The disease-free equilibrium satisfies $\bar{M} = \Omega/\mu_M$, $\bar{X}_U = [\pi(\mu(1-\epsilon p)+\omega)]/[\mu(\mu+\omega)]$, $\bar{X}_V = \epsilon p\pi/(\mu+\omega)$ and $\bar{N} = \bar{Y}_U = \bar{Y}_V = \bar{Q}_U = \bar{Q}_V = 0$.

Thus, the proportion of the population that is successfully vaccinated, S, satisfies $S = \bar{X}_V/(\bar{X}_U + \bar{X}_V) = \epsilon p\mu/(\mu+\omega)$. In particular, $\bar{X}_U = (\pi/\mu)(1-S)$ and $\bar{X}_V = (\pi/\mu)S$.

At the disease-free equilibrium, the Jacobian matrix is $J =$

$$\begin{bmatrix} \mu_M & 0 & 0 & 0 & -\beta_M\bar{M} & -\beta_M\bar{M} & 0 & 0 \\ 0 & -\mu_M & 0 & 0 & \beta_M\bar{M} & \beta_M\bar{M} & 0 & 0 \\ 0 & -\beta_U\bar{X}_U & -\mu & \omega & h_U & 0 & \delta_U & 0 \\ 0 & -(1-\psi)\beta_V\bar{X}_V & 0 & -\mu-\omega & 0 & h_V & 0 & \delta_V \\ 0 & \beta_U\bar{X}_U & 0 & 0 & -\xi_U & \omega & 0 & 0 \\ 0 & (1-\psi)\beta_U\bar{X}_V & 0 & 0 & 0 & -\xi_V-\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_U & 0 & -\mu-\delta_U & \omega \\ 0 & 0 & 0 & 0 & 0 & \alpha_V & 0 & -\mu-\delta_V-\omega \end{bmatrix}.$$

Thus, $\det(J - \Lambda I) = -(\mu_M + \Lambda)(\mu+\Lambda)(\mu+\omega+\Lambda)(\mu+\delta_U+\Lambda)(\mu+\delta_V+\omega+\Lambda)\det M$, where

$$M = \begin{bmatrix} -\mu_M - \Lambda & \beta_M\bar{M} & \beta_M\bar{M} \\ \beta_U\bar{X}_U & -\xi_U - \Lambda & \omega \\ (1-\psi)\beta_V\bar{X}_V & 0 & -\xi_V - \omega - \Lambda \end{bmatrix}.$$

Thus, the largest eigenvalue for J will be the largest eigenvalue for M. The vanishing determinant condition gives $-\mu_M\xi_U(\xi_V+\omega) + (1-\psi)\beta_V\beta_M\omega\bar{X}_V\bar{M} + (1-\psi)\xi_U\beta_V\beta_M\bar{X}_V\bar{M} + (\xi_V+\omega)\beta_U\beta_M\bar{X}_U\bar{M} = 0$. Hence,

$$\frac{(1-\psi)\beta_V\beta_M\bar{M}(\xi_U+\omega)}{\mu_M\xi_U(\xi_V+\omega)}\bar{X}_V + \frac{\beta_U\beta_M\bar{M}}{\mu_M\xi_U}\bar{X}_U = 1\,.$$

If there is no vaccine, $S = 0$, so $\bar{X}_V = 0$, $\bar{X}_U = \pi/\mu$ and hence the vanishing determinant condition gives $R_0 = \pi\Omega\beta_U\beta_M/\mu\mu_M^2\xi_U$. If the entire population is successfully vaccinated, $S = 1$ and $\omega = 0$, so $\bar{X}_V = \pi/\mu$, $\bar{X}_U = 0$ and hence the vanishing determinant condition gives $R_V = (1-\psi)(\pi\Omega\beta_V\beta_M/\mu\mu_M^2\xi_V)$. Thus, the population reproduction number is $R_P = (1-S)R_0 + SR_V$.

To estimate the minimum coverage levels p_c for an imperfect disease-modifying vaccine, when $R_P = 1$, this last equation can be rearranged to produce

$$S = \frac{\epsilon p_c\mu}{\mu+\omega} = \frac{1-R_0}{R_V-R_0}\,.$$

Thus, the threshold disease-modifying vaccine coverage level is

$$p_c = \frac{(\mu+\omega)(\mu+\gamma_V+\alpha_V+h_V)[\mu\mu_M^2(\mu+\gamma_U+\alpha_U+h_U)-\beta_U\beta_M\Omega\pi]}{\epsilon\mu\beta_M\Omega\pi[(1-\psi)\beta_V(\mu+\gamma_U+\alpha_U+h_U)-\beta_U(\mu+\gamma_V+\alpha_V+h_V)]}\,.$$

Vaccination programs whose coverage levels exceed this proportion of the population are likely to eradicate the disease.

Once a vaccine is introduced, the number of secondary infections will increase if $R_P > R_0$ (i.e., if the population reproduction number after the introduction of a vaccine is greater than the reproduction number currently). This occurs when

$$(1-S)R_0 + SR_V > R_0$$
$$\frac{\beta_V}{\beta_U} > \frac{\xi_V}{(1-\psi)\ \xi_U}\,.$$