Eigenvalues

Eigenvalues are numbers that 'represent' a matrix; if we have an $n \times n$ matrix A and can find a number λ and a nonzero vector x such than $Ax = \lambda x$, then λ is an eigenvalue and x is an eigenvector. Thus

$$\begin{aligned} Ax - \lambda x &= 0 \\ Ax - \lambda I x &= 0 \end{aligned}$$

where I is the $n \times n$ identity matrix. We put this in so that λI is a matrix of the same size as the matrix A (the expression $A - \lambda$ would make no sense). Hence

$$(A - \lambda I)x = 0$$

Clearly we want $x \neq 0$ (or else this is all trivial). But if $(A - \lambda I)^{-1}$ exists (i.e. $\det(A - \lambda I) \neq 0$), then the only solution is $x = 0$. So there will only be eigenvalues when $\det(A - \lambda I) = 0$.

Thus, for the matrix

$$J\big|_{(N,0)} = \begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix}$$

we have

$$\begin{aligned} 0 = \det(J - \lambda I) &= \det\left(\begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det \begin{bmatrix} -\lambda & b - aN \\ 0 & aN - b - \lambda \end{bmatrix} \\ &= \lambda(aN - b - \lambda) \end{aligned}$$

How did we get this last line? Eigenvalues of a 2×2 or a 3×3 matrix have a formula. For the former we have

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc\,,$$

and for the latter we have

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = aej + bfg + cdh - ceg - afh - bdj\,.$$

In general eigenvalues are quite hard... unless we have a row or column where all but one entry is zero. In this case we're allowed to reduce the size of the matrix by extracting that entry. But not only do we get to extract the entry, we get to eliminate everything else in that row and column!

Thus

$$\det \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & j & 0 \\ k & m & n & p \end{bmatrix} = p \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}.$$

So not only does the p come out of the determinant (because everything else in the last column was zero), reducing the remaining determinant to a much more manageable 3×3 determinant, but the last row is simply gone. That is, k, m and n are out of the picture.

(Technical note: If you're extracting anything that's not one of the entries along the main diagonal then you may or may not need an extra minus sign when you extract it. We don't do any such extracting here, so you don't need to worry about it, but if you're interested, check out any undergraduate linear algebra textbook.)

Assuming you have matrices with lots of zeros, you can reduce very high order matrices down to 3×3 or 2×2 matrices using this method. Fortunately, the Jacobian matrix almost always has lots of zeros and the things that aren't are usually on the diagonals anyway, so life is a lot easier than it otherwise would be.