_2 s - a_4 s z^2$$
$$\frac{dz}{dt} = -a_3 s z + a_4 s z^2 - a_5 z$$

- A trick for reducing the number of parameters: make time dimensionless
- Divide the first equation by a_2
- Let $\underline{t}=a_2 t$ be dimensionless time
- Also let $s=\underline{S}S$ and $z=\underline{S}Z$, where $\underline{S}=a_1 s_0/a_2$
- Then the model becomes

$$\frac{dS}{d\underline{t}} = 1 - S - \underline{a}_4 S Z^2$$
$$\frac{dZ}{d\underline{t}} = -\underline{a}_3 S Z - \underline{a}_5 Z + \underline{a}_4 S Z^2$$

- *Homework*: Find $\underline{a}_3$, $\underline{a}_4$ and $\underline{a}_5$.

s: susceptibles z: zombies
a_1: human migration
a_2: human death
a_3: zombie kill rate
a_4: zombie conversion
a_5: zombie death

Proportional representation

$$\frac{dS}{d\underline{t}} = 1 - S - \underline{a}_4 S Z^2$$
$$\frac{dZ}{d\underline{t}} = -\underline{a}_3 S Z - \underline{a}_5 Z + \underline{a}_4 S Z^2$$

- This model only has three parameters, not five
- For notational simplicity, we'll drop the underscores
- The three rate constants (a_3,a_4,a_5) can be expressed in terms of the rate of migration of fresh human meat and steady state levels of humans in the absence of attacks
- Eg (S,Z)=(0.25,0.75) means humans are reduced to 25% of the pre-attack population
- Zombies are 75% of pre-attack humans.

Low-density attacks are survivable

$$\frac{dS}{dt} = 1 - S - a_4 S Z^2$$
$$\frac{dZ}{dt} = -a_3 S Z - a_5 Z + a_4 S Z^2$$

- This model has up to three equilibria
- Two of these can be stable
- (S,Z)=(1,0) is always an equilibrium and it is always asymptotically stable
- Thus, for low-density zombie attacks, humans always survive.

S: susceptibles Z: zombies
a_3: zombie kill rate
a_4: zombie conversion
a_5: zombie death

Three equilibria

$$\frac{dS}{dt} = 1 - S - a_4 S Z^2$$
$$\frac{dZ}{dt} = -a_3 S Z - a_5 Z + a_4 S Z^2$$

- Add the equations together:
- $1-S-a_3SZ-a_5Z=0$, so

$$S = \frac{1 - a_5 Z}{1 + a_3 Z}$$

- Substituting, we have $Z(a_3+a_5-a_4Z+a_4a_5Z^2)=0$
- Thus, Z=0 (as expected) and

$$Z = \frac{a_4 \pm \sqrt{a_4^2 - 4a_4a_5(a_3+a_5)}}{2a_4a_5}$$

- If the term inside the radical is positive, we have three positive equilibria.

S: susceptibles Z: zombies
a_3: zombie kill rate
a_4: zombie conversion
a_5: zombie death

Linearising

$$Z = \frac{a_4 \pm \sqrt{a_4^2 - 4a_4a_5(a_3 + a_5)}}{2a_4a_5}$$

- If the natural death rate of zombies (a_5) and the rate at which humans kill zombies (a_3) are small, then the zombies will be able to mount an attack that overwhelms the humans
- We still need to show the equilibria are stable
- To do this, we'll linearise about the equilibrium and examine the eigenvalues of the resulting Jacobian matrix
- If they all have negative real parts, it's stable.

Z: zombies
a_3: zombie kill rate
a_4: zombie conversion
a_5: zombie death

Jacobian

- The Jacobian matrix is

$$J = \begin{bmatrix} -1 - a_4 Z^2 & -2a_4 SZ \\ -a_3 Z + a_4 Z^2 & -a_5 - a_3 S + 2a_4 SZ \end{bmatrix}$$

- For Z=0, the trace is negative and the determinant positive
- This is equivalent to stability in a 2D matrix
- *Homework*: If a_5 is small, show that the middle root is unstable and the large root is stable.

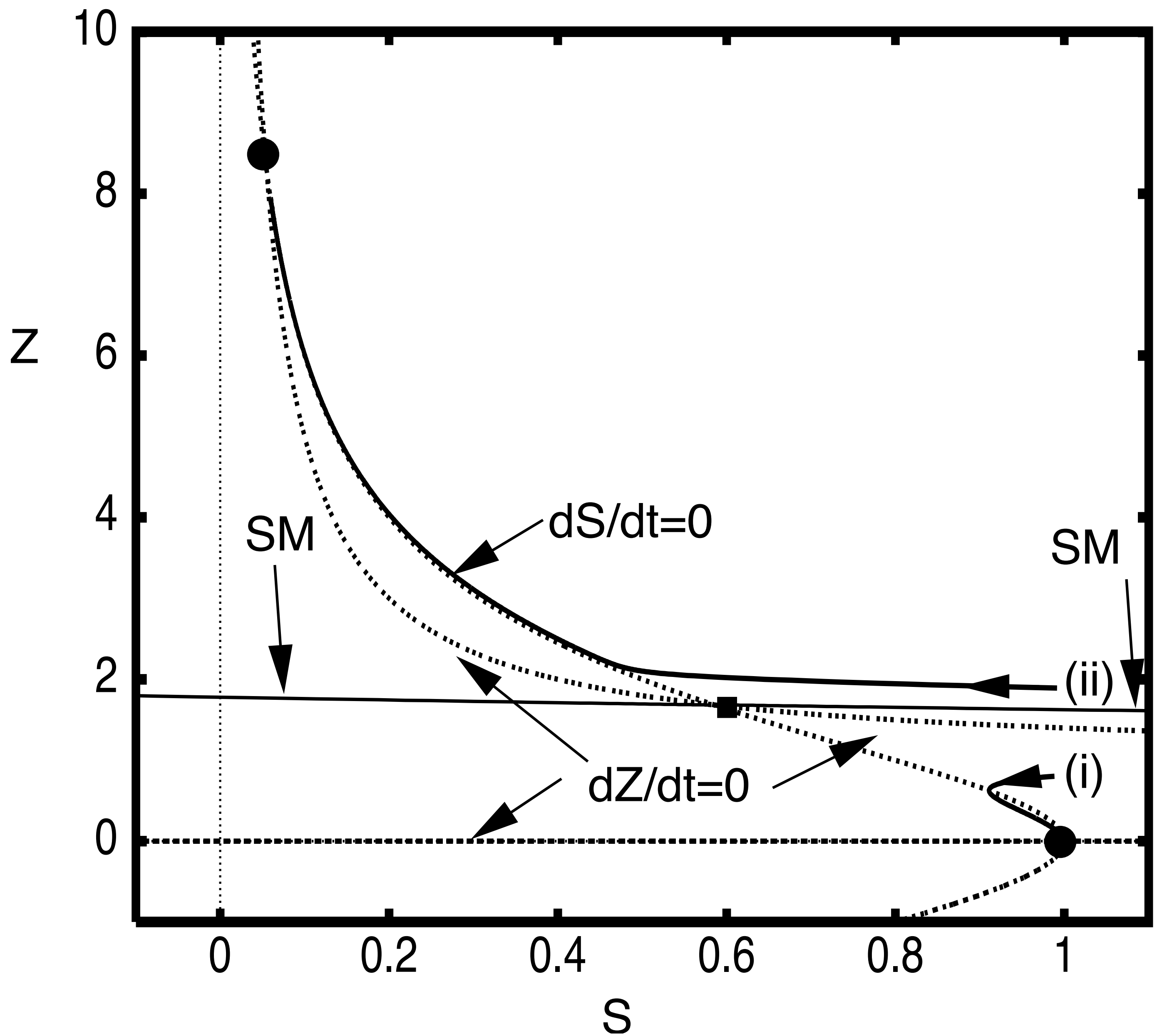
S: susceptibles Z: zombies
a_3: zombie kill rate
a_4: zombie conversion
a_5: zombie death

Bistable zombie model

- To understand the qualitative dynamics, we can sketch the phase plane
- We can also sketch the $S'=0$ and $Z'=0$ nullclines
- Equilibria are at intersections of the nullclines
- Stable equilibria are marked with circles
- Unstable equilibria with square
- Stable manifolds are labelled SM
- Two sample trajectories (i) and (ii) illustrate the bistability in the model.

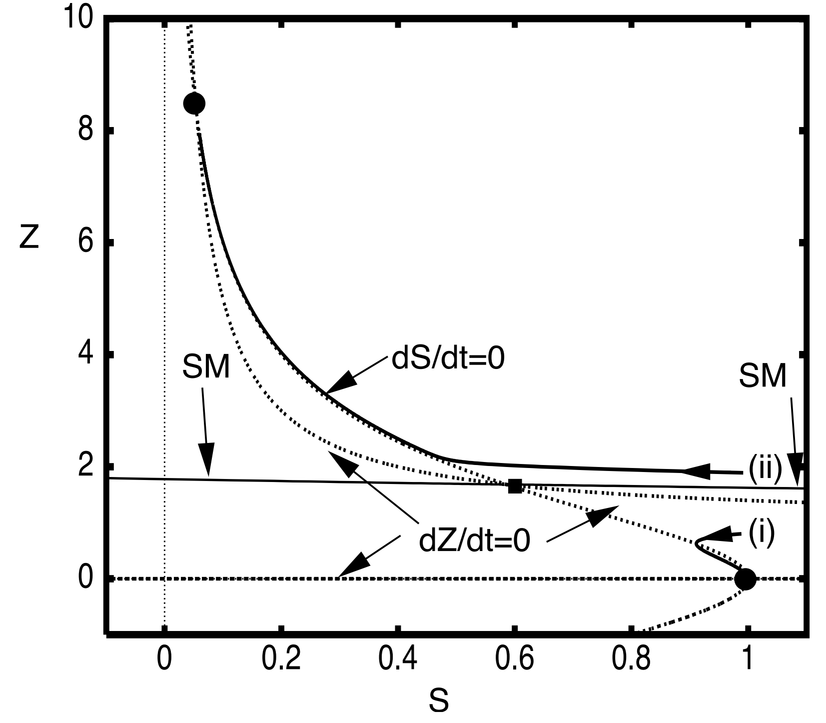
S: susceptibles
Z: zombies

Phase plane



Separatrix

- The unstable equilibrium is a saddle point
- This means there is a pair of trajectories which go to the saddle point as $t \longrightarrow \infty$
- They form a separatrix between the two stable equilibria
- All initial conditions above the curve go to a persistent high zombie state
- Those below go to a zero zombie state.



The key threshold parameter is a_3

- This is the ability of a human to beat a zombie in a one-on-one interaction
- If humans are totally unprepared, then this term could even be negative
- In this case, we would adjust the equations by adding $+\min\{a_3 SZ, 0\}$ to the S' equation
- If $a_3<0$ we subtract from the population
- If $a_4=0$ and $a_3<0$, we recover a simplified version of the original Munz model.

S: susceptibles
Z: zombies
a_3: zombie kill rate
a_4: zombie conversion

One more alteration

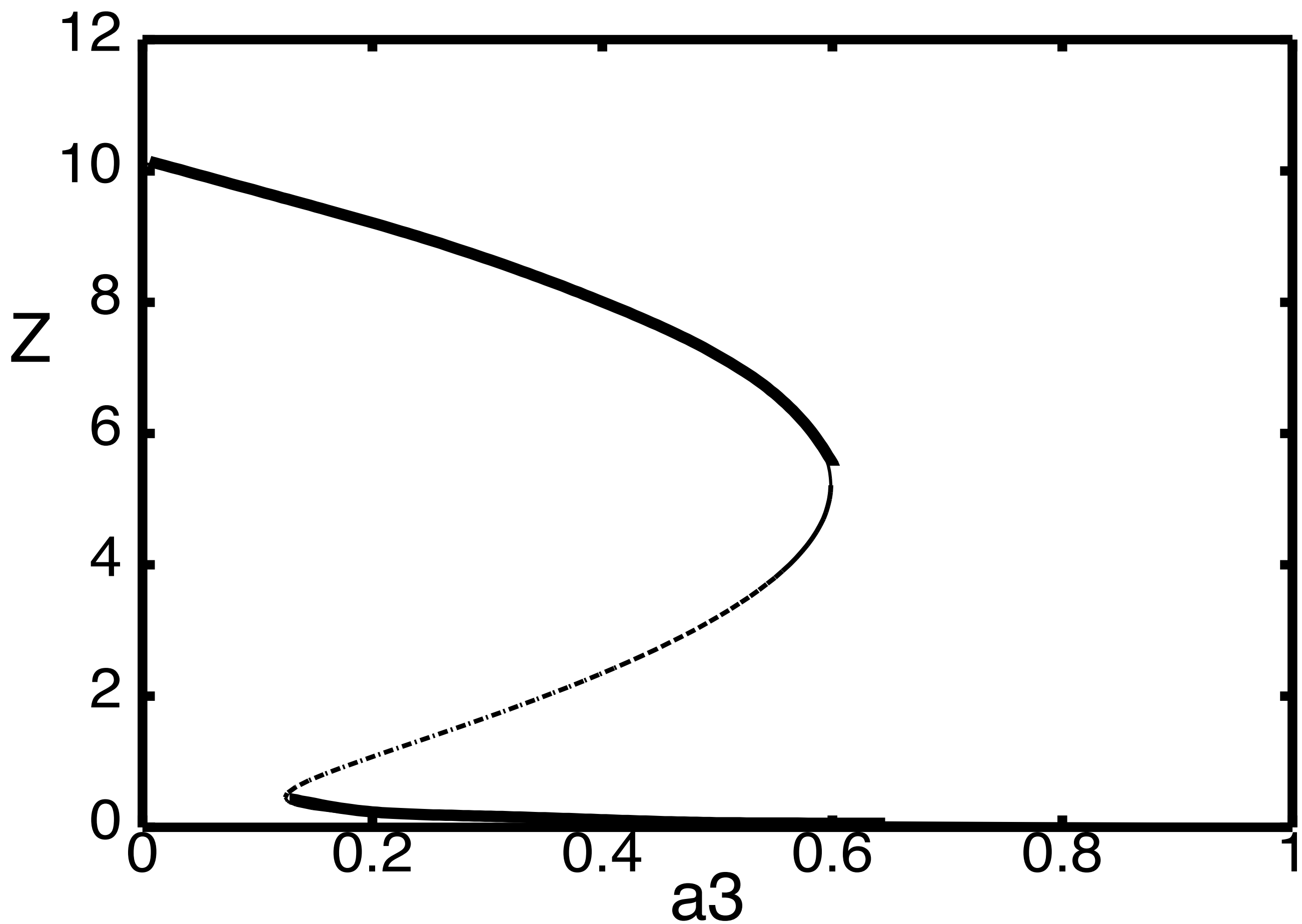
- What if zombies migrate?
- Adding a small source term to the zombie equation and including group attacks by zombies, we have

$$\frac{dS}{dt} = 1 - S - a_4 S Z^2 + \min(a_3 S Z, 0)$$
$$\frac{dZ}{dt} = -a_3 S Z - a_5 (Z - Z_0) + a_4 S Z^2$$

- Finding equilibria is now much harder
- We compute them numerically as a_3 varies
- We illustrate this in a *bifurcation diagram*.

S: susceptibles Z: zombies
a_3: zombie kill rate
a_4: zombie conversion
a_5: zombie death

Bifurcation diagram



- Solid curves are stable, dashed are unstable
- a_4=0.25, a_5=0.1.

Z: Zombies
a_3: zombie kill rate
a_4: zombie conversion
a_5: zombie death

Zombie migration

- Because of the small migration term, there is a positive number such that if a_3 is smaller than this value, the lower equilibrium does not exist and the zombies rule
- If a_3 is large, the zombies can never take hold and the population is largely zombie-free
- Thus, preparedness for zombie attacks can maintain a low or zero population of zombies
- However, if groups of them form, and there are enough, the zombies can overcome the defences and reign supreme.

a_3: zombie kill rate

Adaptive strategies for humans

- Being prepared for a zombie attack at all times is tough
- Carrying an ice pick or cricket bat whenever we go shopping is inconvenient
- Thus, if zombie attacks remain infrequent, a_3 might begin to fall, perhaps even below zero
- In this case, an isolated zombie could attack and kill and citizen
- Eg in *Night of the Living Dead*, Johnny is easily killed by a lone zombie, due to his naïve attitude towards the threat.

Changing the “readiness” parameter

- Eventually the attacks would become common as isolated attacks increase the zombie population
- This is then amplified by group attacks
- As more zombies are created, the populace will step up their readiness and thus increase a_3
- Thus, we’ll examine the effects of adapting the “readiness” parameter a_3 to the zombie population.

a_3: zombie kill rate

The “readiness” variable

- To make “readiness” a dynamic variable, we give it a differential equation:

$$\frac{da_3}{dt} = F(Z, a_3)$$

- How do we choose F?
- We want a_3 to increase if the zombie population is large and decrease if it is small
- How small or large depends on our tolerance to the presence of zombies
- Some people may want zero zombies
- However, that comes at a cost
  - – carrying ice picks about your person.

Z: zombies
a_3: zombie kill rate

A linear ODE

- A simple linear equation will suffice:

$$\tau \frac{da_3}{dt} = Z - \underline{Z} - ca_3$$

- $\underline{Z}$ sets the level of zombies we are willing to tolerate
- c is the decay of a_3 (and could be set to 0)
  - a nonzero c means there is a natural decay of readiness to a neutral value of a_3=0
- τ sets the timescale
- If humans are slow to react, τ is large
- If they react quickly, τ is small.

Z: zombies
a_3: zombie kill rate

Intersections with the bifurcation

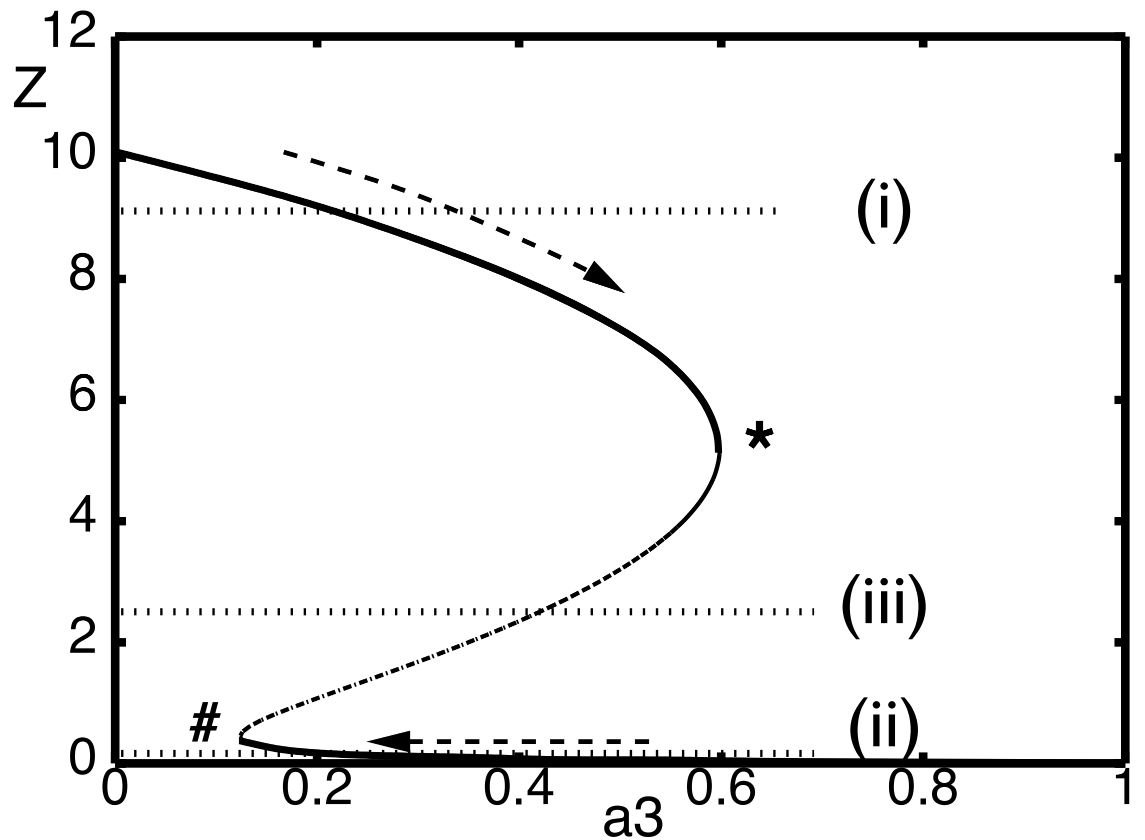
$$\tau \frac{da_3}{dt} = Z - \underline{Z} - ca_3$$

- Equilibria satisfy $Z=\underline{Z}+ca_3$
- This is straight line
- Intersections of this line with the bifurcation diagram give us equilibrium values

  (because a_3 is now a dynamic variable)
- Eg if c=0, then the equilibrium values are found by drawing horizontal lines at $Z=\underline{Z}$.

Z: zombies τ: timescale
$\underline{Z}$: zombie tolerance
a_3: zombie kill rate
c: readiness decay

Acceptable zombie tolerance



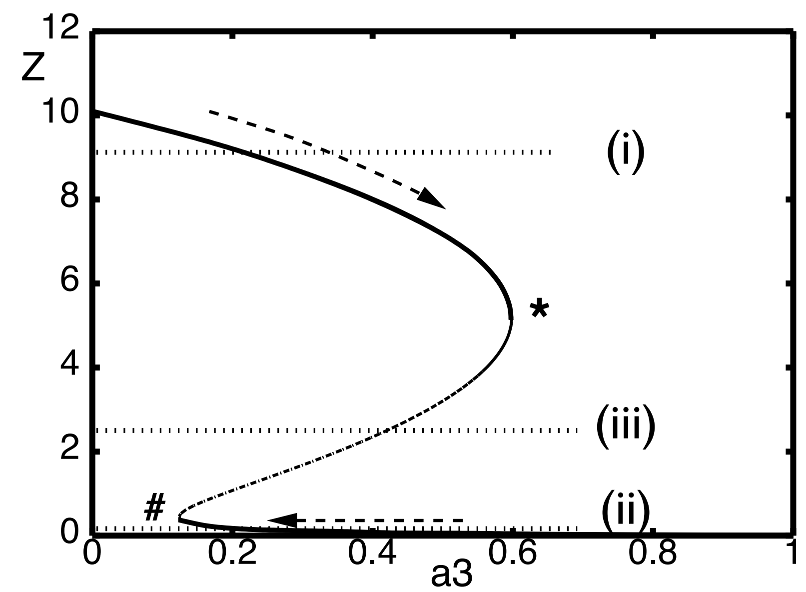
(i) high tolerance

(ii) low tolerance

(iii) intermediate tolerance.

High and low zombie tolerance

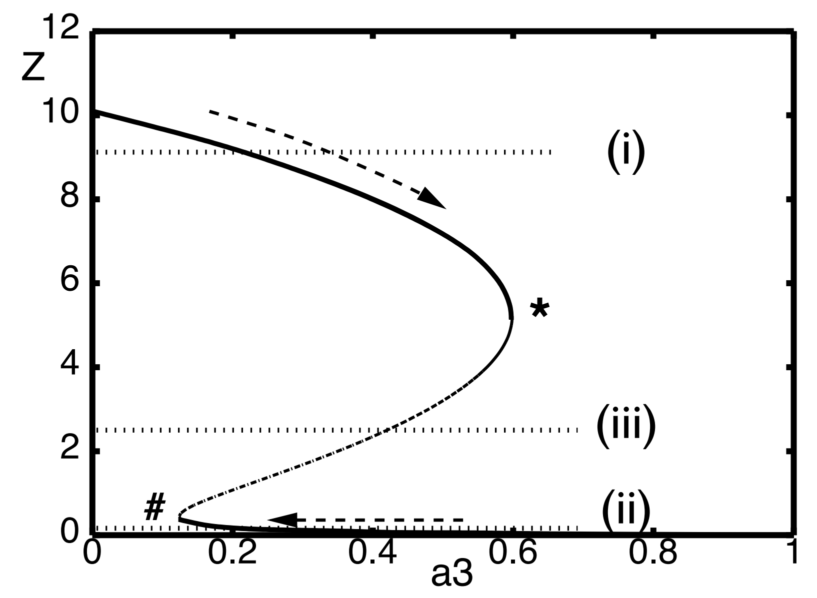
- If we have a high tolerance of zombies (i), then we can choose a_3 quite low
- Admittedly, we'll all be nearly exterminated, but at least we can leave the icepick at home
- If we have a low tolerance of zombies (ii), then we should set a_3 to be larger than about 0.6
- In this case, there will never be a dominant zombie presence.



Z: zombies
a_3: zombie kill rate

Intermediate zombie tolerance

- Suppose we hedge our bets and choose an intermediate tolerance (iii)
- Then the intersection is on the “unstable” part of the zombie equilibrium curve
- In reasonable circumstances, we expect to see periodic fluctuations in the zombie and human populations
- The readiness parameter a_3 will also fluctuate.

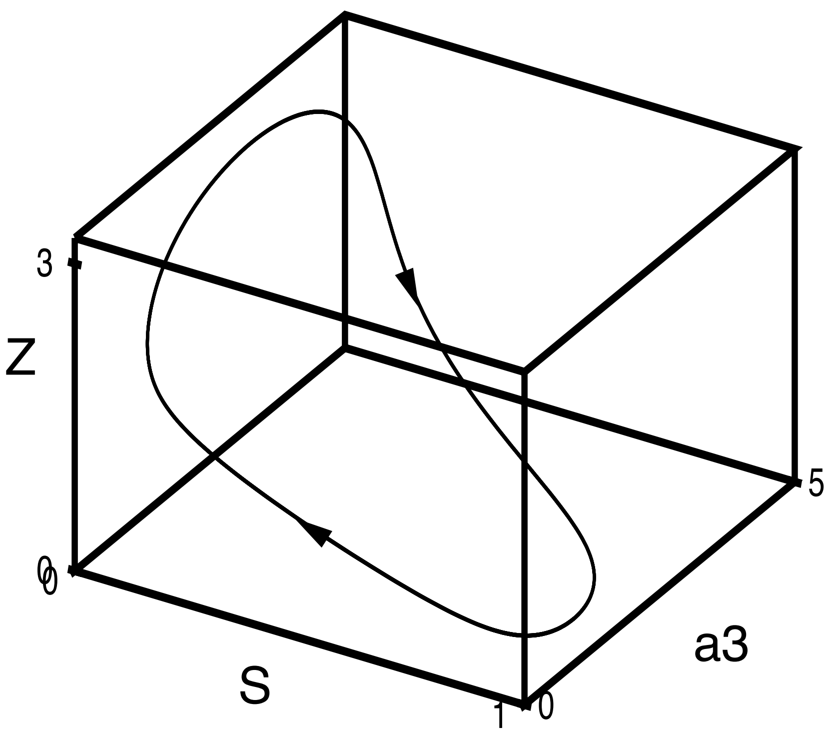


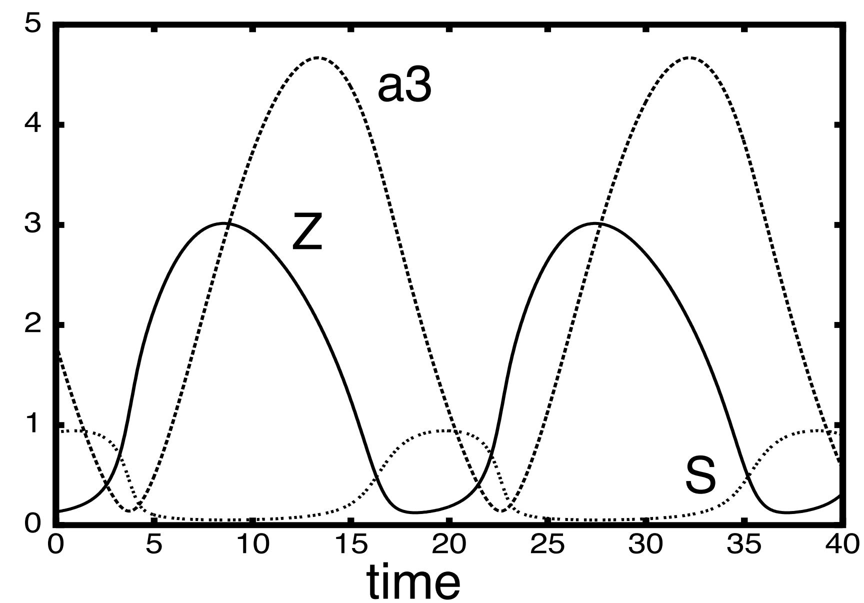
Z: zombies
a_3: zombie kill rate

The rise and fall of zombies

- With intermediate tolerance, the system oscillates
- The zombie population rises and wipes out a substantial fraction of people (a_3 is low)
- The remainder arm themselves to the teeth and cut down the zombies (a_3 is high)
- They then become complacent, allowing the zombie population to rise once more (a_3 decreases).

Oscillations in the system

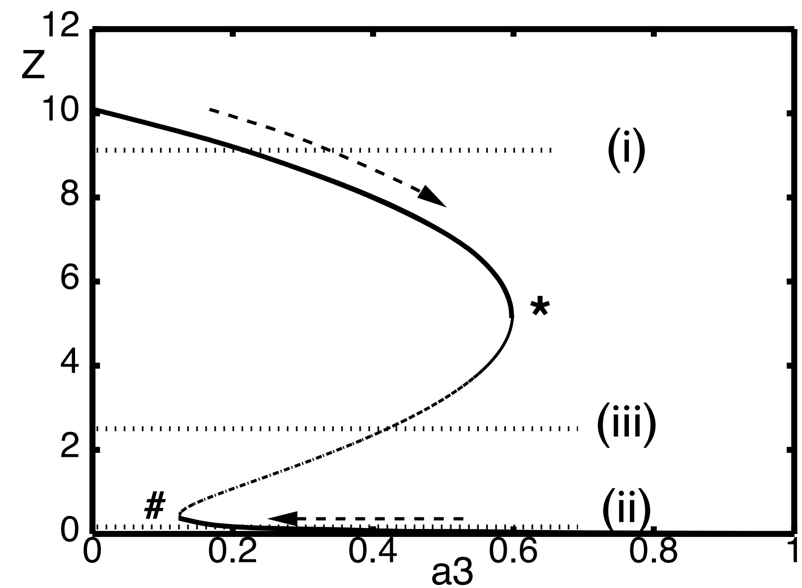




- There is a limit cycle in phase space
- The time series shows the cycling populations.

Population crash and explosion

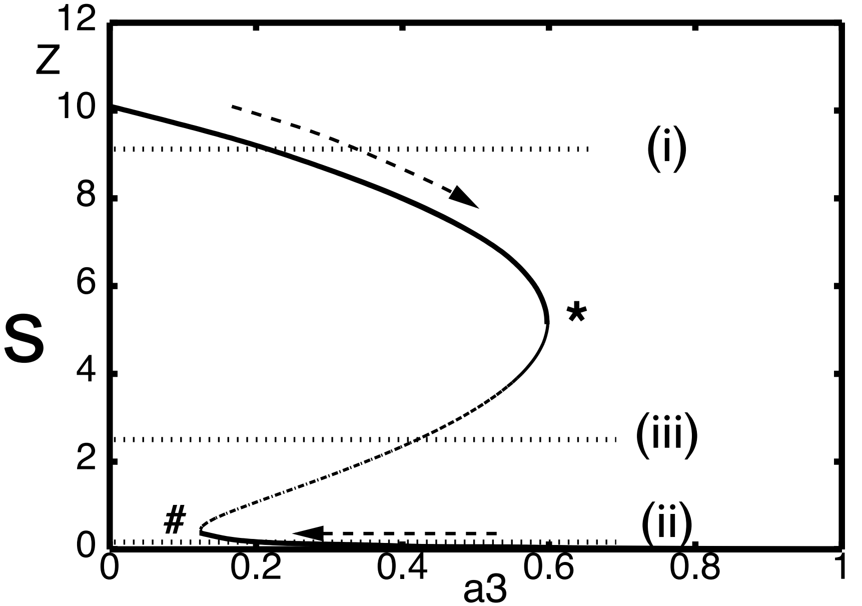
- Suppose τ is large so that people adapt slowly
- At a high zombie population, $Z>\underline{Z}$ so $a_3'(t)>0$
- a_3 increases the use of weaponry, causing the zombie population to slowly decrease until point ★ is reached
- The zombie population crashes to nearly zero
- Now $Z<\underline{Z}$, so $a_3'(t)<0$
- The zombie population rises slowly until #, when the zombie population explodes.



Z: zombies τ: timescale
$\underline{Z}$: zombie tolerance
a_3: zombie kill rate

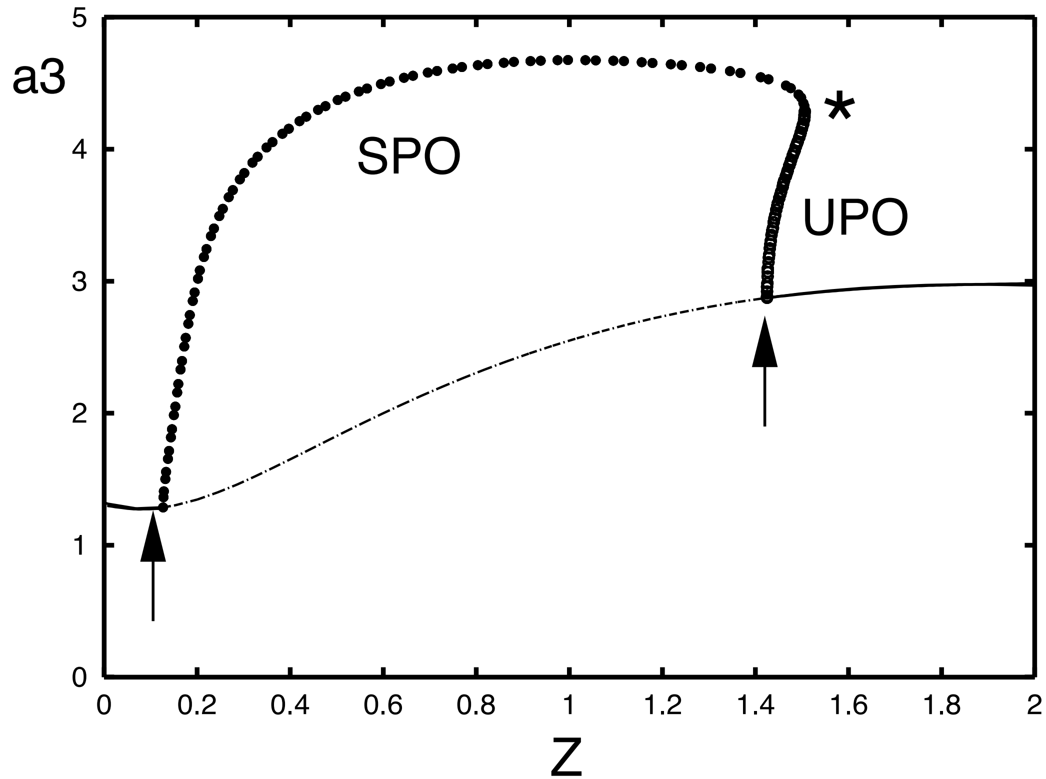
Hopf bifurcation

- Suppose we vary the tolerance (iii) numerically
- The equilibrium value of a_3 is stable for low tolerance, but quickly loses stability
- This will be in the form of a Hopf bifurcation
- The means that damped oscillations become growing oscillations
- A periodic orbit emerges as the only stable behaviour.



Z: zombies
a_3: zombie kill rate

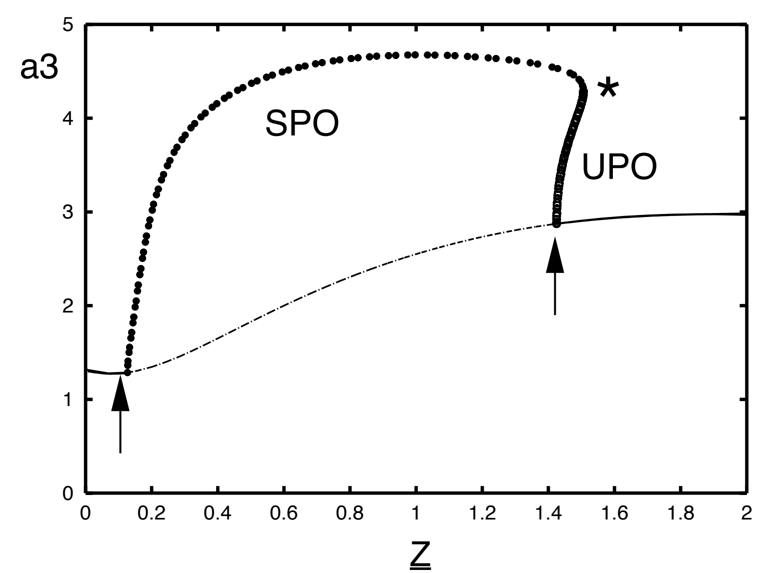
Bifurcations in the adaptive model



- SPO: Stable periodic orbit
- UPO: Unstable periodic orbit
- Arrows: Hopf bifurcations
- ★: collision of stable and unstable orbits.

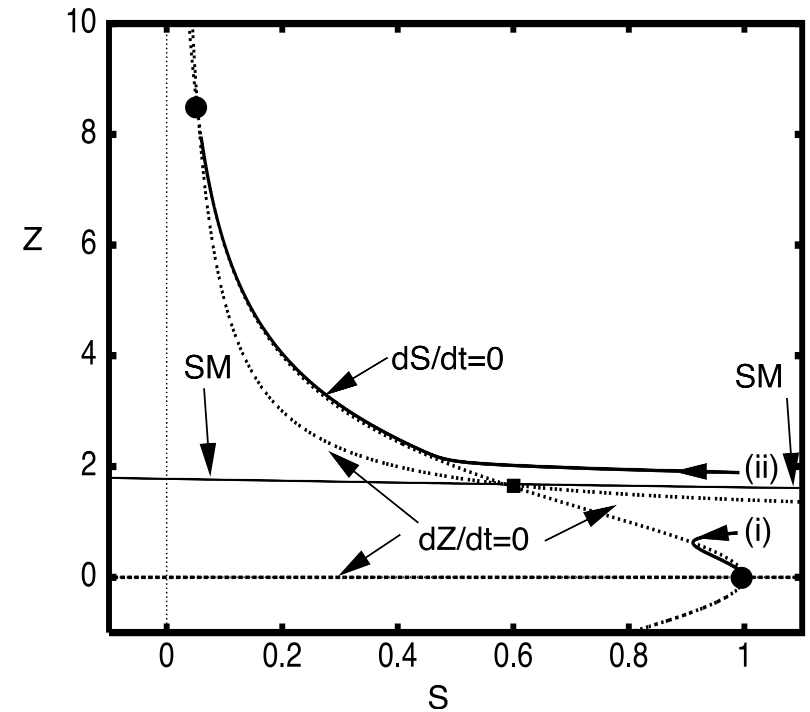
Rhythmicity

- The periodic orbit grows in amplitude until it is abruptly lost at ★
- Trajectories return to a stable equilibrium point
- Thus, if an intermediate tolerance is chosen, then the zombie attacks wax and wane in a rhythmic manner
- Note that there is a region where there is both rhythmicity and stable equilibrium behaviour.



Simplifying the model

- Three-dimensional dynamics are difficult
- Could we reduce the system to a simpler one?
- Consider the original phase-plane diagram when a_3 was fixed
- The two trajectories move horizontally until they hit the S-nullcline
- Then they follow it nearly perfectly to the equilibrium
- Thus, the dynamics of S are much faster than Z.



S: susceptibles Z: zombies
a_3: zombie kill rate

Reducing the dimension

- It makes sense that S dynamics are faster than Z, since classic zombies are slow compared to humans
- Thus, we could let S reach its equilibrium:

$$S = S_{eq}(Z) \equiv \frac{1}{1 + a_4 Z^2 - \min(a_3 Z, 0)}$$

- If we make this substitution, then the 3D model becomes a 2D model:

S: susceptibles
Z: zombies τ: timescale
$\underline{Z}$: zombie tolerance
a_3: zombie kill rate
c: readiness decay
a_4: zombie conversion

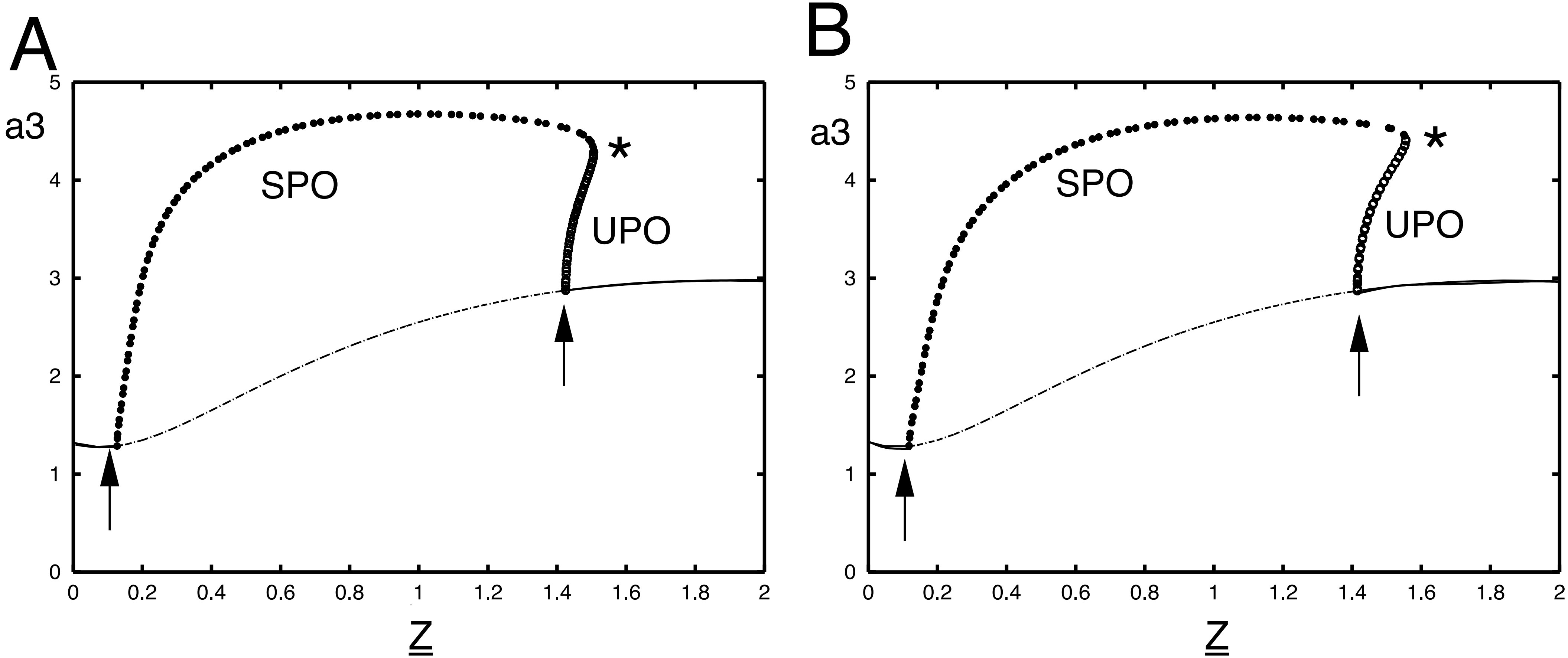
$$\frac{dZ}{dt} = -a_3 S_{eq}(Z) Z + a_4 Z^2 S_{eq}(Z) - a_5 Z$$

$$\tau \frac{da_3}{dt} = \bar{Z} - Z - c a_3.$$

Equilibrium approximation

- Little is lost by making this simplification
- The adaptive bifurcation diagram is almost unchanged
- This is a nice trick that saves a lot of trouble
- Letting fast dynamics go to their equilibria can often be enormously helpful in simplifying the analysis.

Bifurcation diagram in both models



A: 3D model

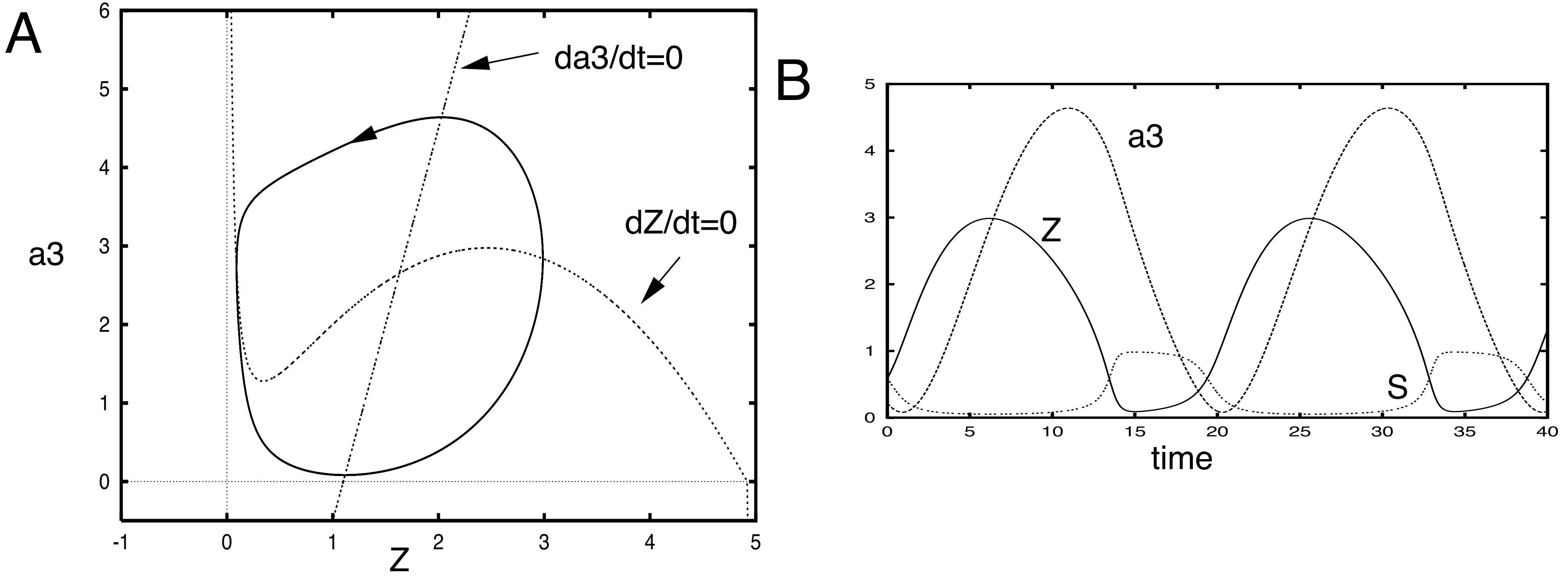
B: 2D model.

Comparison

- We can now examine the phase plane and nullclines for the same parameter values as before
- Z-nullclines and a_3-nullclines can easily be sketched
- The time series shows little difference from the 3D model.

Z: zombies
a_3: zombie kill rate

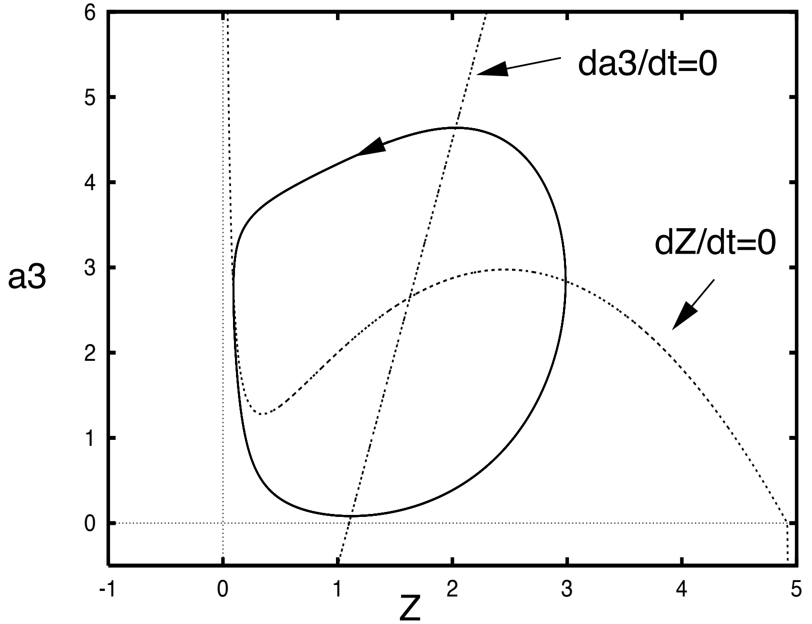
Phase plane and time series for 2D model



- A: Phase plane with nullclines
- B: Time series.

Varying the dynamics

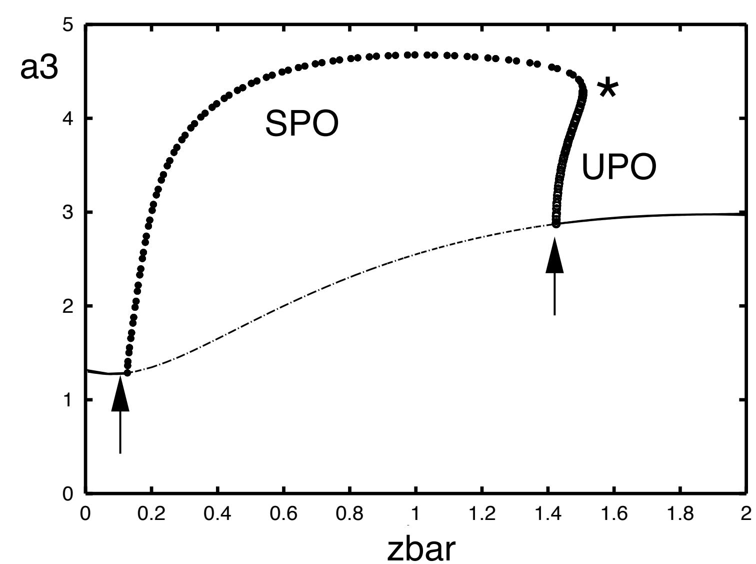
- As <u>Z</u> changes, we can get the a_3-nullcline to intersect in different parts of the Z-nullcline
- This will vary the qualitative dynamics
- Intersections away from the middle part of the Z-nullcline will lead to stable equilibria
- This will be a stable adaptation to zombie attacks.



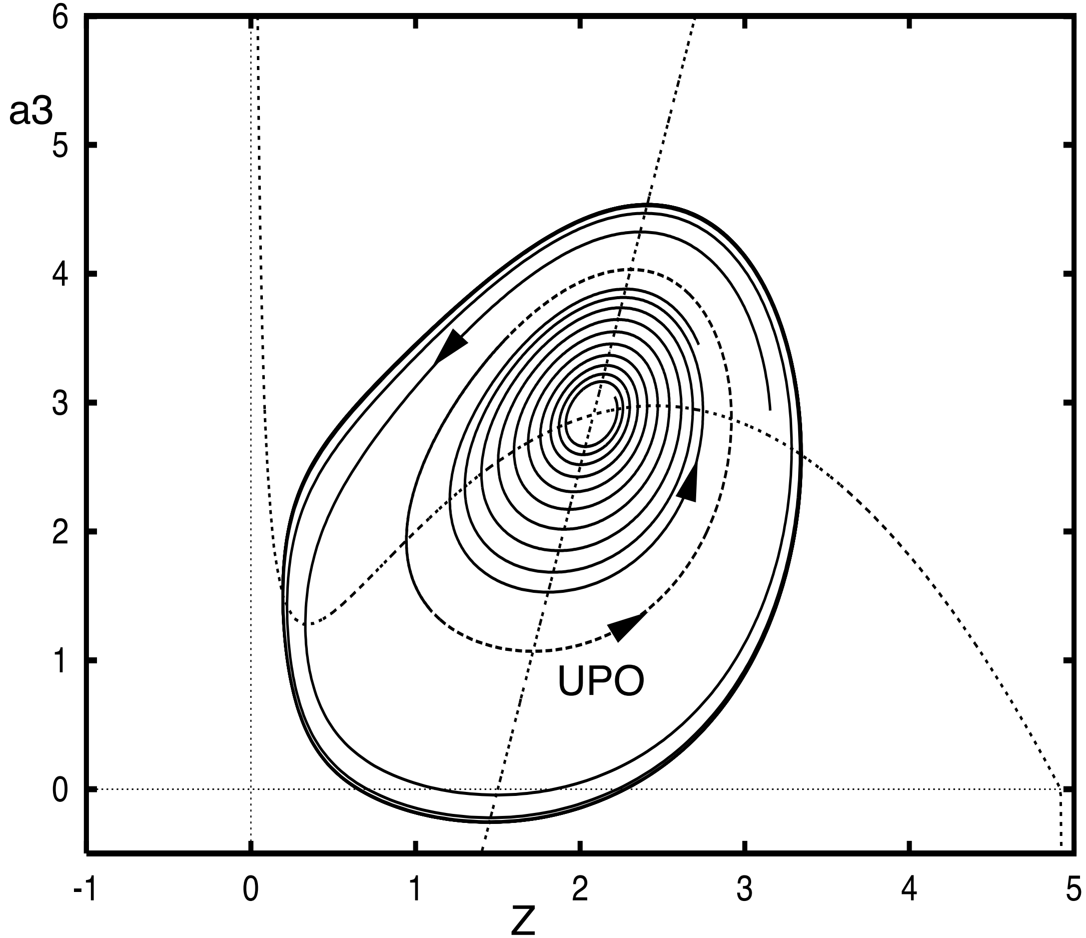
Z: zombies
<u>Z</u>: zombie tolerance
a_3: zombie kill rate

Orbits inside orbits

- Recall that there was a region where a stable equilibrium and stable limit cycle coexisted
- This is the region between the right arrow and the asterisk
- In this case, there is an unstable periodic orbit inside the stable one, surrounding a stable equilibrium.



Bistable region

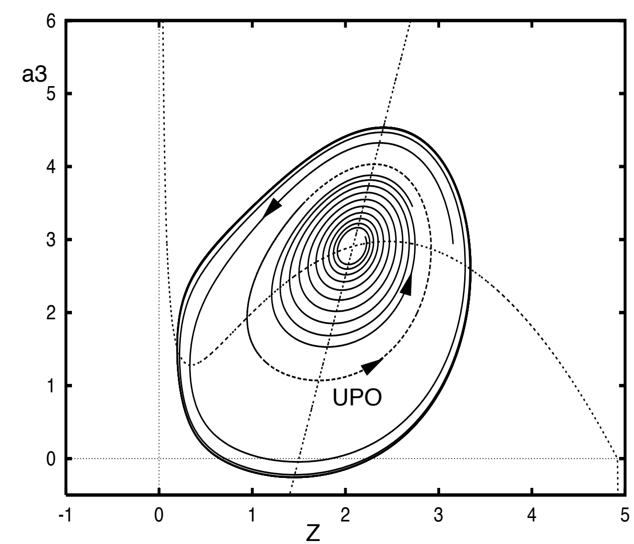


- Reduced adaptive model
- $\underline{Z}$=1.4 in the bistable region.

Z: zombies
$\underline{Z}$: zombie tolerance
a_3: zombie kill rate

Disturbing the equilibrium

- Suppose we are at the stable equilibrium
- A brief influx of new zombies could push the initial conditions to the right, past the UPO
- This would result in a massive depletion of zombies, subsequent complacency and a rebound to oscillation
- Only a carefully timed culling of the zombies would take us back to the equilibrium.



Waxing and waning

- Thus, having a zombie-dependent behaviour on readiness to confront the undead hordes can lead to instabilities
- These result in the waxing and waning of the total zombie population
- This occurs in nature
  - eg population cycles
  - lynx and hares
  - also in many diseases.

Empirical observations

- The results here were based on some empirical observations from classic films
- Specifically, that low populations of zombies are easily overcome by suitably armed humans
- These defences can be overcome when zombies attack in groups
  - we kept this to groups of two for simplicity
- It is well known that zombies like to congregate in groups.

Summary

- We considered two types of interactions:
  - zombies lose at low density
  - but win at high density
- The model system shows that there can be two qualitatively distinct outcomes for the same parameters:
  - humans dominate or zombies dominate
- This winner-take-all behaviour is common in population models with competition
- Here, the prey can become the predators if they are well-armed.

Summary II

- We also introduced adaptive strategies for humans
- ie when zombie levels were low, alertness and readiness of humans became low
- This resulted in a massive amplification of zombie attacks and a near decimation of humans...
- ...until they slowly return to their vigilant and armed condition
- Then comes the inevitable return to complacency and the cycle begins anew.

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B. Ermentrout, K. Ermentrout. When Humans Strike Back! Adaptive Strategies for Zombies Attacks. (In: R. Smith? (ed) Mathematical Modelling of Zombies, University of Ottawa Press, *in press*.)