Vampires: do they want to suck our blood?

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ABSTRACT

The supernatural has always had an impact on the media and one of the most popular embodiments has been the vampire. Infiltrating everything from classic literature to modern teen novels, from children shows to prime-time adult television, the vampire myth has, more or less, stayed the same: these soulless demons are members of the undead who feed on the blood of humans to survive, creating more of their kind by exchanging blood with their victims. Although, there exists various ways to kill these predators – exposure to sunlight, incineration, decapitation, wooden stake through the heart, to name a few – their superhuman strength and considerable intelligence makes them to be formidable enemies, if and when we have to face them. However, the question remains: even if vampires want and need to hunt us for their survival, is coexisting with such creatures possible? Using mathematical models, we will deduce whether vampires can exist and moreover, if we could live alongside a creature, who wants nothing from us but to suck our blood.

INTRODUCTION

In the simplest definition, a vampire is a corpse that returns to life, usually at night, to suck on human blood (Dundes, 1989b; "Vampire," 2011). They require this ritual as they are in need of a constant supply of fresh blood which they obtain by biting the neck of their victims. The subsequent blood loss causes the strength of the victims to wane and, if enough blood is drained, to die. According to tradition if blood is exchanged, the victims become vampires themselves (Skal, 1996). Interestingly enough, superstitions exist that maintain the fact that people who commit suicide, die violently, or are condemned by the Church become vampires without any blood exchange (Dundes, 1989b). Tradition also dictates that to kill a vampire one must drive a wooden stake through its heart; metal cannot hurt them. Modern folklore, however, tells of other ways to kill a vampire: decapitation, incineration, exposure to sunlight, garlic, holy water, to name of few (Dundes, 1989b; Skal, 1996).

Stories of vampire-like creatures have come from many parts of the world. As far back as Europe in the late 1600s to the early 1800s, people dug up graves looking for vampires (Dundes, 1989b). Many of the vampire tales originated in the Balkan countries (Eastern Europe) such as Albania, Greece, Hungary and Romania (Dundes, 1989b; "Vampire," 2011). In fact, literary classic *Dracula* is a horror novel written by Bram Stoker in 1897. Stoker based the character of Count Dracula, a vampire, on an actual villain: Vlad Tepes. Tepes, a cruel prince from Walachia (now part of Romania), has been immortalized to the history books as Vlad (III) the Impaler; a reference to his preferred method of torture and punishment, impaling his enemies on wooden stakes. Furthermore, for the atrocities he committed against his people during his tyrannous rule they gave him the nickname Dracula; Romanian for "son of the devil" (Dundes, 1989a,b; Skal, 1996).

Figure 1 Map of the Balkan Countries of Eastern Europe (coloured). Many of the vampire stories and legends originated in these countries in the 18th century. The most notable is the story of the vampire Count Dracula from Romania, written by Englishman Bram Stoker in 1897. (Photo taken from Laura, 2010)

Interestingly enough, the fear and loathing of the vampires of the original legends does not exist as prominently in modern literature. Rather the idea of the vampire has been romanticized and coexistence with these creatures is portrayed as possible. The question then is: how realistic is this? If vampires are real, can humans and vampires live together harmoniously? If not, and a war were to break out, what would be the best strategy for humans to employ so that the human race survives, or is Doomsday inevitable?

To answer these questions, we shall employ various mathematical principles and devices. Unfortunately, due to the lack of statistics and data on these fictional creatures this paper will be largely theoretical. In any case, one needs a vampire to model and the most famous, and perhaps familiar to the audience, depiction of vampires are those characterized by the hit 90s television show *Buffy the Vampire Slayer*. Therefore before the introduction of the models, it is necessary to explain the characters and setting as to better understand the models.

Buffy the Vampire Slayer

Buffy the Vampire Slayer takes places in the fictional town of Sunnydale, California, in the southern United States of America in the late 20^{th} century. When the first settlers landed in the area from Spain two centuries before, they named their settlement “boca del infierno”. The name, in English, roughly translates to “Hellmouth”, as the locals soon figured it was a unique supernatural portal between two different realities: a “centre of mystical energy that things gravitate towards” (Whedon, 1997). A century later, gold-seeker Richard Wilkins realized the supernatural properties of the area. In exchange for immortality, as a pure demon, Wilkins founded the town of Sunnydale atop of the “Hellmouth” so that the demons could have a “feeding ground” (“Sunnydale,” 2011).

Watching the opening credits of the first season of *Buffy the Vampire Slayer*, one hears the infamous words: “In every generation there is a Chosen One. She alone will stand against the vampires, the demons and the forces of darkness. She is the Slayer” (Whedon, 1997). The Slayer they speak of for this generation is Buffy Anne Summers. The television series starts with Buffy being sent to Sunnydale to combat the growing supernatural activity in the area which includes, but isn’t limited to, vampires. Her appointed Watcher, who acts as a guardian and mentor, is Rupert Giles. As Giles, a third-generation Watcher, is appointed by the Watcher’s Council he comes with much knowledge about magick and various supernatural creatures and he offers up information pertaining to such matters in many episodes.

When asked about how vampires originated Giles explains that “for untold eons, demons walked the Earth. They made it their home... their hell. But in time, they lost their purchase on this reality and the way was made for mortal animals, for man.” He goes on to say that “the books tell that the last demon to leave this reality fed off a human, mixed their blood. He was a human form possessed – infected – by the demon’s soul. He bit another and another. ‘And so they walked the Earth, feeding, killing some, mixing their blood with others to make more of their kind’” (ibid). This leads to several important conclusions. Implicitly stated, vampires only feast on the living. Mathematically, this means that the removed class (that is, the class representing the dead) decouples from the other classes. Immediately, this makes modelling vampires different from modelling other undead creatures, specifically zombies. Explicitly stated, are three conclusions: blood mixing is required to create a vampire, vampires do not beget vampires, and when one becomes “infected” they maintain a human form but are no longer human. Giles further explains that the human host retains their corporeal form and only when feeding time is upon them do they exhibit the face of the creature that killed them: “a vampire isn’t a person at all. It may have the movements, the memories, even the personality of the person it took over, but it’s still a demon at the core. There is no halfway” (ibid). Furthermore, Angel, Buffy’s love interest and vampire, explains that “when you become a vampire, the demon takes your body but it doesn’t take your soul. That’s gone. No conscious, no remorse” (ibid). Therefore, putting all that together places several constraints on our model. There is no vertical transmission of vampirism; that is, there is no birth rate into the vampire class. Moreover, vampires do not kill every human they drink from but rather, sire a new vampire by exchanging blood with a human they have drunk from. Furthermore, once a human is turned they leave the susceptible human class and enter the infected vampire class, maintaining the body and brain they once had. Therefore, not only are vampires smart creatures, capable of thought and speech, but as they look like humans, so it will make it harder for humans to kill something that looks human. Add this to their voracious appetite and the absence of thoughts of guilt when a vampire kills a human, and we have a very dangerous opponent capable of mobilizing an army.

There exists various ways in which to kill this ruthless undead creature. In a conversation between Buffy and her friend Xander, it is remarked that not only are crosses, garlic, and stakes through the heart necessary for “vampire slayage” but so are fire, beheading, sunlight and holy water (ibid). Moreover, since metal cannot hurt a vampire Buffy points out that the police are ineffective as “they would only come with guns” (ibid). However, according to Darla, one of Buffy’s many vampiric enemies: even if bullets can’t kill them they can hurt like hell (ibid). Moreover, Spike, a vampire enemy-turned-friend explains that while vampires can starve, they do not starve to death – they just get really thirsty (ibid). Also, it is pointed out several times that in one-on-one combat only the Slayer has the capacity and strength to kill vampires and that it would require a group of humans to do the same (ibid). This leads to the conclusion that eradication may be difficult, when compared to something like zombies, as vampires are much stronger than other undead creatures.

The Models

For each model, the same set-up has been employed: the parameters and equations are defined, and from there, using a variety of mathematical techniques, a threshold and the stability of the equilibrium is calculated. Moreover, within each section there is flow-chart of the model and any relevant graphs. Discussion is held until the end.

“Bare bones” Model

Current statistics state that the present human population is 6.9 billion and increasing. For that reason, it is often sited that vampires cannot have ever existed as, if vampires were to feed on humans at the rates that literature and film tell us (that is, weekly at best), the human race would have been wiped out within three years. This is the logic used by physicists Dr. Sohang Gandhi and Dr. Costas J. Efthimiou in their paper *Cinema Fiction vs Physics Reality(*2007) where they attempt to debunk the vampire legend.

The basic “bare bones” model (herein SV model) that Gandhi and Efthimiou propose uses the mathematical concept of geometric progression. Such logic is employed in various fields: calculating interest in economics and finance to finding out the quantity of a decaying radioactive element in geology and physics. In simplest terms, geometric progression is the concept that beginning with an initial value, X_0, of a substance X one can calculate future values of the substance if the ratio, r, between successive terms is constant:

$$r = \frac{X_{n+1}}{X_n} \;;$$

the subscript n is a placeholder within the series. Therefore, the general formulation of the geometric series is given as

$$X_n = X_0 r^n$$

and, with some rearranging, the sum of the first n terms of this series is given by

$$S_n = \frac{X_0(1 - r^n)}{1 - r} \,.$$

Introducing the notation of this paper, the total population is given by

$$N = V(t) + S(t)$$

where V is population of the vampire class and S the population of the susceptible class at iteration, month, t. A keen reader will notice that a constraint induced by the model is that the world population is constant (that is, either the birth rate and the background death rate are both zero or the birth rate is equal to the background death rate); there is no way to leave the system. Therefore, to obtain the populations of either class, the formulas are

$$V(t) = V_0 \alpha^t \,,$$
$$S(t) = S_0 - V(t) \,.$$

where α is the rate in which humans are turned into vampires. As we are dealing with people, the only biologically realistic scenarios when V and S are non-negative; $S, V \geq 0$. Moreover, once a class ceases to exist, that is become negative, they will continue to do so for all $t' > t$.

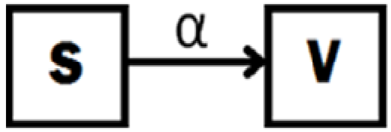

Figure 2 Visual representation of the SV model. In this model, there is assumed no change to either the susceptible human (S) or infected vampire (V) class and the rate at which the humans are turned into vampires is given by the parameter α.

Therefore, taking the total population of the world in 1600 as 536 870 911 as the initial susceptible population, 1 as the initial vampire population and a feeding rate of once per month, that is the vampire population doubles monthly, we arrive at the following equations:

$$V(t) = 2^t$$
$$S(t) = 536870912 - V(t)\ .$$

Using these equations and parameters Figure 3 is obtained.

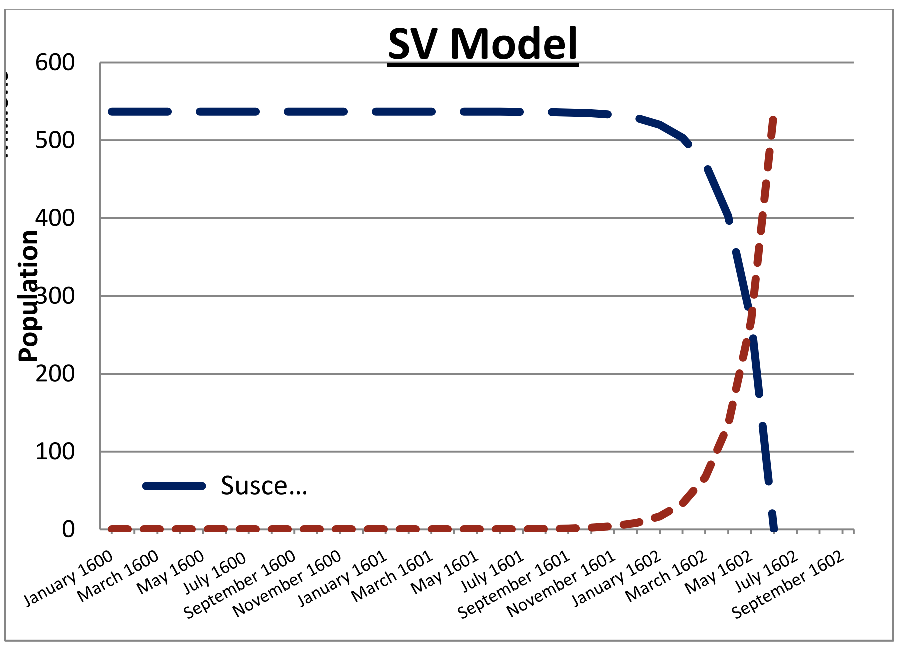

Figure 3 Plot of the populations of the classes against time of the SV model. One can see that the susceptible population declines over time while the vampire population increases by the same amount, maintaining a constant total population of 536,870,912 bodies. Ultimately, if each vampire were to turn one human on the first of each month, this model ends with there being no more humans left on Earth by June 1602.

As one can see in the above graph, there ceases to be any living humans and the entire world population has been become vampires by June 1602. Gandhi and Efthimiou use this fact, coupled with the philosophical anthropic principle[1], to conclude that vampires cannot exist as the "nonexistence of vampires is necessary for human existence." However, although the mathematics is true, one cannot deny the exaggerated simplicity of this model (Sejdinovic, 2008). Therefore, one can better model reality by reintroducing some of the assumptions and taking several of those different factors, on an individual basis, into account and see how it affects the model.

If we introduced a realistic death rate for the human population, it should not come to a surprise when we see that the human population declines even faster. The positive news is that there will not be as many vampires after the demise of human civilization as there were fewer bodies for them to feed on. Future models will include a death rate.

If we introduced a realistic birth rate, the human race would still decline but just at a slightly slower rate. However, it will ultimately end with there being no more humans left on Earth. This is not an unexpected outcome as an exponential function grows much faster than a simple polynomial. Interesting enough, even if an unrealistic birth rate was included the vampires would still take over – it would just take more time to do so. It is important to note though that the Earth has a carrying capacity and this would need to be taken into account as an upper bound on the population. Future models will include a birth rate.

The last assumption has to do with the vampires themselves. It is not in the best interest of the vampire to turn everyone they feed on into a vampire, as this would just mean they are creating another competitor for the resources, humans, they need in order to survive. Therefore, they have the option to kill the human or leave the human wounded, but able to recover, so that they can feed on them again at a later date. This is not the best case for the humans, but this does means that coexistence may be possible. Future models will distinguish between if a vampire turns the human or partially or fully drains them.

Lastly, to increase the likelihood of survival one can introduce a predator for the vampires. This idea will be presented in two ways: via increased knowledge of the population and via a Slayer. The idea behind the former is that, although vampires are strong creatures, the average human can take more precautions to avoid being bitten (such as travelling in the daytime, wearing a cross, carrying garlic) and even kill a vampire in combat if they know what to look for.

The Basic Model (no general knowledge, no Slayer)

For the basic (SVR-) model, three classes were considered; the original two and a third, removed class, R. This removed class represents those who have died and, as they are dead, they do not interact with the

[1] The anthropic principle states that Universe is designed in such a way to positively reinforce, permit and sustain life. Therefore, if something is necessary for life to exist then it too must exist or be true, since life is present.

living. Moreover, they do not carry the disease. Henceforth, although it decouples for the analysis, the removed class will be included in the flow-chart and its differential equation described for completeness sake. Specific to this model, it is important to note that there is no killing of vampires by humans as they are generally unaware of the vampires' existence.

The various parameters involved between classes are defined as follows:

- Birth rate, π;
- Background non-vampire related death rate, μ;
- Rate of infection (ie. vampirism), α;
- Death by vampires (ie. blood draining) rate, β; and,
- Background non-human related death rate, δ.

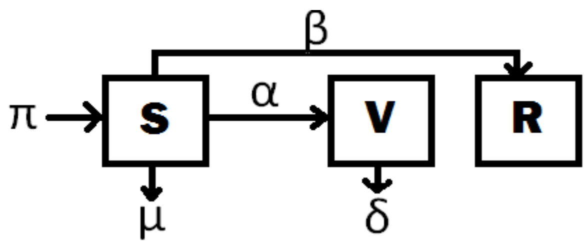

Figure 4 Visual representation of the SVR- model. In this model, the rate that humans (S) are turned into vampires (V) is given by parameter α and the rate that humans are killed by vampires by the parameter β. The parameter π represents the birth rate, μ the background death rate for the humans, and δ the non-human related death rate for vampires. The removed class representing the dead is denoted by the R. It is important to note that there are no humans killing the vampires.

The differential equations that model the SBV- model are:

$$\frac{dS}{dt} = \pi - \alpha SV - \beta SV - \mu S$$
$$\frac{dV}{dt} = \alpha SV - \delta V$$
$$\frac{dR}{dt} = \beta SV + \mu S + \delta V$$

Therefore, the Jacobian is

$$J = \begin{bmatrix} -\alpha V - \beta V - \mu & -\alpha S - \beta S \\ \alpha V & \alpha S - \delta \end{bmatrix}.$$

The disease-free equilibrium is calculated by setting $V = 0$, and so from the first equation it follows that $S = \frac{\pi}{\mu}$. Therefore, the Jacobian matrix evaluated at the disease-free equilibrium is

$$J_{DFE} = J|_{\left(\frac{\pi}{\mu},0,0\right)} = \begin{bmatrix} -\mu & -\frac{\alpha\pi}{\mu} - \frac{\beta\pi}{\mu} \\ 0 & \frac{\alpha\pi}{\mu} - \delta \end{bmatrix}$$

where the determinant is

$$\det(J_{DFE} - \lambda I) = (-\mu - \lambda)\left(\frac{\alpha\pi}{\mu} - \delta - \lambda\right)$$

and thus, the eigenvalues are

$$\lambda_1 = -\mu\,,$$
$$\lambda_2 = \frac{\alpha\pi}{\mu} - \delta\,.$$

Therefore, if the second eigenvalue is positive the disease-free equilibrium is unstable. Moreover, as it is the only possible non-negative eigenvalue, the threshold parameter can be calculated as

$$\frac{\alpha\pi}{\mu} - \delta > 0$$
$$\Longrightarrow \mathbb{R}_0^A = \frac{\pi\alpha}{\mu\delta}\,.$$

The endemic equilibrium exists when the disease-free equilibrium is unstable. Therefore, setting the differential equations to zero and solving for S, V, R we find that the endemic equilibrium is defined to be

$$(\bar{S}, \bar{V}, \bar{R}) = \left(\frac{\mu}{\alpha}, \frac{\pi\alpha - \delta\mu}{\delta(\alpha + \beta)}, 0\right).$$

The existence of this endemic equilibrium can also determine the threshold parameter. One has to look when the equilibrium value governing the infected class is greater than zero:

$$\bar{V} > 0$$
$$\frac{\pi\alpha - \delta\mu}{\delta(\alpha + \beta)} > 0$$
$$\Longrightarrow \mathbb{R}_0^B = \frac{\pi\alpha}{\delta\mu}.$$

If we compare $\mathbb{R}_0^A$ and $\mathbb{R}_0^B$, we see that, we arrive at the same threshold value. Therefore we will define the following

$$\mathbb{R}_0^{SVR-} = \frac{\pi\alpha}{\delta\mu}$$

such that, if $\mathbb{R}_0^{SVR-} < 1$ there will be no outbreak but if, however, $\mathbb{R}_0^{SVR-} > 1$ the disease becomes endemic.

The Basic Model (with general knowledge, with Slayer)

This model is the based on the SVR- model, and so the parameters in common are as defined above. The difference in the model lies in the fact that humans are aware of the existence of vampires and they actively hunt and kill them at a rate, γ. Moreover, there is a Slayer and their efficacy of eradicating vampires is given by the parameter σ. As the Slayer is stronger, faster and, overall, better equipped for killing the undead, $\sigma > \gamma$. This model shall be referred to as SVR+ model.

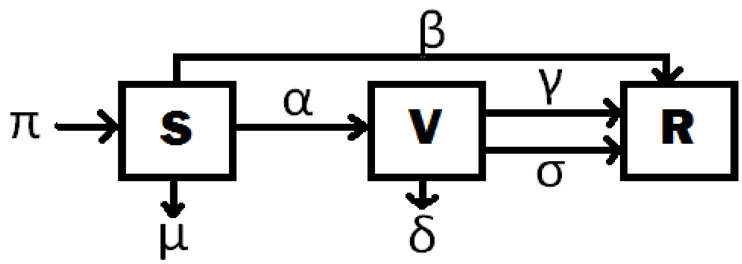

Figure 5 Visual representation of the SVR+ model. In this model, the rate that humans (S) are turned into vampires (V) is given by parameter α and the rate that humans are killed by vampires by the parameter β. The parameter π represents the birth rate, μ the background death rate for the humans, and δ the non-human related death rate for vampires. Moreover, humans and Slayer are killing the vampires at rate γ and σ, respectively, such that σ > γ. The removed class representing the dead is denoted by the R.

The differential equations that model the SVR+ model are:

$$\frac{dS}{dt} = \pi - \alpha SV - \beta SV - \mu S$$
$$\frac{dV}{dt} = \alpha SV - \delta V - \gamma V - \sigma V$$
$$\frac{dR}{dt} = \beta SV + \mu S + \delta V + \gamma V + \sigma V$$

The Jacobian is

$$J = \begin{bmatrix} -\alpha V - \beta V - \mu & -\alpha S - \beta S \\ \alpha V & \alpha S - \gamma - \delta - \sigma \end{bmatrix}.$$

As before, the disease-free equilibrium is found to be $(\bar{S}, \bar{V}, \bar{R}) = (\frac{\pi}{\mu}, 0,0)$. Therefore, the Jacobian matrix, evaluated at the disease-free equilibrium, is

$$J_{DFE} = J|_{\left(\frac{\pi}{\mu},0,0\right)} = \begin{bmatrix} -\mu & -\frac{\alpha\pi}{\mu} - \frac{\beta\pi}{\mu} \\ 0 & \frac{\alpha\pi}{\mu} - \gamma - \delta - \sigma \end{bmatrix}$$

where the determinant is

$$\det(J_{DFE} - \lambda I) = (-\mu - \lambda)\left(\frac{\alpha\pi}{\mu} - \gamma - \delta - \sigma - \lambda\right)$$

and thus, the eigenvalues are

$$\lambda_1 = -\mu,$$
$$\lambda_2 = \frac{\alpha\pi}{\mu} - \gamma - \delta - \sigma\,.$$

Therefore, if the second eigenvalue is positive the disease-free equilibrium is unstable. Moreover, as it is the only possible non-negative eigenvalue the threshold parameter can be calculated as follows

$$\frac{\alpha\pi}{\mu} - \gamma - \delta - \sigma > 0$$

$$\Longrightarrow \mathbb{R}_0^C = \frac{\alpha\pi}{\mu(\gamma + \delta + \sigma)}.$$

Again as explained before, the endemic equilibrium is found to be

$$(\bar{S}, \bar{V}, \bar{R}) = \left(\frac{\delta + \sigma + \gamma}{\alpha}, \frac{\alpha\pi - \mu(\gamma + \delta + \sigma)}{(\gamma + \delta + \sigma)(\alpha + \beta)}, 0\right),$$

and the threshold parameter is given by

$$\bar{V} > 0$$

$$\frac{\alpha\pi - \mu(\gamma + \delta + \sigma)}{(\gamma + \delta + \sigma)(\alpha + \beta)} > 0$$

$$\Longrightarrow \mathbb{R}_0^D = \frac{\alpha\pi}{\mu(\gamma + \delta + \sigma)}.$$

If we compare $\mathbb{R}_0^C$ and $\mathbb{R}_0^D$ we see that we arrive at the same threshold value. Hence, we will define

$$\mathbb{R}_0^{SVR+} = \frac{\pi\alpha}{\mu(\gamma + \delta + \sigma)}$$

such that if $\mathbb{R}_0^{SVR+} < 1$ there will be no outbreak but if, however, $\mathbb{R}_0^{SVR+} > 1$ the disease becomes endemic.

Model with Latent Infected (Bitten) Class

This model (SBVR) differs from the basic models as there is a fourth class, the bitten class B, and a modification of parameters. To clarify, the susceptibles become vampires in two steps: first they get bitten at a rate ε, and second they drink a vampire's blood at a rate ζ. Moreover, those in the bitten class die by vampires draining their blood at a rate η. The remaining parameters are as defined in the basic model.

The motivation behind adding this fourth class was to make the model more realistic. As explained before, it is not in the best interest of the vampire to drain or turn every human they drink from. The two primary reasons are that too many dead bodies would arouse suspicion among the humans and too many vampires would increase competition for the resources, human blood. It is important to note those bitten are not infectious (i.e. vampires) since blood was not mutually exchanged.

The assumptions of this model are the same as before with one additional constraint: that there are no individuals who have drank a vampire's blood and have not been bitten themselves; such an individual is a statistical anomaly.

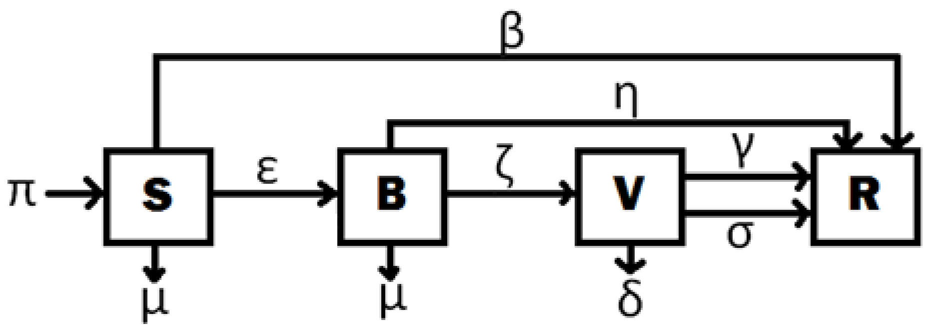

Figure 6 Visual representation of the SVBR model. In this model, there four classes: susceptible humans (S), non-infectious bitten humans (B), infectious vampires (V) and removed dead class (R). Humans are turned into vampires in two steps as they are first bitten at a rate ε and then blood is exchanged with a vampire at a different rate ζ. As before, a vampire could drain a human of their blood at a rate β if they have never been bite, and at a different rate η if they have been bit. The parameters π represents the birth rate, μ the background death rate for the humans, and δ the non-human related death rate for vampires. Moreover, humans and Slayer are killing the vampires at rate γ and σ, respectively, such that σ > γ.

The differential equations that model the SBVR model are:

$$\frac{dS}{dt} = \pi - \varepsilon SV - \beta VS - \mu S$$
$$\frac{dB}{dt} = \varepsilon SV - \zeta VB - \eta VB - \mu B$$
$$\frac{dV}{dt} = \zeta BV - \delta V - \gamma V - \sigma V$$
$$\frac{dR}{dt} = \mu S + \beta SV + \mu B + \eta VB + \delta V + \gamma V + \sigma V$$

The Jacobian is

$$J = \begin{bmatrix} -\varepsilon V - \beta V - \mu & 0 & -\varepsilon S - \beta S \\ \varepsilon V & -\zeta V - \eta V - \mu & \varepsilon S - \zeta B - \eta B \\ 0 & \zeta V & \zeta B - \delta - \gamma - \sigma \end{bmatrix}.$$

The disease-free equilibrium again when $V = 0$ is yields

$$(\bar{S}, \bar{B}, \bar{V}, \bar{R}) = \left(\frac{\pi}{\mu}, 0,0,0\right).$$

The Jacobian evaluated at the disease-free equilibrium is

$$J_{DFE} = J|_{\left(\frac{\pi}{\mu},\frac{\rho}{\mu},0,0\right)} = \begin{bmatrix} -\mu & 0 & -\varepsilon\frac{\pi}{\mu} - \beta\frac{\pi}{\mu} \\ 0 & -\mu & \varepsilon\frac{\pi}{\mu} \\ 0 & 0 & -\delta - \gamma - \sigma \end{bmatrix}.$$

Thus the determinant is

$$\det(J_{DFE} - \lambda I) = (-\mu - \lambda)^2(-\delta - \gamma - \sigma - \lambda),$$

and the eigenvalues are

$$\lambda_1 = \lambda_2 = -\mu,$$
$$\lambda_3 = -\delta - \gamma - \sigma\,.$$

Therefore, as all the parameters are positive the disease-free equilibrium is stable; there is no threshold value. Moreover, this means there does not exist and endemic equilibrium. As this is unexpected, we shall look at this from a different point of view.

Predator Prey Model

Another approach to answering the question concerning the coexistence of humans and vampires lies in the field of predator-prey population dynamics. The mathematics is more or less the same as with the epidemiological models but there are some extra terms with respect to the carrying capacity of the prey in a given environment. In biology, the carrying capacity of a given environment is the maximum number of a particular species that can be supported and sustained indefinitely given that food, water and other necessities are available in that environment.

This approach was looked at in a mock unpublished paper by Dr. Brian Thomas, who was a PhD candidate in ecology at the time, titled *Vampire Population Ecology* (2002). The inspiration for some of this mathematics comes from that paper; it has been cleaned up, corrected and expanded upon for its presentation in this paper. Herein, this model is referred to as the PP model.

The differential equations that model the PP model are:

$$\frac{dS}{dt} = \chi S - \frac{\chi SS}{\kappa} - \phi VS$$
$$\frac{dV}{dt} = \alpha\phi SV + \omega V - \gamma V$$

The various parameters involved between classes are defined as follows:

- Net growth rate of the human population, χ;
- Net migration rate of the vampire population, ω;
- Carrying capacity of the human population, κ;
- Coefficient that relates human of vampire encounters to number of actual feedings, ϕ;
- Rate of infection (ie. vampirism), α; and,
- Death by humans rate, γ;

To obtain the nullclines of these differential equations we set them to zero and we see what values of H and V make each equation is true. Therefore, the H- nullclines (blue in Figure 7) are

$$S = 0\,, \qquad V = \frac{\kappa\chi - \chi S}{\phi\kappa}$$

and the V nullclines (green) are

$$V = 0\,, \qquad S = \frac{\gamma - \omega}{\alpha\phi}\,.$$

As we are interested in the equilibrium points and they are defined as where the H and V nullclines intersect. It is important to note that, if the trajectory starts at an equilibrium point, it stays there forever since both derivatives vanish at that point (Taubes, 2001). Therefore, plotting the nullclines we see there are three equilibria (in red where the labels correspond to the bullet numbering, Figure 7) and they are when:

1) Both the human and vampire populations are extinct;
2) The vampire population is extinct and the human population is at, or near, carrying capacity; and,
3) Vampires and humans coexist at some value.

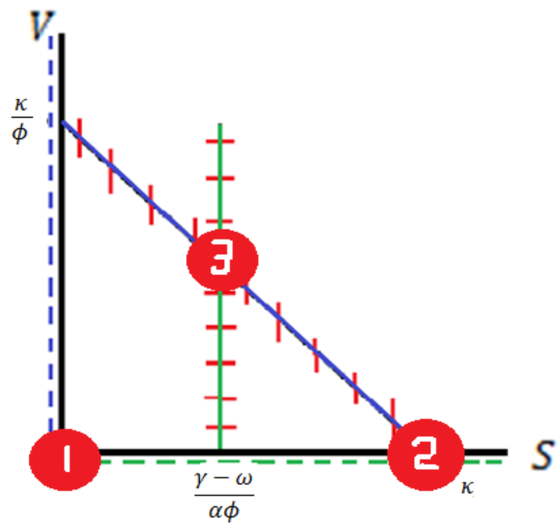

Figure 7 Plot of the nullclines and equilibria of the PP model. The equilibria are defined as the intersection between two different nullclines; there are three. The first case (1) represents when both the human and vampire populations are extinct; the second case (2) when the vampires are extinct but the human population hovers near carrying capacity; and, the third case (3) is where the human and vampire population coexist.

The equilibrium point given by the third case is given by the coordinate $\left(\frac{\gamma-\omega}{\alpha\phi}, \frac{\chi}{\phi}+\frac{\omega-\gamma}{\alpha\phi\kappa}\right)$ or equivalently,

$$\bar{S} = \frac{\gamma-\omega}{\alpha\phi}$$
$$\bar{V} = \frac{\chi}{\phi}+\frac{\omega-\gamma}{\alpha\phi\kappa}.$$

The important thing to notice is that the human equilibrium population does not depend on its own carrying capacity but that the vampire population does depend on it. Between what we know from the SV model and the existence of the other two equilibrium points, the human population is not large enough to support a vampire population. However, as long as the carrying capacity is higher than the equilibrium population vampires and humans can coexist.

From the figure we can see that the nullclines cut the SV-plane into four regions. By plugging in values for the parameters we can grasp a general direction of the motion in each region (Figure 8). The parameters used to determine the direction of motion in each region is pulled from Thomas' paper. Vampires migration rate to Sunnydale is 10% annually $\{\omega = 1.1\}$. Moreover, it is assumed that a vampire feeds every 3 days, encountering 100 possible victims during that time (that is, drinks 1 out of 300 encounters $\{\phi = 0.00333\}$) and sires a victim every other year (that is, turning 1 out of 240 feedings $\{\alpha = 0.00417\}$). Sunnydale's net population growth – after taking into account birth, date, immigration, and migration rates – is 10% annually $\{\chi = 1.1\}$; however, the carrying capacity is about 100,000 people $\{\kappa = 100\ 000\}$. Buffy and her friends stake a third of the vampire population each year $\{\gamma = 0.6\}$. Plugging these parameters into the equilibrium values we find that that $\bar{S} = 36{,}346$ and $\bar{V} = 18$. This seems quite realistic as the initial population of Sunnydale is given as 38, 500 in an early episode of *Buffy the Vampire Slayer*.

However we do not know that stability of that equilibrium point and hence have to look at the Jacobi for that. There are two scenarios. The first is if the trajectory approaches a closed loop trajectory encircling the equilibrium point (Figure 8, top right panel), meaning the eigenvalues of the Jacobi matrix are imaginary. The second is if the trajectory spirals slowly into the equilibrium point (Figure 8, bottom right panel), meaning the eigenvalues of the Jacobi are negative. The latter case corresponds with stable cyclic behavior of the populations, while the former describes an unstable cyclic oscillation of the populations (Hartl, 1992; Perko, 2001).

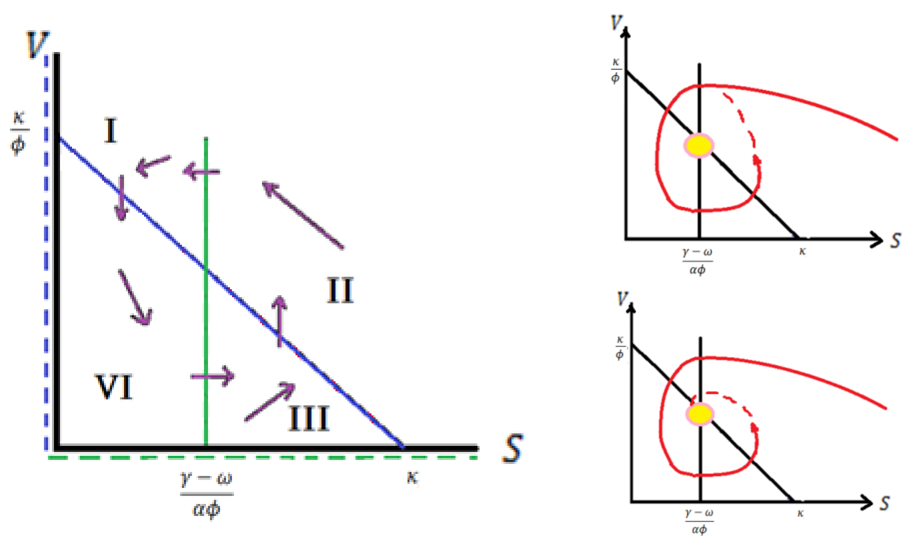

Figure 8 Plots showing the possible trajectories of the PP model. (Left panel) Given the values for the parameters chosen, we can determine the behavior in each of the regions; $\{\omega = 1.1, \phi = 0.00333, \alpha = 0.00417, \chi = 1.1, \kappa = 100\ 000, \gamma = 0.6\}$. A description of the parameters and how they are chosen, is provided in the body of the paper. The blue lines represent the S nullclines, green the V nullclines, and purple the trend of the model. (Right panel) The top panel represents a closed loop trajectory encircling the equilibrium point, corresponding with unstable cyclic oscillations of the population. The bottom panel represents the trajectory slowly spiraling into the equilibrium point, indicating stable behaviour of the populations.

The Jacobi is

$$J = \begin{bmatrix} \chi - \frac{2\chi S}{\kappa} - \phi V & -\phi S \\ \phi \alpha V & \alpha \phi S + \omega - \gamma \end{bmatrix}$$

and the eigenvalues are

$$\lambda_1 = 2(\alpha\phi S + \omega - \gamma)$$
$$\lambda_2 = 2(2\frac{\chi}{\kappa}S - \phi V - \chi)$$

Thus, plotting the results of the model for various possible combinations of populations we see that the trajectories spiral into the equilibrium point over time, indication that is a stable point. Therefore, this shows that humans can coexist with vampires (Figure 9). However, although this is stable it does not take into consideration large scale supernatural events (like the Apocalypse) that would perturb the system into chaos.

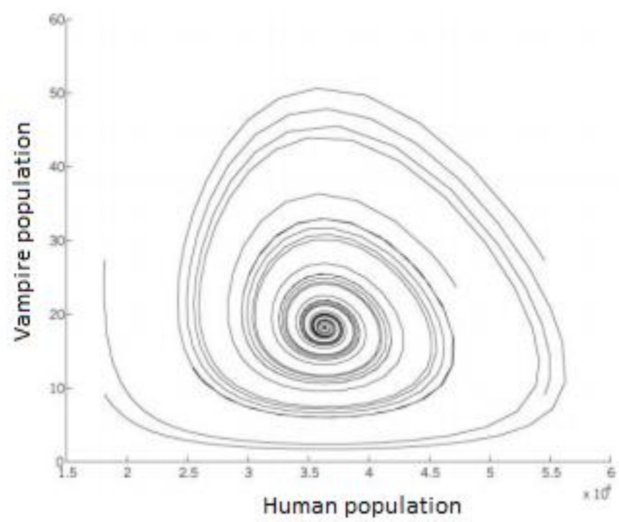

Figure 9 Graph of several trajectories through time Each line represents a trajectory for a different combination of human and vampire population sizes through time. The important thing to notice is that each trajectory spirals into the equilibrium point and therefore is an indication of stability. Moreover, it shows that the two species can coexist.

Discussion

As previously mentioned there is not a lot of data readily available on the feeding habits of vampires. However, perhaps given sufficient time a compilation of data could be obtaining from watching episodes of Buffy the Vampire Slayer and taking note of the frequency in which a given vampire encounters, feeds, kills and turns humans. Unfortunately, though this would, of course, still be an estimate as there would be many off camera encounters that would not be accounted for. Moreover, time would be a difficult variable to gauge.

Moreover, due to the large uncertainty on each parameter, preliminary sensitivity trials of the variables indicated that any graph of the differential equations can be tweaked to show a variety of outcomes. For this reason graphs of these models were excluded from this paper and a comparison of the results is purely theoretical.

Comparison of the SVR- and SVR+ models

If one recalls the reproductive ratios of the SVR- and SVR+ models were calculated to be:

$$\mathbb{R}_0^{SVR^-} = \frac{\pi\alpha}{\delta\mu} \text{ and } \mathbb{R}_0^{SVR^+} = \frac{\pi\alpha}{\mu(\gamma+\delta+\sigma)}.$$

Knowing that all the parameters are positive, but fractions less than one, realistically we have parameters that satisfy

$$0 < \gamma + \sigma < 1$$

and hence,

$$\frac{1}{\delta} > \frac{1}{\gamma + \delta + \sigma}.$$

Thus, we arrive at the conclusion that

$$\mathbb{R}_0^{SVR^-} < \mathbb{R}_0^{SVR^+} .$$

Moreover, if we are interested if an endemic will occur, then

$$\mathbb{R}_0^{SVR^+} > \mathbb{R}_0^{SVR^-} > 1 .$$

This means that if an outbreak of vampires were to break out in an area, like Sunnydale, where the people were generally aware and there was a Slayer, it would less likely be endemic than in a population, like Ottawa, where there is no Slayer and the population is generally unaware. Intuitively, looking at what the parameters actually mean this makes sense as the parameters γ and σ represent the affect humans have on reducing the vampire population. That is, the more positive these terms are then the less the vampire population can grow. However, that being said there is an important oversight in the model: location is not accounted for.

The G factor

If a vampire population tried to live in Ottawa, they would find it very difficult. The reason is that many places in the city the scent of garlic can be smelt wafting from the many shawarma shops, where it is used as a sauce (tuum). Hence, during a feeding time vampires would find it difficult to maneuver their way through the city, especially downtown, as they would be assaulted by the aroma of garlic and forced to turn back.

However not every place in Ottawa is equipped which such a natural defense strategy. Expanding on this idea requires Google Maps and searching for the nearest shawarma places to a given location. However, location is not enough as garlic has an potent odor that is noticeable a several meters away, sometimes even a block to an acute sniffer, from the shop itself. Therefore, each shawarma shop also has an "effective radius" attached to it which is determined by the number of stars it has received from on-line reviews: the more stars, the larger the radius. The logic behind this is twofold. One, more stars means the more popular the restaurant is and hence the more people will eat there. Therefore, this aids in the circulation of garlic (directly, by ingestion on-site or take-out, or indirectly, by simply opening the door and letting wind currents do the rest). Second, more stars indicate better quality food and hence, by Middle Eastern standards, the more garlic-y the tuum. This means that even after one leaves the store, the garlic will be in their system and act as a natural vaccine as they go about their business. If more time was permitted, this would have been something to look into perhaps using periodic forcing.

In any case, two examples are drawn out: Carleton University and the University of Ottawa. Both of these are likely areas for vampires to attack as there is a large population is such small area. Moreover, if the vampires are like those seen in *Buffy the Vampire Slayer,* they are of the same age bracket and demographic of many of the students at either university. Therefore, they are better able to assimilate themselves into that sub-population. Figure 10 shows that near Carleton campus there is one shawarma shop and it is insufficient to ward off any attack; brute force along the northeast corridor and south passageway on Colonel By Drive may be the best defense option. However, Figure 11 tells a different story at the University of Ottawa, as this campus is located downtown and near the Byward Market. Hence, in case of an attack the vampires are forced to funnel themselves through certain passages where a small number of troops can be waiting. Incidentally, in both cases, both campuses are located near water. Therefore, in case of an assault an additional response would be to bless the water, thereby making it holy, and impossible for the vampires to cross.

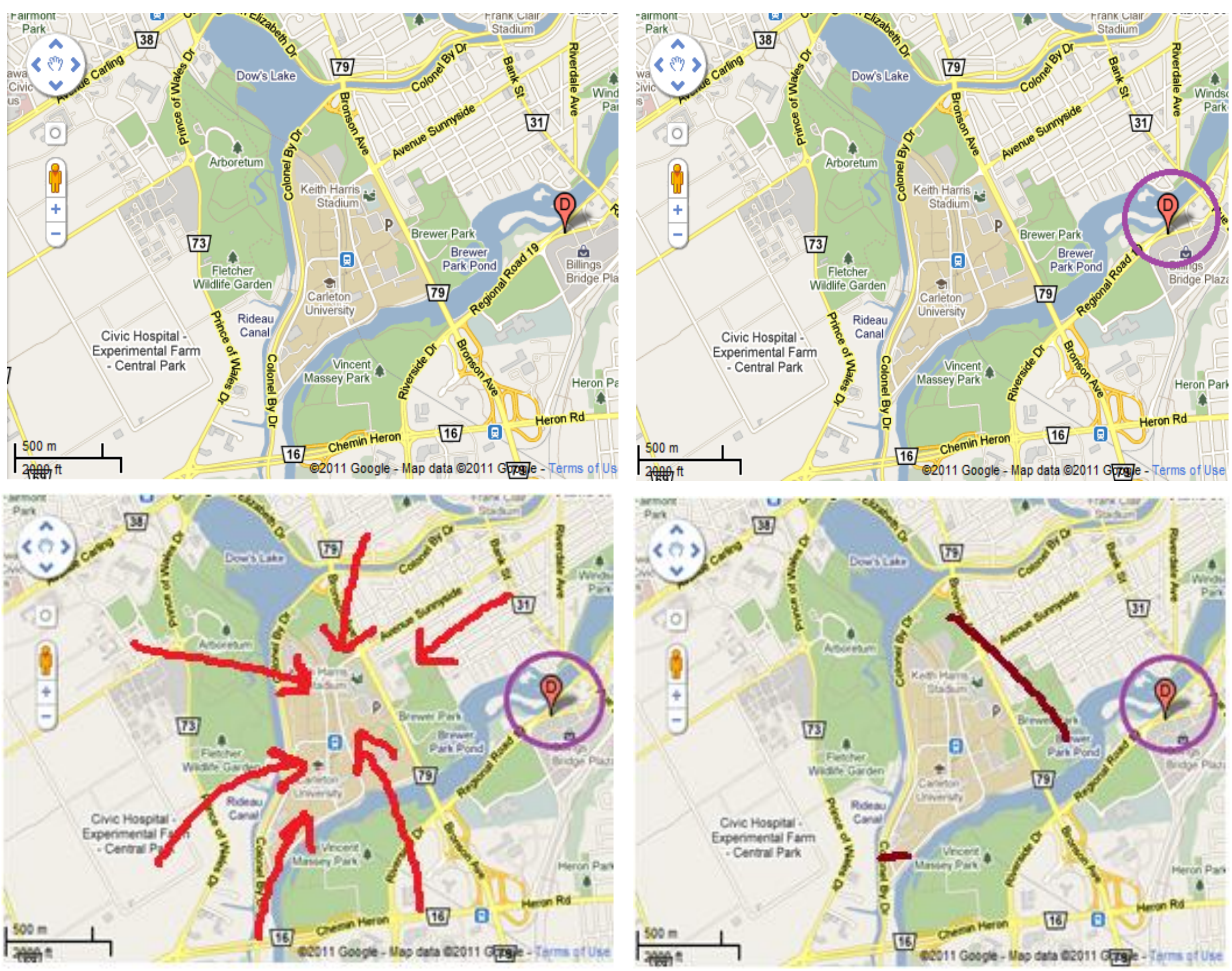

Figure 10 Maps of the Carleton University campus. (Top left panel) The marker indicates that there is only one shawarma store near Carleton. (Top right panel) The purple circle surrounding the marker, measures the “effective radius” the garlic smell produced by shop. (Bottom left panel) As one can see, the campus itself is open for attack in various directions, indicated by the red arrows. (Bottom right panel) It is suggested that defending the campus from the land, which is not immediately adjacent to water, is the best tactic. Such locations are drawn in burgundy. Additional methods include blessing the water to prevent vampire crossing on those fronts. (Photo taken from Google Maps, 2011a)

Figure 11 Maps of the University of Ottawa (main) campus). (Top left panel) The markers indicate that there are several shawarma shops in the downtown vicinity, many which are situated in the area to the northwest of campus in the Byward Market. (Top right panel) The purple circle surrounding the markers, measures the “effective radius” the garlic smell produced by the shop at its centre. (Bottom left panel) As one can see, this campus is well guarded with only a few avenues open for attack, indicated by the red arrows. (Bottom right panel) It is suggested that defending from the land, in stations marked in burgundy, is the best option. Additional preventative methods include blessing the water, as to make it impossible to cross for vampires. (Photo taken from Google Maps, 2011b)

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