The model is

$$\begin{aligned}
\frac{dS}{dt} &= \pi - \beta_H SN + hI + \delta R - \mu_H S \\
\frac{dI}{dt} &= \beta_H SN - hI - \alpha I - (\mu_H + \gamma) I \\
\frac{dR}{dt} &= \alpha I - \delta R - \mu_H R \\
\frac{dM}{dt} &= \Lambda - \mu M - \beta_M M I \\
\frac{dN}{dt} &= \beta_M M I - \mu N
\end{aligned} \tag{1}$$

for $t \neq t_k$, with impulsive conditions given by

$$\begin{aligned}
\Delta M &= -rM^- \\
\Delta N &= -rN^-
\end{aligned}$$

for $t = t_k$, where $\Delta M = M^+ - M^-$, $M^- \equiv M(t_k^-)$ and, equivalently, $M^+ \equiv M(t_k^+)$.

The disease-free equilibrium for the nonimpulsive model is given by

$$\begin{aligned}
E_0 &= (\bar{S}, \bar{I}, \bar{R}, \bar{M}, \bar{N}) \\
&= \left(\frac{\pi}{\mu_H}, 0, 0, \frac{\Lambda}{\mu}, 0 \right) .
\end{aligned}$$

The endemic equilibrium is given by

$$E_1 = (S^*, I^*, R^*, M^*, N^*) ,$$

where

$$\begin{aligned}
S^* &= \frac{\pi}{\mu_H} + \frac{\delta \alpha I^*}{\mu_H(\delta + mu_H)} - \frac{\alpha + \mu_H + \gamma}{\mu_H} I^* \\
R^* &= \frac{\alpha I^*}{\delta + \mu_H} \\
M^* &= \frac{\Lambda}{\mu + \beta_M I^*} \\
N^* &= \frac{\beta_M \Lambda I^*}{\mu(\mu + \beta_M I^*)}
\end{aligned}$$

and

$$I^* = \frac{[\beta_H \beta_M \Lambda \pi - (h + \alpha + \mu_H + \gamma)\mu^2 \mu_H](\delta + \mu_H)}{\beta_M[(\mu_H + \gamma)(\beta_H \Lambda + \mu)(\delta + \mu_H) + (\beta_M \Lambda + \mu)\alpha \mu_H + \mu h(\delta + \mu_H) + \mu \alpha \delta]} .$$

It can be seen that E_0 attracts the region

$$\Omega_0 = \{(S, I, R, M, N) : I = R = N = 0\} .$$

Theorem 1 *The basic reproductive ratio for model (1) is given by*

$$R_0 \quad = \quad \frac{\beta_H \beta_M \Lambda \pi}{\mu^2 \mu_H (\mu_H + \alpha + \gamma + h)} \,.$$

The disease-free equilibrium is stable if and only if $R_0 < 1$. Furthermore, the endemic equilibrium is positive if and only if $R_0 > 1$.

Proof. The Jacobian matrix for model (1) is

$$J \quad = \quad \begin{bmatrix} -\beta_H N - \mu_H & h & \delta & 0 & -\beta_H S \\ \beta_H N & -(h + \alpha + \mu_H + \gamma) & 0 & 0 & \beta_H S \\ 0 & \alpha & -(\delta + \mu_H) & 0 & 0 \\ 0 & -\beta_M M & 0 & -\mu - \beta_M I & 0 \\ 0 & \beta_M M & 0 & \beta_M I & -\mu \end{bmatrix} .$$

At the disease-free equilibrium,

$$J\Big|_{I=N=0} \quad = \quad \begin{bmatrix} -\mu_H & h & \delta & 0 & -\beta_H \bar{S} \\ 0 & -(h + \alpha + \mu_H + \gamma) & 0 & 0 & \beta_H \bar{S} \\ 0 & \alpha & -(\delta + \mu_H) & 0 & 0 \\ 0 & -\beta_M \bar{M} & 0 & -\mu & 0 \\ 0 & \beta_M \bar{M} & 0 & 0 & -\mu \end{bmatrix} .$$

The eigenvalues of this matrix satisfy the characteristic equation

$$(-\mu_H - \lambda)(-\delta - \mu_H - \lambda)(-\mu - \lambda) \det \begin{bmatrix} -(h + \alpha + \mu_H + \gamma) - \lambda & \beta_H \bar{S} \\ \beta_M \bar{M} & -\mu - \lambda \end{bmatrix} \quad = \quad 0 \,.$$

The only change in sign from the eigenvalues can occur from this last determinant, which satisfies

$$\lambda^2 + \lambda(\mu + h + \alpha + \mu_H + \gamma) + \mu(h + \alpha + \mu_H + \gamma) - \beta_H \beta_M \bar{S} \bar{M} \quad = \quad 0 \,.$$

This equation will have negative roots if $\mu(h + \alpha + \mu_H + \gamma) - \beta_H \beta_M \bar{S} \bar{M} > 0$; or, equivalently, if and only if

$$R_0 \quad \equiv \quad \frac{\beta_H \beta_M \Lambda \pi}{\mu^2 \mu_H (\mu_H + \alpha + \gamma + h)} \; < \; 1 \,.$$

Finally, I^* is clearly positive if and only if $R_0 > 1$.

□

When spraying events are included, the system will undergo an instantaneous jump when IRS is applied. We thus analyse model (1) when impulses are included. However, the mosquito dynamics shall prove to be far more important in the model than those of humans.

If we define the total mosquito population by

$$\Psi = M + N \,, \tag{2}$$

then we have the decoupled impulsive differential equation

$$\begin{aligned} \frac{d\Psi}{dt} &= \Lambda - \mu\Psi & t &\neq t_k \\ \Delta\Psi &= -r\Psi & t &= t_k\,. \end{aligned} \tag{3}$$

Thus,

$$\begin{aligned} \Psi^+ - \Psi^- &= -r\Psi^- \\ \Psi^+ &= (1-r)\Psi^-\,. \end{aligned} \tag{4}$$

Hence, for $t_k \leq t < t_{k+1}$,

$$\begin{aligned} \Psi'(t) + \mu\Psi(t) &= \Lambda \\ \frac{d}{dt}\left(e^{\mu t}\Psi\right) &= \Lambda e^{\mu t} \\ e^{\mu t}\Psi - e^{\mu t_k}\Psi(t_k^+) &= \frac{\Lambda}{\mu}e^{\mu t} - \frac{\Lambda}{\mu}e^{\mu t_k} \\ \Psi(t) &= \frac{\Lambda}{\mu}\left(1 - e^{\mu(t_k - t)}\right) + \Psi(t_k^+)e^{\mu(t_k - t)}\,. \end{aligned} \tag{5}$$

It follows that

$$\begin{aligned} \Psi_{k+1}^- &= \frac{\Lambda}{\mu}\left(1 - e^{-\mu(t_{k+1} - t_k)}\right) + \Psi_k^+ e^{-\mu(t_{k+1} - t_k)} \\ &= \frac{\Lambda}{\mu}\left(1 - e^{-\mu(t_{k+1} - t_k)}\right) + (1-r)\Psi_k^- e^{-\mu(t_{k+1} - t_k)}\,, \end{aligned} \tag{6}$$

using (4).

We thus have a recurrence relation for the total number of mosquitos immediately before spraying. This relation depends on the birth and death rates of mosquitos, the spraying times and the spraying effectiveness.

Theorem 2 *If spraying occurs at fixed times, satisfying $t_{k+1} - t_k = \tau$, then*

$$\tilde{\Psi}^-(r) = \frac{\Lambda}{\mu} \cdot \frac{1 - e^{-\mu\tau}}{1 + (r-1)e^{-\mu\tau}}\,. \tag{7}$$

is a globally asymptotically stable fixed point of the recurrence relation

$$\Psi_{k+1}^- = \frac{\Lambda}{\mu}\left(1 - e^{-\mu(t - t_k)}\right) + (1-r)\Psi_k^- e^{-\mu(t - t_k)}\,,$$

Proof. For completeness, define Ψ_0 to be the pre-image of $\Psi(0)$ under the impulsive condition. That is, $\Psi_0 = \frac{1}{1-r}\Psi(0)$. Then we have

$$\Psi_1^- = \frac{\Lambda}{\mu}\left(1 - e^{-\mu(t_1 - t_0)}\right) + (1-r)\Psi_0 e^{-\mu(t_1 - t_0)}$$

$$
\begin{aligned}
\Psi_2^- &= \frac{\Lambda}{\mu}\left(1-e^{-\mu(t_2-t_1)}\right)+(1-r)\Psi_1^- e^{-\mu(t_2-t_1)} \\
&= \frac{\Lambda}{\mu}\left(1-e^{-\mu(t_2-t_1)}\right)+(1-r)\frac{\Lambda}{\mu}\left(1-e^{-\mu(t_1-t_0)}\right)e^{-\mu(t_2-t_1)} \\
&\quad +(1-r)^2\Psi_0 e^{-\mu(t_1-t_0)}e^{-\mu(t_2-t_1)} \\
&= \frac{\Lambda}{\mu}\left(1-re^{-\mu(t_2-t_1)}-(1-r)e^{-\mu(t_2-t_0)}\right)+(1-r)^2\Psi_0 e^{-\mu(t_2-t_0)} \\
\Psi_3^- &= \frac{\Lambda}{\mu}\left(1-e^{-\mu(t_3-t_2)}\right)+(1-r)\Psi_2^- e^{-\mu(t_3-t_2)} \\
&= \frac{\Lambda}{\mu}\left(1-e^{-\mu(t_3-t_2)}\right)+(1-r)\frac{\Lambda}{\mu}\left(1-re^{-\mu(t_2-t_1)}\right. \\
&\quad \left.-(1-r)e^{-\mu(t_2-t_0)}\right)e^{-\mu(t_3-t_2)}+(1-r)^3\Psi_0 e^{-\mu(t_2-t_0)}e^{-\mu(t_3-t_2)} \\
&= \frac{\Lambda}{\mu}\left(1-re^{-\mu(t_3-t_2)}-r(1-r)e^{-\mu(t_3-t_1)}-(1-r)^2e^{-\mu(t_3-t_0)}\right) \\
&\quad +(1-r)^3\Psi_0 e^{-\mu(t_3-t_0)} \\
\Psi_4^- &= \frac{\Lambda}{\mu}\left(1-re^{-\mu(t_4-t_3)}-r(1-r)e^{-\mu(t_4-t_2)}-r(1-r)^2e^{-\mu(t_4-t_1)}\right. \\
&\quad \left.-(1-r)^3e^{-\mu(t_4-t_0)}\right)+(1-r)^4\Psi_0 e^{-\mu(t_4-t_0)} \\
&\vdots \\
\Psi_n^- &= \frac{\Lambda}{\mu}\left(1-\sum_{i=1}^{n-1} r(1-r)^{n-i-1}e^{-\mu(t_n-t_i)}-(1-r)^{n-1}e^{-\mu(t_n-t_0)}\right) \\
&\quad +(1-r)^n\Psi_0 e^{-\mu(t_n-t_0)}\,.
\end{aligned}
$$

For regular spraying, $t_n - t_i = (n-i)\tau$, so we have

$$
\begin{aligned}
\Psi_n^- &= \frac{\Lambda}{\mu}\left(1-\frac{re^{-\mu\tau}-(1-r)^{n-1}re^{-\mu\tau}}{1-(1-r)e^{-\mu\tau}}-(1-r)^{n-1}e^{-\mu n\tau}\right) \\
&\quad +(1-r)^n\Psi_0 e^{-\mu n\tau} \\
&\to \frac{\Lambda}{\mu}\left(1-\frac{re^{-\mu\tau}}{1-(1-r)e^{-\mu\tau}}\right)
\end{aligned}
$$

as $n \to \infty$, since $0 < r < 1$.

□

Corollary 1 *1. To reduce the total mosquito population below a desired threshold $\tilde{\Psi}$, the minimum spraying effectiveness satisfies*

$$
\tilde{r} = 1-\left[1-\frac{\Lambda}{\mu\tilde{\Psi}}(1-e^{-\mu\tau})\right]e^{\mu\tau}\,.
$$

2. To reduce the mosquito population below a desired threshold $\tilde{\Psi}$, the minimum *spraying period satisfies*

$$\tilde{\tau} \;=\; -\frac{1}{\mu}\ln\left[\frac{\Lambda-\mu\tilde{\Psi}}{\Lambda+\mu\tilde{\Psi}(r-1)}\right]. \qquad (8)$$

Proof. 1. Since $\Psi(t) \le \Psi^-$ for $t_k \le t \le t_{k+1}$, the maximum within each cycle occurs immediately before spraying is undertaken, so we can set $\tilde{\Psi} = \Psi^-$. By Theorem 2, we have

$$\begin{aligned}
\tilde{\Psi} &= \frac{\Lambda}{\mu}\cdot\frac{1-e^{-\mu\tau}}{1+(\tilde{r}-1)e^{-\mu\tau}}\\
1+(\tilde{r}-1)e^{-\mu\tau} &= \frac{\Lambda}{\mu\tilde{\Psi}}(1-e^{-\mu\tau})\\
\tilde{r} &= 1-\left[1-\frac{\Lambda}{\mu\tilde{\Psi}}(1-e^{-\mu\tau})\right]e^{\mu\tau}.
\end{aligned}$$

2. Similarly, we have

$$\begin{aligned}
\left(r-1+\frac{\Lambda}{\mu\tilde{\Psi}}\right)e^{-\mu\tilde{\tau}} &= \frac{\Lambda}{\mu\tilde{\Psi}}-1\\
\tilde{\tau} &= -\frac{1}{\mu}\ln\left[\frac{\Lambda-\mu\tilde{\Psi}}{\Lambda+\mu\tilde{\Psi}(r-1)}\right]
\end{aligned}$$

□

It follows that we can find the minimal spraying effectiveness or the minimal spraying period, in terms of the birth and death rates of mosquitos and the spraying effectiveness.

Theorem 3 *If spraying occurs at non-fixed times, then, assuming the two previous spraying events are known, the population of mosquitos can be reduced below the threshold $\tilde{\Psi}$ if the next spraying event satisfies*

$$t_{n+1} \;\le\; t_n-\frac{1}{\mu}\ln\left[\frac{2-r-\mu\tilde{\Psi}/\Lambda}{1+r(1-r)e^{-\mu(t_n-t_{n-1})}}\right].$$

Proof. For n large,

$$\Psi_n^- \;\approx\; \frac{\Lambda}{\mu}\left(1-\sum_{i=1}^{n-1} r(1-r)^{n-i-1}e^{-\mu(t_n-t_i)}\right),$$

since $(1-r)^{n-1} \approx 0$ and $e^{-\mu(t_n-t_0)} \approx 0$. If we assume $e^{-\mu(t_n-t_{n-2})}$ is small, then, using (6), we have

$$\begin{aligned}
\Psi_n^- &< \frac{\Lambda}{\mu}\left(1-re^{-\mu(t_n-t_{n-1})}\right)\\
\Psi_{n+1}^+ &< \frac{\Lambda}{\mu}\left(1-re^{-\mu(t_n-t_{n-1})}\right)+(1-r)\Psi_n^- e^{-\mu(t_{n+1}-t_n)}\\
&< \frac{\Lambda}{\mu}\left(1-re^{-\mu(t_n-t_{n-1})}\right)+(1-r)\frac{\Lambda}{\mu}\left(1-re^{-\mu(t_n-t_{n-1})}\right)e^{-\mu(t_{n+1}-t_n)}.
\end{aligned}$$

Define

$$\tilde{\Psi} \equiv \frac{\Lambda}{\mu}\left(1 - re^{-\mu(t_n - t_{m-1})}\right) + (1-r)\frac{\Lambda}{\mu}\left(1 - re^{-\mu(t_n - t_{n-1})}\right) e^{-\mu(t_{n+1}-t_n)} .$$

Thus,

$$\begin{aligned}
\frac{\Lambda}{\mu}\left(1 + (1-r)\right) - \tilde{\Psi} &= e^{-\mu(t_{n+1}-t_n)}\frac{\Lambda}{\mu}\left(1 + r(1-r)e^{-\mu(t_{n+1}-t_n)}\right) \\
e^{-\mu(t_{n+1}-t_n)} &= \frac{2 - r - \mu\tilde{\Psi}/\Lambda}{1 + r(1-r)e^{-\mu(t_n - t_{n-1})}} \\
t_{n+1} &= t_n - \frac{1}{\mu}\ln\left[\frac{2 - r - \mu\tilde{\Psi}/\Lambda}{1 + r(1-r)e^{-\mu(t_n - t_{n-1})}}\right] .
\end{aligned}$$

Hence, if spraying occurs at t_{n+1} or earlier, then the number of mosquitos will be less than or equal to $\tilde{\Psi}$, immediately after the $(n+1)^{th}$ spraying event. □

Thus, we can derive the "next best" spraying events for non-fixed spraying, by assuming that the time between the current spraying and that of two sprayings ago is sufficiently large.

Theorem 4 *If non-fixed spraying occurs indefinitely, then there exists a minimum spraying effectiveness r_0, satisying $0 < r_0 < 1$, such that variable spraying is only effective for $r_0 \le r \le 1$. Furthermore, on this interval, the minimum spraying interval for indefinite non-fixed spraying is always less than the minimum spraying interval for regular spraying.*

Proof. First, note that, for regular spraying, we have

$$\begin{aligned}
\tilde{\tau} &= -\frac{1}{\mu}\ln\left[\frac{\Lambda - \mu\tilde{\Psi}}{\Lambda + \mu\tilde{\Psi}(r-1)}\right] \\
\tilde{\tau}\big|_{r=0} &= -\frac{1}{\mu}\ln\left[\frac{\Lambda - \mu\tilde{\Psi}}{\Lambda - \mu\tilde{\Psi}}\right] \qquad (9) \\
&= 0 \\
\tilde{\tau}\big|_{r=1} &= -\frac{1}{\mu}\ln\left[\frac{\Lambda - \mu\tilde{\Psi}}{\Lambda}\right] \\
&= -\frac{1}{\mu}\ln\left[1 - \frac{\mu\tilde{\Psi}}{\Lambda}\right] . \qquad (10)
\end{aligned}$$

However, $\Psi = M + N$, so, if there is no impulse, then, from (3), $\lim_{t\to\infty}\Psi(t) = \Lambda/\mu$. Thus, we can assume that $\tilde{\Psi} < \Lambda/\mu$. Hence,

$$0 < 1 - \mu\tilde{\Psi}/\Lambda < 1 \qquad (11)$$

and thus $\tilde{\tau}\big|_{r=1} > 0$.

If non-fixed spraying occurs indefinitely, then let $\tau_{\rm nf} \equiv t_{n+1} - t_n = t_n - t_{n-1}$. The minimum spraying effectiveness then satisfies

$$\tau_{\rm nf} = -\frac{1}{\mu}\ln\left[\frac{2 - r - \mu\tilde{\Psi}/\Lambda}{1 + r(1-r)e^{-\mu\tau_{\rm nf}}}\right].$$

If $\tau_{\rm nf} = 0$, then

$$\begin{aligned} -\frac{1}{\mu}\ln\left[\frac{2 - r - \mu\tilde{\Psi}/\Lambda}{1 + r(1-r)}\right] &= 0 \\ 2 - r - \mu\tilde{\Psi}/\Lambda &= 1 + r(1-r) \\ r^2 - 2r + 1 - \mu\tilde{\Psi}/\Lambda &= 0 \\ r &= 1 \pm \sqrt{\mu\tilde{\Psi}/\Lambda} \end{aligned}$$

Clearly, the larger root exceeds unity and can hence be discounted. The smaller root, $r_0 \equiv 1 - \sqrt{\mu\tilde{\Psi}/\Lambda}$ satisfies $0 < r_0 < 1$, by (11). It follows that spraying is only effective in the range $r_0 \le r \le 1$.

Next, we have

$$\begin{aligned} \tau_{\rm nf}\Big|_{r=1} &= -\frac{1}{\mu}\ln\left[\frac{1 - \mu\tilde{\Psi}/\Lambda}{1}\right] \\ &= \tilde{\tau}\Big|_{r=1}, \end{aligned}$$

from (10).

Since $e^{-\mu\tau_{\rm nf}} < 1$, we have

$$\frac{\Lambda - \mu\tilde{\Psi}}{\Lambda + \mu\tilde{\Psi}(r-1)} \cdot \frac{1 + r(1-r)e^{-\mu\tau_{\rm nf}}}{2 - r - \mu\tilde{\Psi}/\Lambda} < \Lambda(\Lambda - \mu\tilde{\Psi})f(r)$$

where

$$f(r) = \frac{1 + r - r^2}{[\Lambda + \mu\Psi(r-1)][(2-r)\Lambda - \mu\Psi]}.$$

We can write

$$\begin{aligned} f(r) &= \frac{1 + r - r^2}{r^2\Lambda\mu\Psi - r(\Lambda^2 + 3\Lambda\mu\Psi - \mu^2\Psi^2) + 2\Lambda^2 + 3\Lambda\mu\Psi - \mu^2\Psi^2} \\ f'(r) &= \frac{\alpha}{[r^2\Lambda\mu\Psi - r(\Lambda^2 + 3\Lambda\mu\Psi - \mu^2\Psi^2) + 2\Lambda^2 + 3\Lambda\mu\Psi - \mu^2\Psi^2]^2}, \end{aligned}$$

where

$$\begin{aligned} \alpha &= [-r^2\Lambda\mu\Psi - r(\Lambda^2 - 3\Lambda\mu\Psi + \mu^2\Psi^2) + 2\Lambda^2 - 3\Lambda\mu\Psi + \mu^2\Psi^2](1-2r) \\ &\quad -(1 + r - r^2)[-2r\Lambda\mu\Psi - (\Lambda^2 - 3\Lambda\mu\Psi + \mu^2\Psi^2)] \\ &= r^2(-\Lambda^2 + 4\Lambda\mu\Psi - \mu^2\Psi^2) + 2r(-\Lambda^2 + \Lambda\mu\Psi) + 3\Lambda^2 + 2\mu^2\Psi^2 \\ &= \Lambda^2(-r^2 - 2r + 3) + \Lambda\mu\Psi(4r^2 + 2r) + \mu^2\Psi^2(2 - r^2). \end{aligned}$$

On the interval $0 < r < 1$, $-r^2 - 2r + 3 > 0$, $4r^2 + 2r > 0$ and $2 - r^2 > 0$. Thus, $\alpha > 0$ on $0 < r < 1$. It follows that $f(r)$ is increasing on this interval.

Furthermore,

$$f(1) \quad = \quad \frac{1}{\Lambda(\Lambda - \mu\Psi)}\,.$$

Consequently,

$$\frac{\Lambda - \mu\tilde{\Psi}}{\Lambda + \mu\tilde{\Psi}(r-1)} \cdot \frac{1 + r(1-r)e^{-\mu\tau_{\mathrm{nf}}}}{2 - r - \mu\tilde{\Psi}/\Lambda} \quad < \quad 1$$

and hence

$$\tilde{\tau} - \tau_{\mathrm{nf}} \quad = \quad -\frac{1}{\mu}\ln\left[\frac{\Lambda - \mu\tilde{\Psi}}{\Lambda + \mu\tilde{\Psi}(r-1)} \cdot \frac{1 + r(1-r)e^{-\mu\tau_{\mathrm{nf}}}}{2 - r - \mu\tilde{\Psi}/\Lambda}\right] > 0\,.$$

Thus, $\tilde{\tau} > \tau_{\mathrm{nf}}$ for $0 < r < 1$.

□

It follows that non-fixed spraying is always worse than regular spraying – even in the best-case scenario that such spraying is applied at regular intervals – and is only defined for a sufficiently effective insecticide.

As global temperatures increase, one of the major impacts will be an increase in the birth rate of mosquitos. If the mosquito birth rate is increased from Λ to $\Lambda + \Lambda_1$, then the recursion relation (6), with regular spraying, becomes

$$\Psi^-_{k+1} \quad = \quad \frac{\Lambda + \Lambda_1}{\mu}\left(1 - e^{-\mu\tau}\right) + (1-r)\Psi^-_k e^{-\mu\tau}\,.$$

This has solution

$$\tilde{\Psi}^- \quad = \quad \frac{\Lambda + \Lambda_1}{\mu} \cdot \frac{1 - e^{-\mu\tau}}{1 + (r-1)e^{-\mu\tau}}\,.$$

Rearranging, we have

$$\tilde{\tau} \quad = \quad -\frac{1}{\mu}\ln\left[\frac{\Lambda + \Lambda_1 - \mu\tilde{\Psi}}{\Lambda + \Lambda_1 + \mu\tilde{\Psi}(r-1)}\right]\,.$$

It follows that

$$\begin{aligned}\frac{\partial\tilde{\tau}}{\partial\Lambda_1} \quad &= \quad -\frac{r\tilde{\Psi}}{(\Lambda + \Lambda_1 - \mu\tilde{\Psi})(\Lambda + \Lambda_1 - \mu\tilde{\Psi} + r\mu\tilde{\Psi})} \\ &< \quad 0\,,\end{aligned}$$

since $\tilde{\Psi} < \frac{\Lambda}{\mu}$. Thus, as the mosquito birth rate increases, the minimal effective spraying period will always be reduced, for a fixed mosquito threshold $\tilde{\Psi}$. In particular, we have

$$\lim_{\Lambda_1 \to \infty} \tilde{\tau} \quad = \quad 0\,.$$