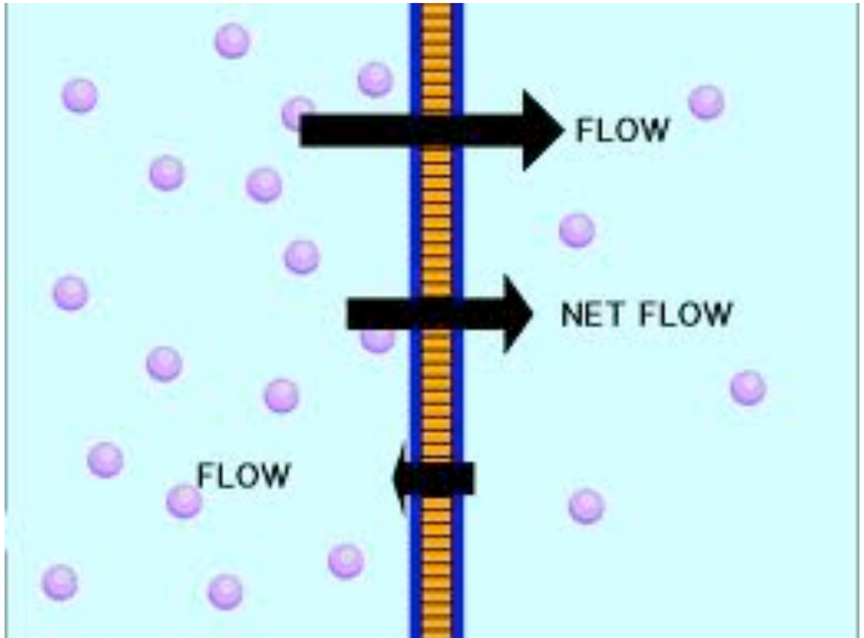
How long can we survive?

Modelling zombie movement with diffusion



The make-up of the zombie horde

- Zombies may not be well-mixed
- The initial horde will be localised to areas containing dead humans
  - hospitals
  - cemeteries
  - etc
- Zombies are also slow-moving
- As a result, humans can outrun them
- We can also build defensible blockades.

Spatial dimension

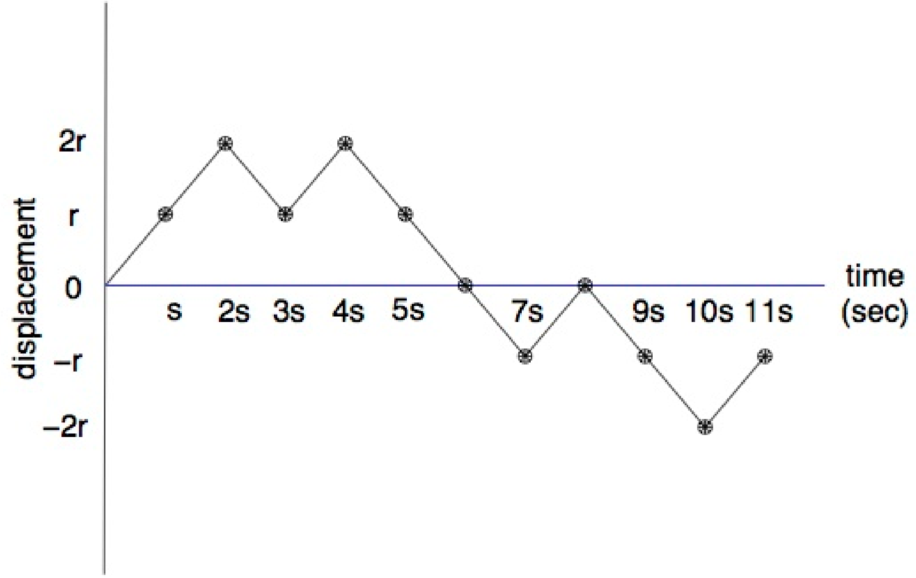
- Thus, we would like to know how long we have until zombies reach our defences
  - this will help in the gathering of supplies, weapons etc
- Another question to ask is:
- If a zombie were to infect a group of humans, could we slow down the rate of infection?
- To answer these questions, we add a spatial dimension to a simple infection model.

One-dimension

- Most results hold in one or two spatial dimensions
- However, we limit the mathematics to one spatial dimension, x
- This means there is only one possible access point to our defences
- Thus, the shortest distance between the humans and the zombies is in a straight line
- This also makes the mathematics more tractable.

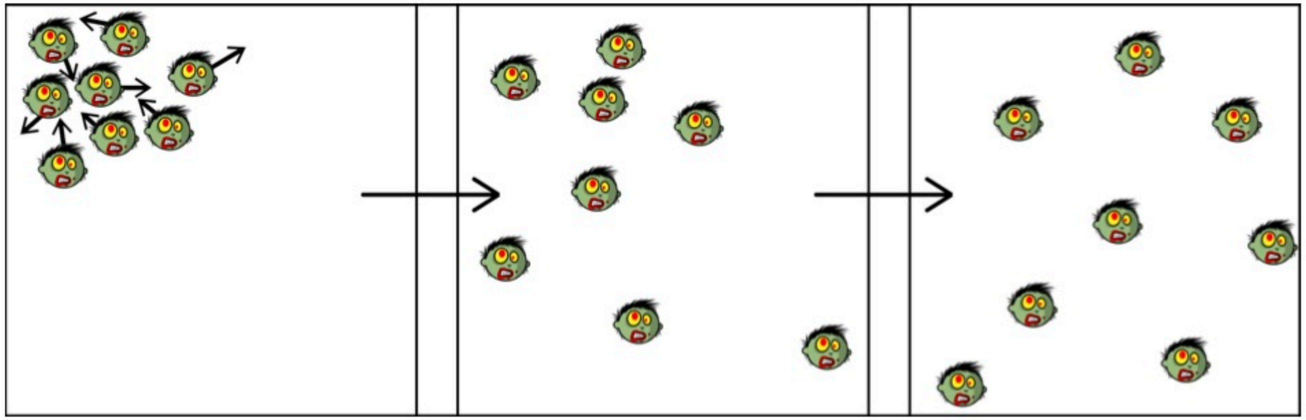
Random walks

- Zombies move in small, irregular steps
- This makes their individual movement a perfect model of a random walk
- The “drunkard’s walk”, or random walk, is a mathematical description of movement in which no direction is favoured
- This also describes the property of airborne particles or the stock market.



Diffusion

- Zombies will move around randomly, bumping into each other
- Thus, they spread over the domain
- Diffusion has two properties:
  - it is random in nature
  - movement is from regions of high density to low densities
- Thus, zombies will spread throughout the space.



Alternatives to diffusion

- We could track the motion of each zombie individually
- However, since their movement is random, unless we track them all exactly, their position will not be certain
- Probabilistic determinations can be used, but the computational power is immense if the number of zombies is initially large
- This may be a problem when running for our lives.

A large number of zombies initially

- Instead, we consider the case where there is initially a large number of zombies
- This produces a continuous model of diffusion
- This needs less computational power
- Analytical solutions are also available (in one dimension, at any rate)
- Which is nice.

Non-integer zombies

- However, what we gain in simplicity, we lose in accuracy
- Realistically, the density of zombies in a region is a discrete integer value
  - eg 5 zombies/metre
- Since we are assuming a large number of zombies, we are forced to allow the density to take any value, including non-integers.
- Thus, instead of density being a discrete-valued function, it has been “smoothed out” to form a continuous function.

Diffusion equation

- Let the density of zombies at a point x and time t be Z(x,t)
- The density has to satisfy the diffusion equation

$$\frac{\partial Z}{\partial t}(x,t) = D\frac{\partial^2 Z}{\partial x^2}(x,t)$$

- Explicitly,

$$\frac{\partial Z}{\partial t}(x,t) = \text{rate of change of } Z \text{ over time at a point } x$$

- If $\partial Z/\partial t$ is positive, then Z is increasing at that point in time and space
- This allows us to consider how Z(x,t) evolves over time.

The diffusion constant

$$\frac{\partial Z}{\partial t}(x,t) = D\frac{\partial^2 Z}{\partial x^2}(x,t)$$

- The factor D is a positive constant the controls the speed of movement
- The units of D are chosen to make the equation consistent
- $\partial Z/\partial t$ has units of [density]/[time]
- $\partial^2 Z/\partial x^2$ has units of [density]/[space]^2
- Thus, D must have units of [space]^2/[time]
- Eg kilometres^2/hour or nanometres^2/second
- However, for scales relevant to a zombie invasion, we choose metres^2/minute.

Z: density
x: space
t: time

Density at the peak

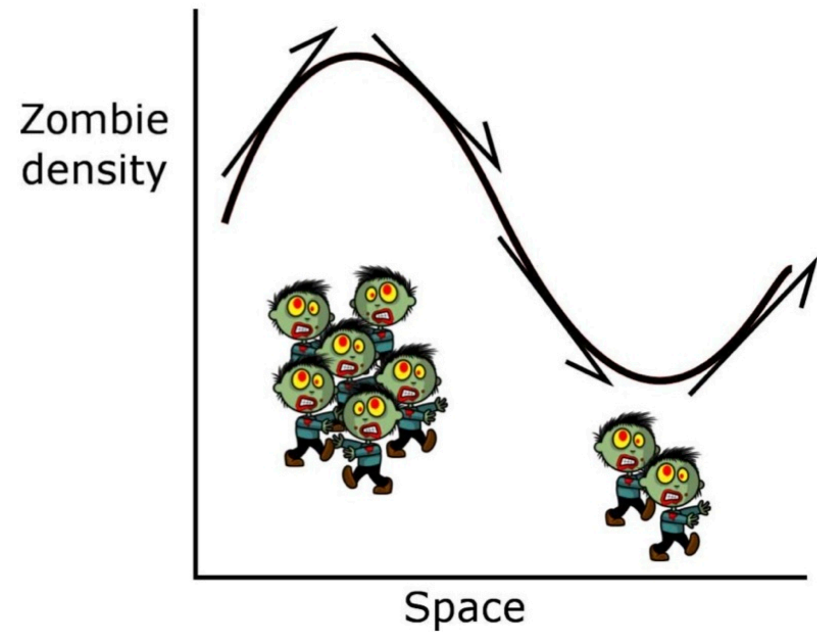
- The second-order term encapsulates the idea that zombies move from high to low densities
- Consider this curve of zombie density Z
- Initially, there are more zombies on the left
- Just before the peak,

$$\frac{\partial Z}{\partial x} = \text{rate of change of } Z \text{ as } x \text{ increases} > 0$$

- Just after the peak,

$$\frac{\partial Z}{\partial x} = \text{rate of change of } Z \text{ as } x \text{ increases} < 0$$

- Thus, at the peak, ∂Z/∂x is decreasing as x increases.



Z: density
x: space

Density at the trough

$$\frac{\partial Z}{\partial t}(x,t) = D\frac{\partial^2 Z}{\partial x^2}(x,t)$$

- From the diffusion equation, at the peak, $\partial^2 Z/\partial x^2 < 0$
- This means the peak of zombie density is decreasing over time
- Similarly, $\partial Z/\partial x < 0$ just before the trough and $\partial Z/\partial x > 0$ just after it
- Thus, at the trough, $\partial^2 Z/\partial x^2 > 0$
- Hence, the density of zombies is increasing at the trough
- Overall, zombies move to from regions of high density to regions of low density.

Z: density
x: space
t: time

Examples of diffusion

- The diffusion equation applies whenever the movement of a modelled species can be considered random and directionless
- Examples include
  - heat conduction through solids
  - gases (eg smells) spreading throughout a room
    - “Who diffused?”
  - proteins moving around the body
  - molecule transportation in chemical reactions
  - rainwater seeping through soil
  - predator-prey interactions.

Solving the diffusion equation

- We have a initial density of Z_0 zombies/metre
- All zombies start in the region $0 \leq x \leq 1$
  - this implies that the undead will originate in one place, such as a mortuary in the hospital
- Zombies cannot move out of the region $0 \leq x \leq L$
- We can assume $L>1$ (we aren't in the morgue)
- This creates a theoretical boundary at $x=0$ that the zombies cannot cross
  - they bounce off $x=0$ and back into the domain
- We place our defences at $x=L$, creating another boundary there.

Z: density
x: space

Zero flux

- Thus, at $x=0$ and $x=L$, the flux of zombies is zero
- Various other boundary conditions are possible, but these are the simplest
- The flux at $x=0$ and $x=L$ is the rate of change of zombies through these boundaries
- Mathematically, this is the spatial derivative at these points
- Thus $\partial Z/\partial x=0$ at $x=0$ and $x=L$.

Z: density
x: space
L: length

Solution of the system

- The system is then fully described as

$$\frac{\partial Z}{\partial t}(x,t) = D\frac{\partial^2 Z}{\partial x^2}(x,t) \qquad \text{(the partial differential equation)}$$

$$Z(x,0) = \begin{cases} Z_0 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases} \qquad \text{(the initial condition)}$$

$$\frac{\partial Z}{\partial x}(0,t) = 0 = \frac{\partial Z}{\partial x}(L,t) \qquad \text{(the zero-flux boundary conditions)}$$

- This is a PDE, so it needs both an initial condition and a boundary condition
- We'll solve this using separation of variables.

Z: density x: space
t: time L: length

Separation of variables

- Since our system depends on both x and t, we assume the solution is of the form $Z(x,t)=f(x)g(t)$
- This will produce a solution, but note that not every function can be decomposed in this way
- Eg if $Z(x,t)=e^{xt}$ is a solution, we'd never know it
- We want to derive constraints on the forms of f and g.

Z: density
x: space
t: time

Two independent equations

$$\frac{\partial Z}{\partial t}(x,t) = D\frac{\partial^2 Z}{\partial x^2}(x,t)$$

- After substituting into the differential equation and rearranging,

$$\frac{1}{g}\frac{dg}{dt} = \frac{D}{f}\frac{d^2 f}{dx^2}$$

- The left-hand side does not depend on x
- The right-hand side does not depend on t
- Thus, both sides must equal a constant
- Call this constant -α
- The negative is arbitrary, but will be useful later on.

Z: density D: diffusion f: space function g: time function x: space t: time

Determining the sign of α

$$\frac{1}{g}\frac{dg}{dt} = \frac{D}{f}\frac{d^2 f}{dx^2}$$

- Our rearranged equation becomes

$$\frac{dg}{dt} = -\alpha g$$

$$\frac{d^2 f}{dx^2} = -\frac{\alpha}{D} f$$

- These are standard differential equations with fairly straightforward solutions
- The first equation has solution $g(t)=C_1e^{-\alpha t}$
- This implies the sign of α
- Since zombies are spreading out, the solution should not increase
- Thus α>0.

D: diffusion α:constant
f: space function
g: time function
x: space t: time

Oscillating spatial solution

$$\frac{d^2 f}{dx^2} = -\frac{\alpha}{D} f$$

- Knowing the sign of α, the second equation has the general solution

$$f(x) = C_2 \cos\left(\sqrt{\frac{\alpha}{D}} x\right) + C_3 \sin\left(\sqrt{\frac{\alpha}{D}} x\right)$$

where C_1 and C_2 are constants

- To find C_1 and C_2, we use the boundary conditions
- Since Z(x,t)=f(x)g(t), the boundary conditions simplify to

$$\frac{df}{dx}(0) = 0 = \frac{df}{dx}(L).$$

D: diffusion α:constant
f: space function
x: space L: length

Applying boundary conditions

- We have

$$\frac{df}{dx} = -C_2 \sin\left(\sqrt{\frac{\alpha}{D}}x\right) + C_3 \cos\left(\sqrt{\frac{\alpha}{D}}x\right)$$

  so

$$\frac{df}{dx}(0) = C_3 = 0$$

$$\frac{df}{dx}(L) = -C_2 \sin\left(\sqrt{\frac{\alpha}{D}}L\right) + C_3 \cos\left(\sqrt{\frac{\alpha}{D}}L\right) = 0$$

$$\Longrightarrow \frac{\partial f}{\partial x}(L) = -C_2 \sin\left(\sqrt{\frac{\alpha}{D}}L\right) = 0$$

- We can't have $C_2=0$ or else $Z(x,t)\equiv 0$
- This is the trivial solution
- It also fails the initial condition.

Z: zombie density
D: diffusion α:constant
f: space function
x: space L: length

A linear combination of solutions

$$\sin\left(\sqrt{\frac{\alpha}{D}}L\right) = 0$$

- Thus, the only nontrivial solutions satisfy

$$\sqrt{\frac{\alpha_n}{D}}L = n\pi \implies \alpha_n = D\left(\frac{n\pi}{L}\right)^2$$

- Defining $C_n = C_1 C_2$ gives

$$Z(x,t) = C_n \cos\left(\frac{n\pi}{L}x\right)\exp\left(-D\left(\frac{n\pi}{L}\right)^2 t\right)$$

- Since this is a solution for all values of n=0,1,2,3,..., we can take a linear combination of them and still have a solution
- Thus

$$Z(x,t) = \sum_{n=0}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right)\exp\left(-D\left(\frac{n\pi}{L}\right)^2 t\right).$$

Z: zombie density
D: diffusion α:constant
x: space t: time L: length

Fourier series

- Note that

$$\int_0^L \cos\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & \text{if } m = 1, 2, 3, \ldots \\ L & \text{if } m = 0 \end{cases}$$

$$\int_0^L \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n \end{cases}$$

- From this, we can deduce that

$$Z(x,t) = \frac{Z_0}{L} + \sum_{n=1}^{\infty} \frac{2Z_0}{n\pi} \sin\left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi}{L}x\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 Dt\right)$$

- *Homework*: show this.

Z: zombie density
D: diffusion constant
x: space t: time L: length

Even spread

$$Z(x,t) = \frac{Z_0}{L} + \sum_{n=1}^{\infty} \frac{2Z_0}{n\pi} \sin\left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi}{L}x\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 Dt\right)$$

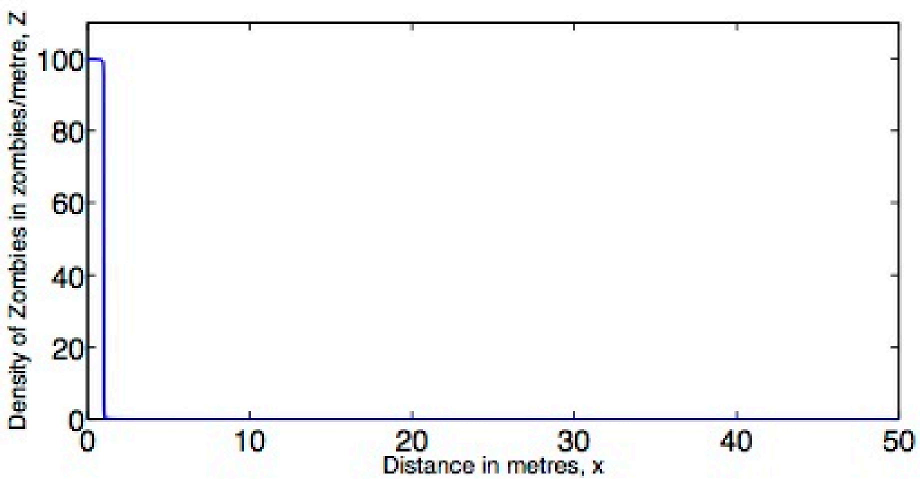
- Because of the exponential term, the solution approaches

$$Z(x,t) \approx \frac{Z_0}{L}$$

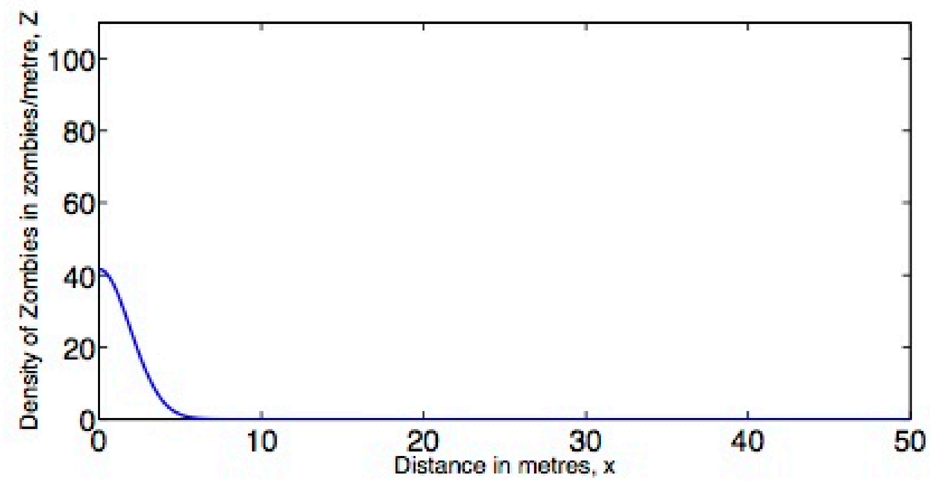
  for large values of t
- Thus, as time increases, zombies spread out evenly across the available space
- The average density is Z_0/L everywhere
- In our example, $Z_0=100$ and $L=50$, so we expect 2 zombies per metre
- We also set $D=100$ metres^2/minute.

Z: zombie density
D: diffusion L: length
x: space t: time

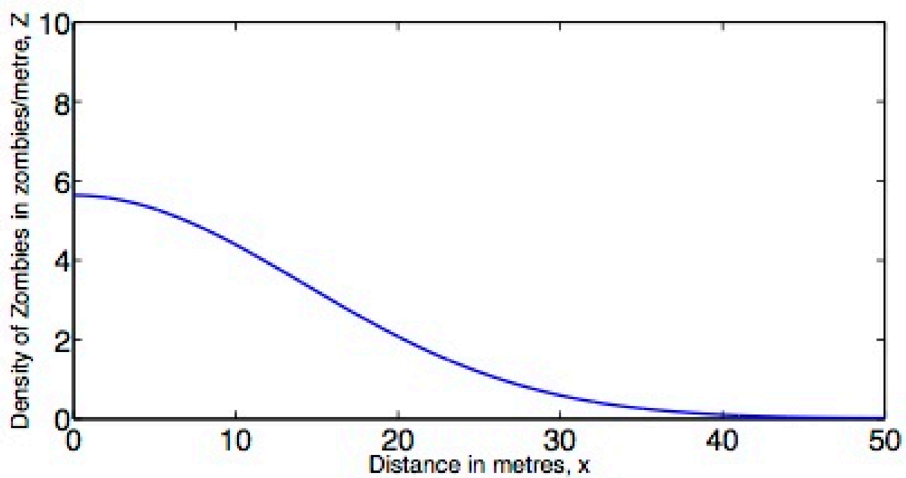
Evolution of the system



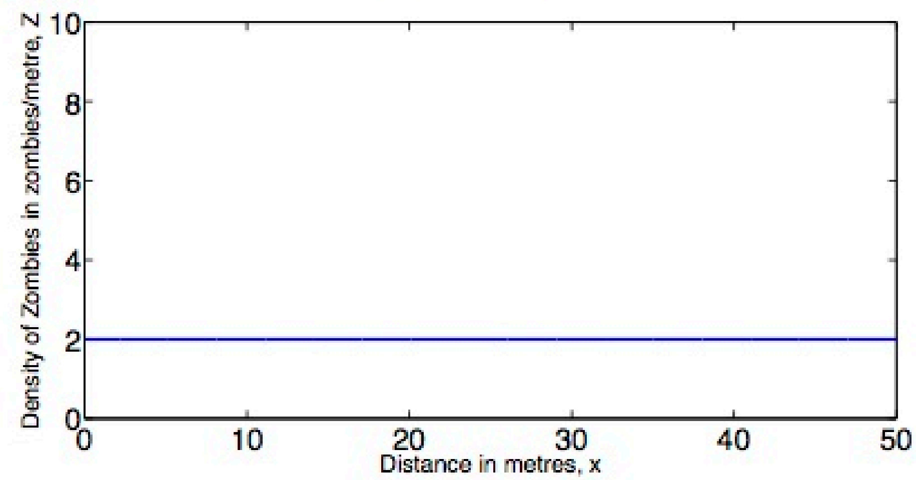
(a) t=0 minutes



(b) t=1 second



(c) t=1 minute



(d) t=100 minutes

Note the change of scale in the lower two figures.

What do we do with the solution?

- Now that we have the density of zombies for all time t and at all places x, what do we do with it?
- One question to address is: How long do we have before the first zombie arrives?
- That is: At what time t_Z does $Z(L,t_Z)=1$?
- The time t_Z is the average time taken for a zombie to reach $x=L$.

Z: zombie density
x: space t: time
L: length

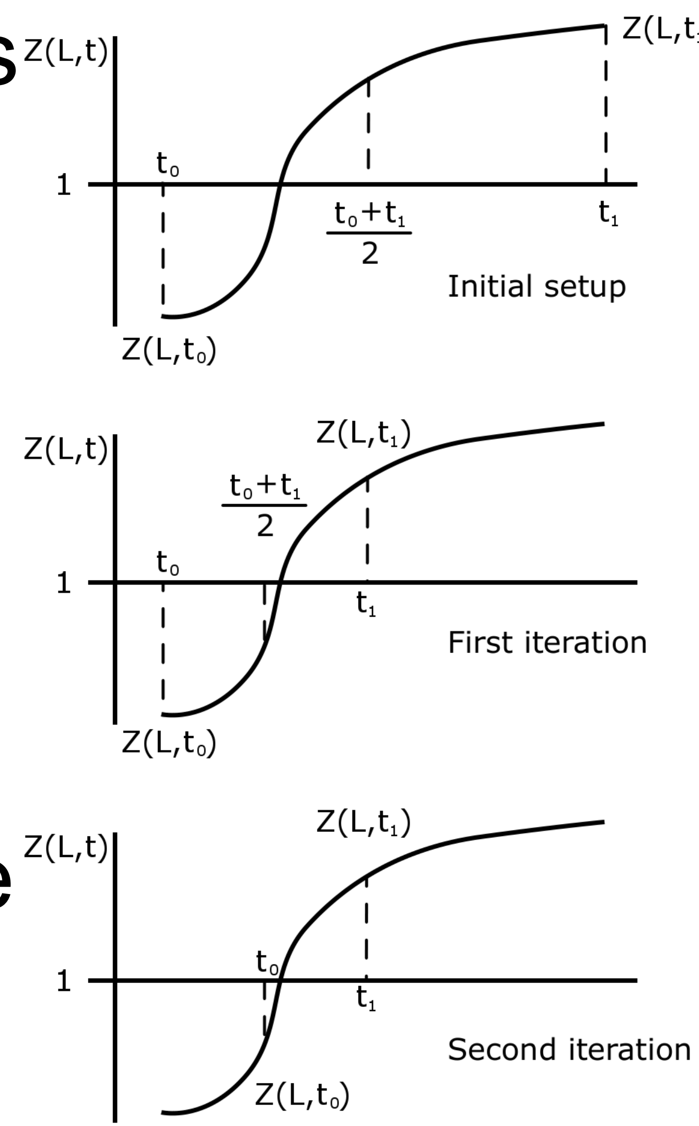
Bisection search technique (outline)

- The bisection search technique allows us to find the solution, approximated to any degree of accuracy we desire
- We could start by substituting a value of t
- If $Z(L,t)<1$ then we double t and consider $Z(L,2t)$
- If $t_1<t_2$, then $Z(L,t_1)<Z(L,t_2)$ and so $Z(L,t)<Z(L,2t)$
- Then we keep doubling t until $Z(L,2^n t)>1$
- If $t_0=2^{n-1}t$ and $t_1=2^n t$, then $Z(x,t)=1$ for some $t\in[t_0,t_1]$.

Z: zombie density
x: space t: time
L: length

More formal version

- This figure illustrates the process
- In the initial setup, t_Z is in the left half, so $t_Z \in [t_0,(t_0+t_1)/2]$
- We redefine $t_1 \equiv (t_0+t_1)/2$
- Repeating the process finds the root in the right half, so we redefine $t_0 \equiv (t_0+t_1)/2$
- After each iteration, we halve the size of the interval
- By design, $t_Z \in [t_0,t_1]$ so we can estimate t_Z to any accuracy.



Z: zombie density L: length
x: space t: time t_z: zombie arrival

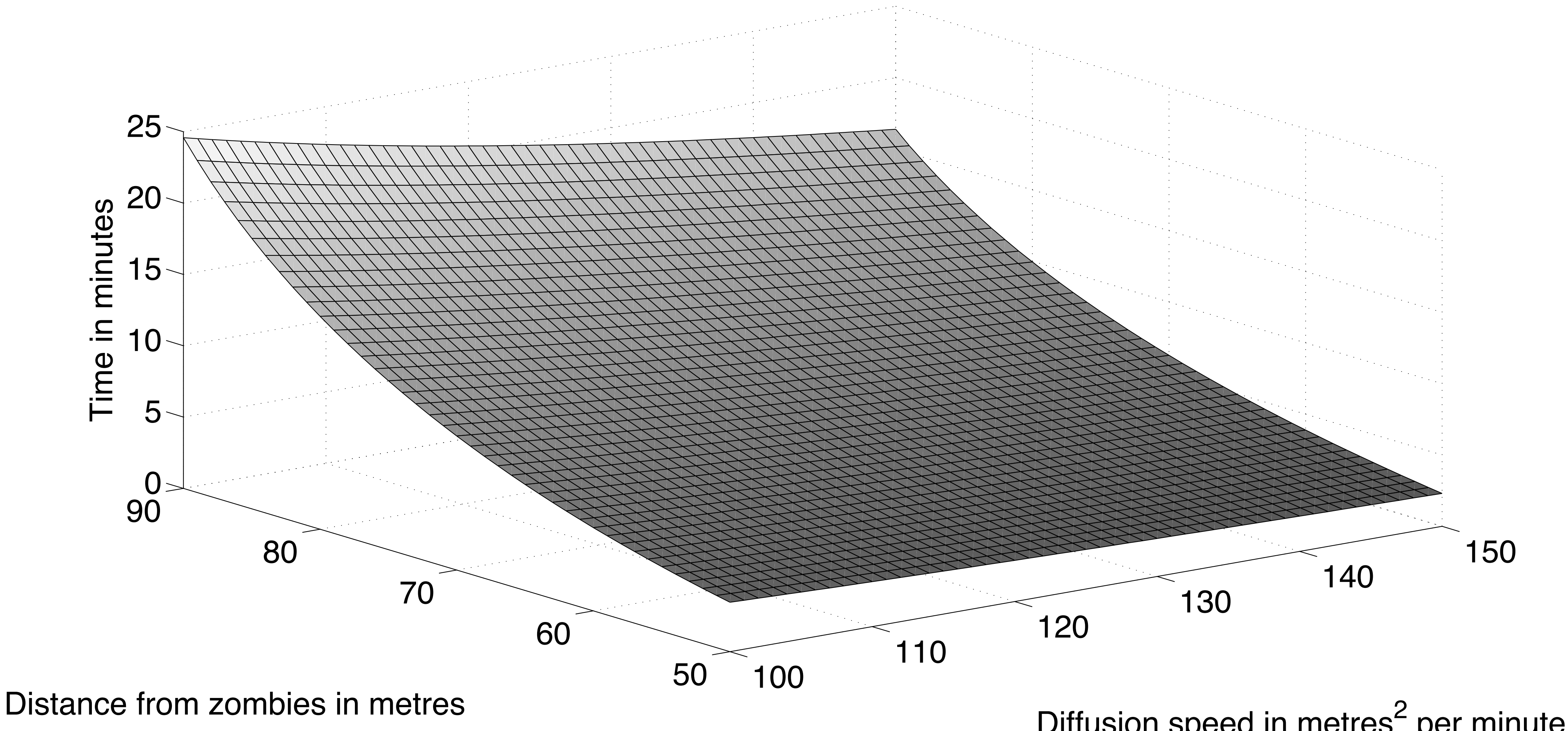
Pros and cons of bisection

- The benefit of this method is in its simplicity and reliability; it will always work
- However, the cost of reliability comes at the price of speed
- If the initial searching method is big, it may take a large number of repeats before the process produces an answer to an accuracy we are happy with
- There are quicker methods, but they are more complex and do not always work.

Interaction time versus diffusion

- Using the bisection technique, we can now vary the distance and speed of zombies to calculate the various interaction times
- In the next figure, we plot the interaction time of an initial density of 100 zombies against a diffusion rate ranging from 100-150m^2/min
- This is realistic for a slow shuffling motion and for distances between 50 and 90 metres.

Time until the first zombie arrives



One caveat

- For distances greater than $L=100$m, the average density of zombies is less than 1 zombie/metre
- Thus, there will be no solution to $Z(L,t_z)=1$, since $Z(L,t_z)<1$ for all $t\geq 0$
- In this situation, the assumption that we have a large number of zombies has failed
- This means the continuous model would be an inadequate description.

Z: zombie density L: length
t: time t_z: zombie arrival time

Diffusive time scale

$$Z(x,t) = \frac{Z_0}{L} + \sum_{n=1}^{\infty} \frac{2Z_0}{n\pi} \sin\left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi}{L}x\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 Dt\right)$$

- Consider the time

$$t = \frac{1}{D}\left(\frac{L}{\pi}\right)^2 \approx 0.32\frac{L^2}{D}$$

- This is the time it takes for the first term of the infinite sum to fall to e^{-1} of its original value
- The factor e^{-1} is used due to its convenience
- The exponential function is monotonically increasing, so the first term gives an approximation to the total solution.

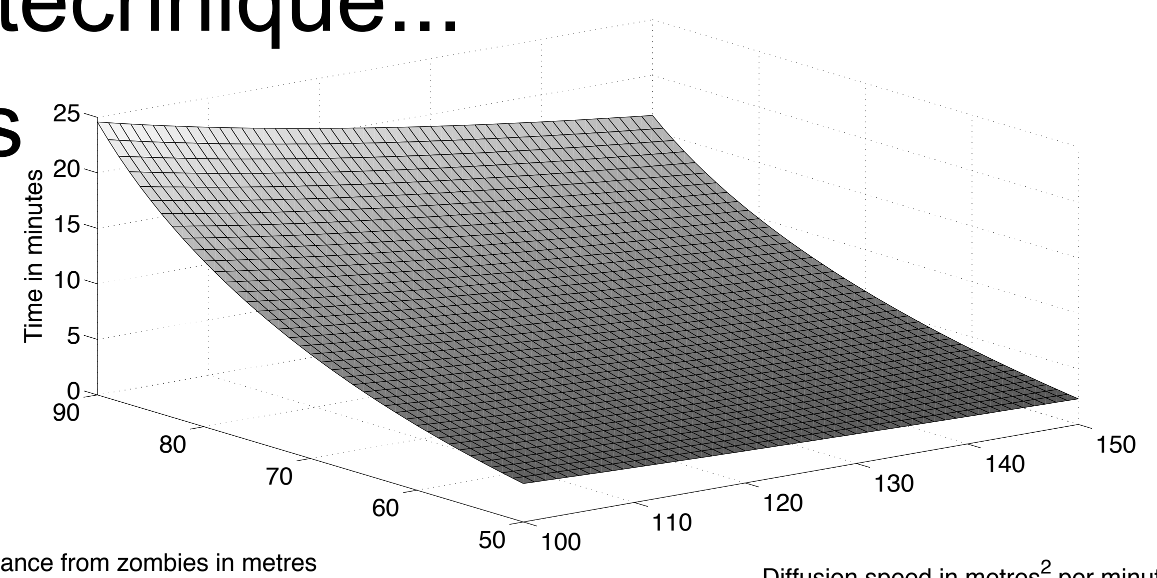
Z: zombie density
D: diffusion α:constant
x: space t: time L: length

We have less than half an hour

$$t = \frac{1}{D}\left(\frac{L}{\pi}\right)^2 \approx 0.32\frac{L^2}{D}$$

- This gives us a rough estimate of how quickly the zombies will reach us
- Eg being 90 metres away and with zombies diffusing at 100m^2/min, t≈26 minutes
- This is roughly comparable with the estimate from the bisection technique...

  ...but with a lot less effort.



D: diffusion constant
t: time L: length

How to buy more time?

- There are two ways to increase the time:
  1. We could run further away
     - since the time taken is proportional to the length squared, L^2
  2. We could try and slow the zombies down
     - since the time taken is inversely proportional to the diffusion speed, D
- If we doubled the distance between us and them, the time taken would quadruple
- However, if we slowed them down by half, the time would only double.

D: diffusion constant
L: length

Fight or flight?

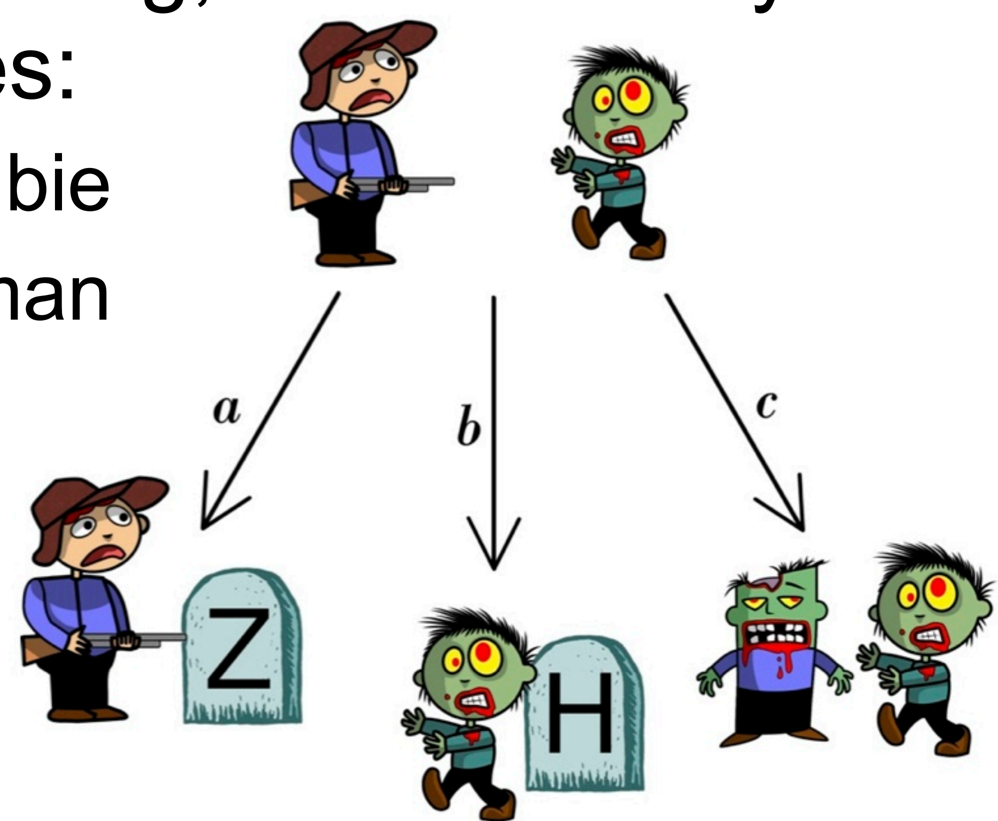
- To delay the zombies, it is better to expend energy travelling away from them rather than trying to slow them down
- The latter involves projectile weaponry, but is difficult to achieve anyway
- Also note that the time derived here is a lower bound
- The zombies would be spreading out in two dimensions and would be distracted by obstacles and victims along the way
- A conservative estimate is important, though.

Slowing the infection

- A victim, once bitten, who survives the attack will eventually turn into a zombie themselves
- This idea is not restricted to fiction
- *Cordyceps unilateralis* is a fungus that infects ants and alters their behaviour
- The infected ants are then recruited in the effort to distribute fungus spores as widely as possible to other ants
- These are found in the Brazilian rainforest.

Interaction kinetics

- We now model the interaction between populations, not just diffusing zombies
- Initially, we'll consider a non-spatial model
- In a human/zombie meeting, there are only three possible outcomes:
  - the human kills the zombie
  - the zombie kills the human
  - the zombie infects the human and the human becomes a zombie.



The three outcomes

- We can write these rules as though they were chemical reactions:

$$
\begin{aligned}
H + Z &\xrightarrow{a} H && \text{(humans kill zombies)}\\
H + Z &\xrightarrow{b} Z && \text{(zombies kill humans)}\\
H + Z &\xrightarrow{c} Z + Z && \text{(humans become zombies).}
\end{aligned}
$$

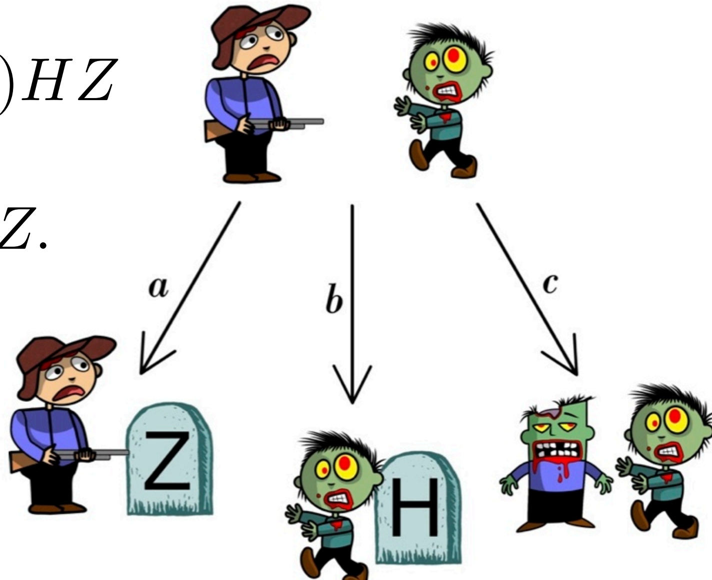
- a,b and c are the (positive) rates at which the transformation happens
- To transform these reactions into equations, we use the law of mass action:
  - the rate of reaction is proportional to the product of the active populations.

H: human density
Z: zombie density

Population dynamics

- Simply put, the law of mass action says that the reactions are more likely to occur if we increase the number of humans and/or zombies
- The population dynamics are thus governed by the system

$$\frac{dH}{dt} = -bHZ - cHZ = -(b+c)HZ$$
$$\frac{dZ}{dt} = cHZ - aHZ = (c-a)HZ.$$



Z: zombies H: humans
a: zombie death
b: human death
c: zombie infection

Bad news for the humans

$$\frac{dH}{dt} = -(b+c)HZ$$
$$\frac{dZ}{dt} = (c-a)HZ$$

- Since b and c are positive, the number of humans will always decrease
- We could add a birth term to allow the population to increase in the absence of zombies
- However, the timescale is short, so we can ignore births
  - a zombie invasion takes days or weeks, much shorter than the 9 months it takes for humans to reproduce.

Z: zombies H: humans
a: zombie death
b: human death
c: zombie infection

Is it possible to survive?

- The (c-a) term may be positive or negative
- If c-a>0, then zombies are being created faster than we can destroy them
- In this case, humans will be wiped out
- If c-a<0, then we are killing zombies faster than they are infecting humans
- In this case, both populations are decreasing
- Our survival will come down to a matter of which species becomes extinct first.

$$\frac{dH}{dt} = -(b+c)HZ$$
$$\frac{dZ}{dt} = (c-a)HZ$$

Z: zombies H: humans
a: zombie death
b: human death
c: zombie infection

A constant relationship

- Let α=b+c be the net removal rate of humans
- Let β=c-a be the net zombie creation rate
- Then we have

$$\frac{d(\beta H + \alpha Z)}{dt} = \beta \frac{dH}{dt} + \alpha \frac{dZ}{dt} = -\beta\alpha HZ + \alpha\beta HZ \equiv 0$$

- This means that (βH+αZ) does not change over time
- Thus, although H(t) and Z(t) do change over time, they do so in a way that keeps the value of (βH+αZ) constant.

Z: zombies H: humans
a: zombie death
b: human death
c: zombie infection

Determining the long-term outcome

- We know the initial populations of humans (H_0) and zombies (Z_0)
- Thus, we can define the constant exactly:

$$\beta H(t) + \alpha Z(t) = \beta H_0 + \alpha Z_0$$

- To survive, we need to be more deadly than the zombies, with $\beta<0$
- Thus, let $\beta=-\gamma$, with $\gamma>0$
- If all the zombies are wiped out, then $Z(\infty)=0$
  - this is the population of zombies after a long time has passed
  - similarly for $H(\infty)$ and humans.

Z: zombies H: humans
α: net zombie creation
β: net human removal

Uncle Z needs you!

$$\boxed{\beta H(t) + \alpha Z(t) = \beta H_0 + \alpha Z_0}$$

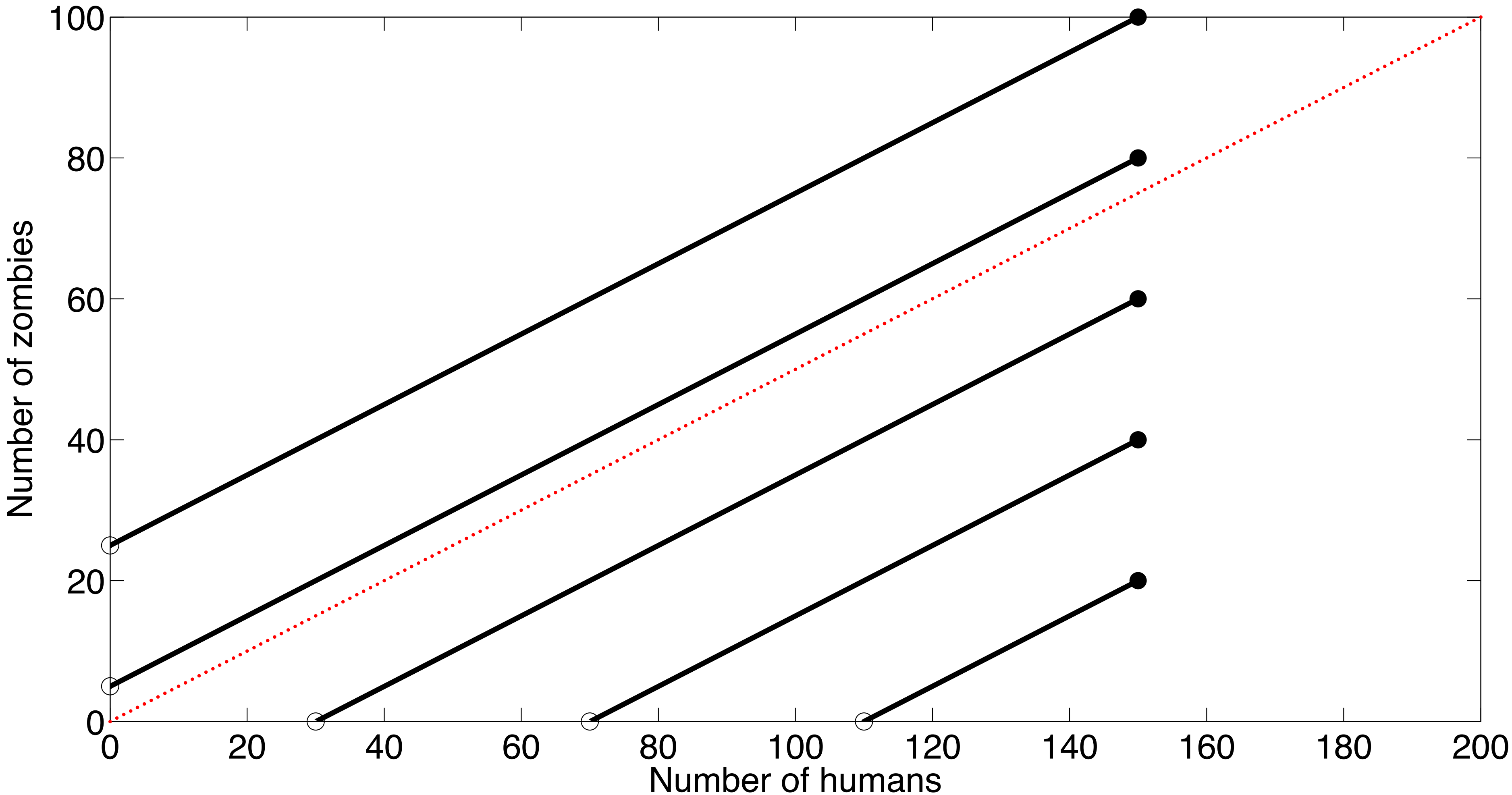
- We want Z(∞)=0, so

$$\gamma H(\infty) = \gamma H_0 - \alpha Z_0$$

- For the humans to exist, H(∞)>0 so to survive against the zombies the initial population must satisfy $\gamma H_0 > \alpha Z_0$
- Thus, to survive extinction, the humans need a large enough initial population
- This initial population must be capable of being more deadly than the zombies
- Otherwise, the zombie revolution is certain.

Z: zombies H: humans
α: net zombie creation
β: net human removal
γ=-β

Evolution of the populations

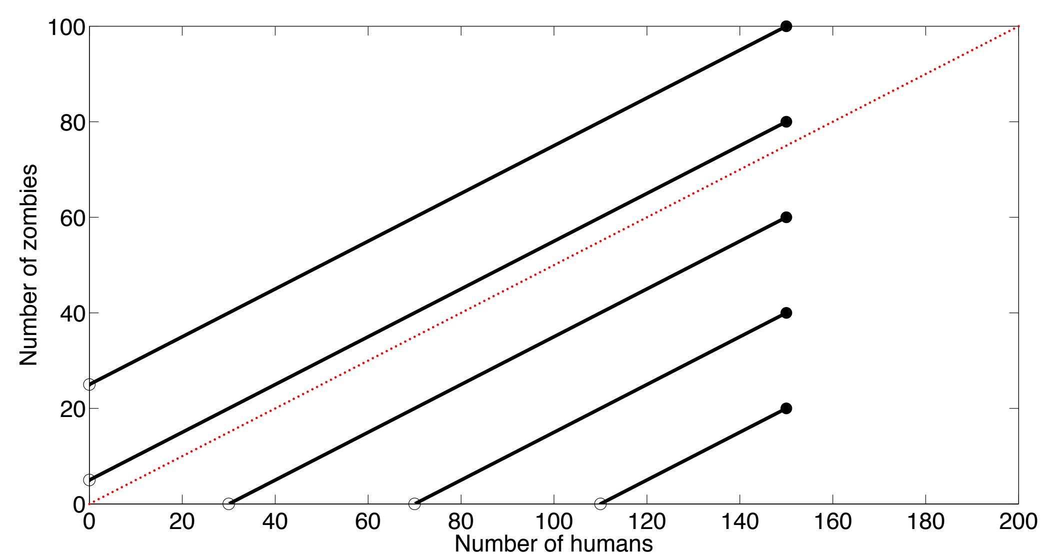


- γ=0.5 human^{-1}min^{-1}, α=1 zombie^{-1}min^{-1}
- Above the dotted line, the zombies win
- Below it, the humans win.

α: net zombie creation
γ: -net human removal

Varying the parameters

- If γ increases (zombies are removed quicker than they are created), then the lines would become steeper
- It would be easier for the population to be below the line and hence survive
- If α increases (human net death rate goes up), the lines become shallower
- Then the zombies would wipe us out.



α: net zombie creation
γ: -net human removal

Infection wave

- We now include spatial effects
- Earlier, we considered only the time before the zombies reached the human population
- Once this occurs, we need to consider how fast the zombies are moving...

  ...and also how fast they infect people
- Intuitively, if the infection rate was very small, then it would not matter if they reached us, because it would be hard for them to infect us, but easy for us to pick them off one by one.

Adding diffusion

- We add the diffusion term to each equation:

$$\frac{\partial H}{\partial t} = D_H \frac{\partial^2 H}{\partial x^2} - \alpha H Z$$
$$\frac{\partial Z}{\partial t} = D_Z \frac{\partial^2 Z}{\partial x^2} + \beta H Z$$

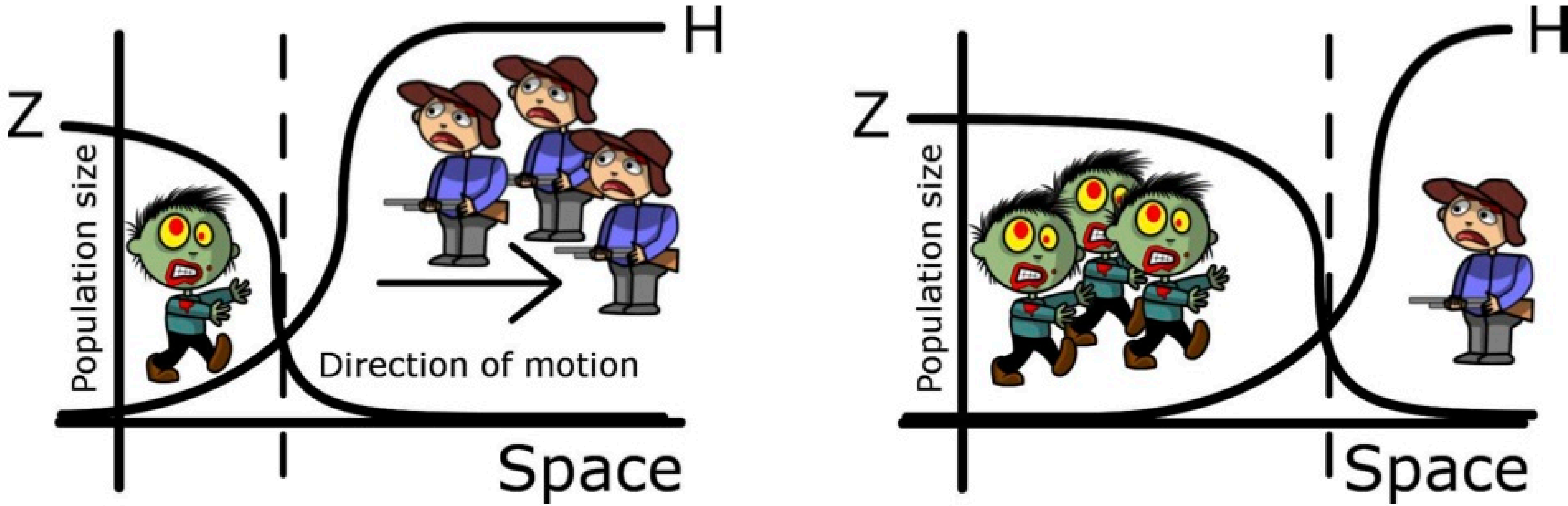
- Diffusion of humans (D_H) is much smaller than the zombies (D_Z)
  - because humans do not usually move randomly
- These still contain the interaction terms, as before
- Remember that β could be positive or negative.

Z: zombies H: humans
α: net zombie creation
β: net human removal
x: space t: time

Infection wave

- To see how quickly infection moves through the humans, we look for an *infection wave*
- This infection wave will move at a certain speed v
- Space, x, and time, t, will be linked through this speed
- We can thus reduce the dimension of the system from two coordinates (x,t) to the single coordinate u=x-vt
- These types of waves are called *Fisher's waves*.

Fisher’s waves



- Ahead of the wavefront, the human population is high and the zombie population is low
- The wave causes the zombie population to increase, while the humans decrease.

Constant speed

- We are interested in waves that move at constant speed and which do not change their shape as they move
- The shape of the wave at each space-time point is $F(x,t)$
- Since the wave does not change shape, it can also be described by $F(u=x-vt) \equiv F(x,t)$ where v is the speed of the wave
- In any interval of length t, a point on the curve at a position $u=x$ will move a distance vt
- The new position is $u+vt=x$ so $u=x-vt$.

x: space t: time

Large domain

- By looking for Fisher waves, we implicitly assume
  – the domain is infinite in size
  – with zero flux boundary conditions
- Since we are using a large, finite domain, the derived results represent a good approximation to those actually seen
- At least when the wave is not close to either boundary.

Consequences of changing variables

$$u = x - vt$$

- Using the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial}{\partial u}$$
$$\implies \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial u^2}$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial u}\frac{\partial u}{\partial t} = -v\frac{\partial}{\partial u}$$

- Then our model becomes

$$0 = D_H H'' + vH' - \alpha HZ$$
$$0 = D_Z Z'' + vZ' + \beta HZ$$

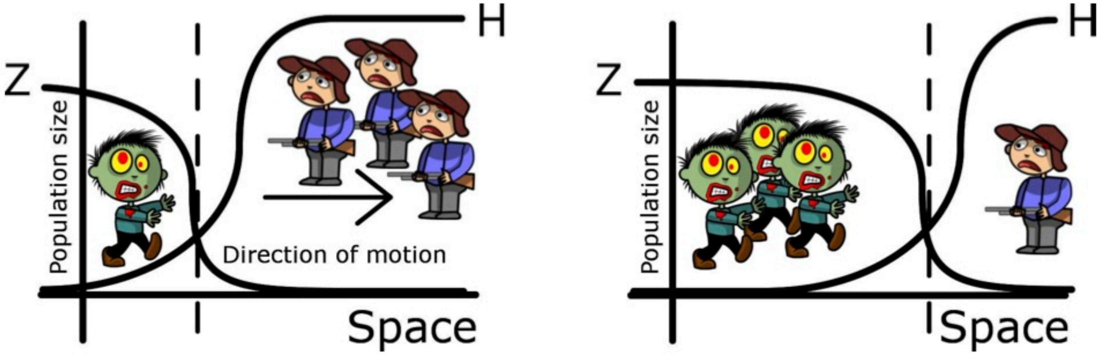
where differentiation is now with respect to u.

Z: zombies H: humans
α: net zombie creation
β: net human removal
x: space t: time v: wave speed

Speed bound

- What happens to a small perturbation ahead of the wave?
- Either it dies out or it will increase
- In front of the wave, $H=H_0$ and $Z=0$
- Let h and z be small perturbations and substitute

$$H(u) = H_0 + h\exp(\lambda u)$$
$$Z(u) = z\exp(\lambda u)$$



- The outcome will depend on the sign of λ.

Z: zombies H: humans
u=travelling wave

The danger of oscillations

$$H(u) = H_0 + h\exp(\lambda u)$$
$$Z(u) = z\exp(\lambda u)$$

- If $\lambda<0$, the perturbations will die out
- If $\lambda>0$, the perturbations grow and the infection is able to take hold
- If λ is complex, then the system would oscillate like a sinusoidal curve and Z(u) would become negative
- This is unrealistic, so we need λ to be real.

Z: zombies u: travelling wave
λ: perturbation eigenvalue

Minimum wave speed

- We now substitute $H(u)=H_0+he^{\lambda u}$ and $Z(u)=ze^{\lambda u}$ into $D_Z Z''+vZ'+\beta HZ=0$
- We can ignore any terms like hz since h and z are both small, so $hz \ll h$ or z
- Upon simplification, we are left with

$$D_z\lambda^2+v\lambda+\beta H_0=0$$

- Solving, we have

$$\lambda=\frac{-v\pm\sqrt{v^2-4D_z\beta H_0}}{2D_z}$$

- Since λ must be real, the wave speed has a minimum value of

$$v_{\min}^2=4D_Z\beta H_0.$$

Z: zombies H: humans
β: net human removal
u: travelling wave v: wave speed
h,z: perturbations
λ: perturbation eigenvalue
D_Z: zombie diffusion

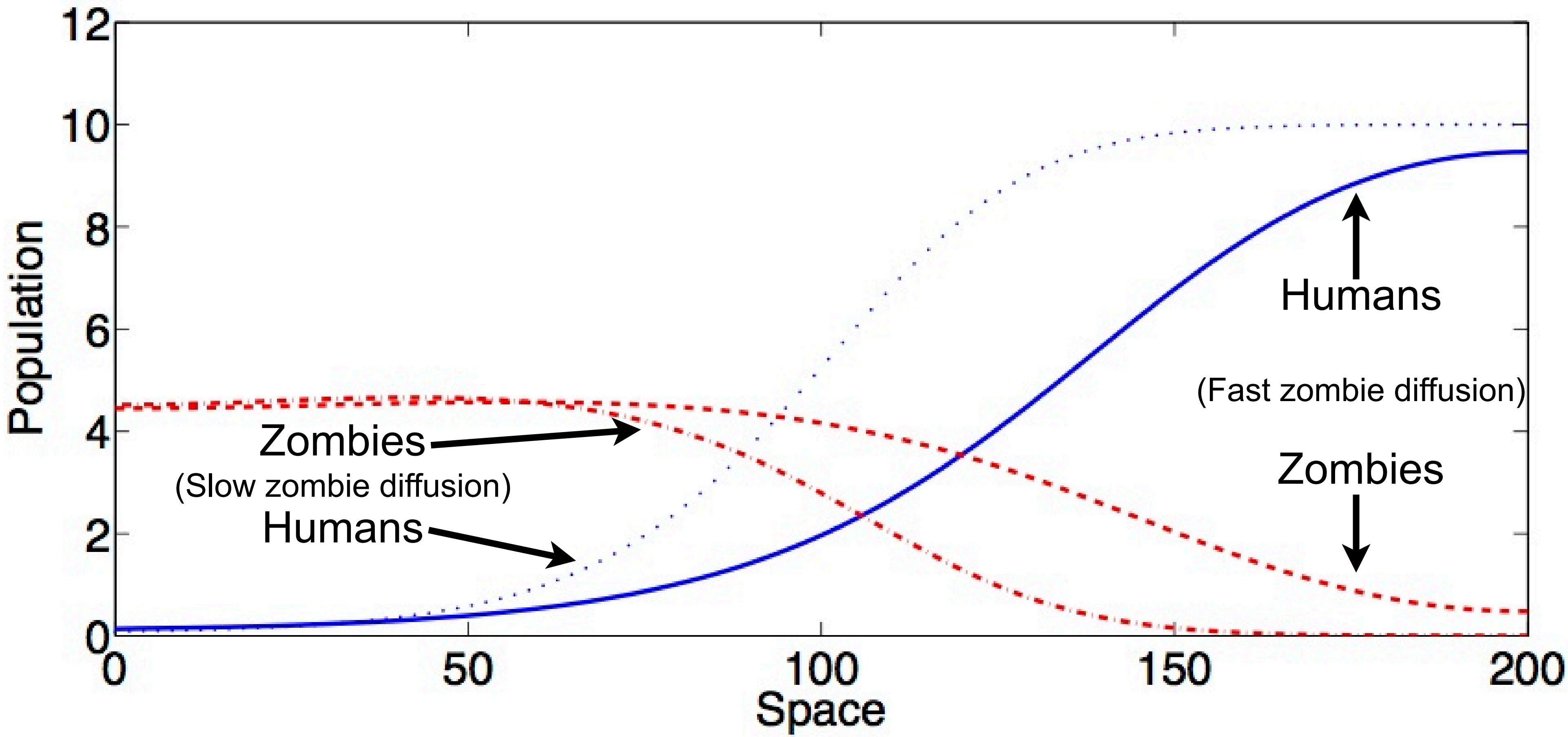
Slowing the zombies down

$$v_{\min}^2 = 4 D_Z \beta H_0$$

- To slow the infection, we should try and reduce the minimum speed for v
- We can reduce either D_Z, β or H_0
- We have already discussed reducing β
- Reducing D_Z amounts to slowing the zombies down
- Thus, effective fortifications should have plenty of obstacles that a human could navigate but a decaying zombie would find challenging.

H: humans
β: net human removal
D_Z: zombie diffusion

Reducing zombie diffusion by half



- D_H=0.1m^2/min, α=0.1zombie^{-1}min^{-1}, β=-0.05 human^{-1}min^{-1}
- D_Z is reduced from 100m^2/min to 50m^2/min.

α: net zombie creation
β: net human removal
D_H: human diffusion
D_Z: zombie diffusion

A controversial tactic

- Finally, we could slow zombies down by reducing H_0
- However, this is the population of humans
- This leads to the controversial tactic of removing your fellow survivors
  - if zombies can't infect them, then their population cannot increase
- However, this also reduces the number of people available to fight the zombies
- It also hastens human extinction
- This course of action isn't recommended.

Conclusions

- When spatial considerations are included, the best outcome is to run
- An island would be a great refuge
  - so long as the bay around the island is sufficiently steep
- Obstacles in the zombies’ path can also help
- Only fight if you are sure you can win
- It’s also best not to kill your fellow survivors...
- ...no matter what the mathematics says.

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T.E. Woolley, R.E. Baker, E.A. Gaffney, P.K. Maini. How Long Can We Survive? (In: R. Smith? (ed) Mathematical Modelling of Zombies, University of Ottawa Press, *in press*.)