Mathematically, our model is represented as follows:

$$\frac{dC_U}{dt} = \pi_W(1 - \epsilon p) - \alpha C_U - \mu_C C_U$$

$$\frac{dC_V}{dt} = \pi_W \epsilon p - \alpha C_V - \mu_C C_V$$

$$\frac{dA_U}{dt} = \alpha C_U - f(\bar{\epsilon}\bar{p})A_U - \mu A_U - \beta_N A_U N$$

$$\frac{dA_V}{dt} = \alpha C_V + f(\bar{\epsilon}\bar{p})A_U - \mu A_V - (1 - \psi)\beta_N A_V N$$

$$\frac{dI_U}{dt} = \beta_N A_U N - \mu I_U$$

$$\frac{dI_V}{dt} = (1 - \psi)\beta_N A_V N - \mu I_V$$

$$\frac{dM}{dt} = \pi_M - \beta_M I_U M - \mu M - \beta_M I_V M$$

$$\frac{dN}{dt} = \beta_M I_U M - \mu N + \beta_M I_V M \, .$$

with the function f given by

$$f(\bar{\epsilon}\bar{p}) = \frac{c\bar{\epsilon}\bar{p}}{1 - \bar{\epsilon}\bar{p} + \gamma} \, .$$

The disease-free equilibrium satisfies

$$\bar{C}_U = \frac{\pi_W(1 - \epsilon p)}{\alpha + \mu_C}$$

$$\bar{C}_V = \frac{\pi_W \epsilon p}{\alpha + \mu_C}$$

$$\bar{A}_U = \frac{\alpha C_U}{f + \mu}$$

$$\bar{A}_V = \frac{\alpha C_V + f A_U}{\mu}$$

$$\bar{M} = \frac{\pi_M}{\mu}$$

and $\bar{I}_U = \bar{I}_V = \bar{N} = 0$. We assume $0 \le \epsilon, \bar{\epsilon}, p, \bar{p} \le 1$. All other parameters are assumed to be positive.

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At the disease-free equilibrium, the Jacobian matrix is

$$
J = \begin{bmatrix}
-\mu_C - \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\mu_C - \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & 0 & -f - \mu & 0 & 0 & 0 & 0 & -\beta_N A_U \\
0 & \alpha & f & -\mu & 0 & 0 & 0 & -(1-\psi)\beta_N A_V \\
0 & 0 & 0 & 0 & -\mu & 0 & 0 & \beta_N A_U \\
0 & 0 & 0 & 0 & 0 & -\mu & 0 & (1-\psi)\beta_N A_V \\
0 & 0 & 0 & 0 & -\beta_M M & -\beta_M M & -\mu & 0 \\
0 & 0 & 0 & 0 & \beta_M M & \beta_M M & 0 & -\mu
\end{bmatrix} .
$$

Then $\det(J - \lambda I) = (-\mu_C - \alpha - \lambda)^2(-\mu - \lambda)^2(-f - \mu - \lambda)\det M$, where

$$
M = \begin{bmatrix}
-\mu - \lambda & 0 & \beta_N A_U \\
0 & -\mu - \lambda & (1-\psi)\beta_N A_V \\
\beta_M M & \beta_M M & -\mu - \lambda
\end{bmatrix} .
$$

Thus, the largest eigenvalue for J will be the largest eigenvalue for M and so we can reduce the problem to solving

$$
\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma = 0
$$

where

$$
\begin{aligned}
\alpha &= 3\mu \\
\beta &= 3\mu^2 - (1-\psi)\beta_M\beta_N M A_V + \beta_N A_U \beta_M M \\
\gamma &= \mu^3 - (1-\psi)\mu\beta_M\beta_N M A_V + \mu\beta_N A_U \beta_M M .
\end{aligned}
$$

By the Routh-Hurwitz condition, all roots will have negative real parts if $\alpha > 0$, $\gamma > 0$ and $\alpha\beta - \gamma > 0$. Clearly $\alpha > 0$. We can write the third condition as

$$
\alpha\beta - \gamma = 6\mu^3 + 2\gamma ,
$$

which will be positive if $\gamma > 0$. Thus, all roots will have negative real part if

$$
\mu^2 - \beta_N\beta_M M[A_U + (1-\psi)A_V] > 0 .
$$

Solving for ϵp in terms of $\bar{\epsilon}\bar{p}$ and substituting equilibrium values, our eradication threshold is thus

$$
\epsilon p = \frac{1}{\psi\mu}\left[\mu + f(\bar{\epsilon}\bar{p})(1-\psi) - \frac{\mu^4(\mu + f(\bar{\epsilon}\bar{p}))(\alpha + \mu_C)}{\beta_M\beta_N\pi_M\pi_W\alpha}\right] , \qquad (1)
$$

with

$$f(\bar{\epsilon}\bar{p}) \;=\; \frac{c\bar{\epsilon}\bar{p}}{1 - \bar{\epsilon}\bar{p} + \gamma}\;.$$

Differentiating (1), we have

$$\frac{\partial p}{\partial \bar{p}} \;=\; \frac{c\bar{\epsilon}(1+\gamma)}{\epsilon\psi\mu(1-\bar{\epsilon}\bar{p}+\gamma)^2}\left[(1-\psi) - \frac{\mu^4(\alpha+\mu_C)}{\beta_N\beta_M\pi_M\pi_W\alpha}\right]\;.$$

It follows that there is a critical vaccine efficacy ψ^*, satisfying

$$\psi^* \;=\; 1 - \frac{\mu^4(\alpha+\mu_C)}{\beta_M\beta_N\pi_M\pi_W\alpha}$$

such that if $\psi < \psi^*$, then eradication of targeted types is not possible. Thus, even if the vaccine mounts an immune response 100% of the time and we can vaccinate 100% of the population, if the efficacy is below this threshold, then the disease will persist.

By setting $p = 1$ and $\bar{p} = 0$, it also follows from (1) that there is a critical immunogenicity value ϵ^*, satisfying

$$\epsilon^* \;=\; \frac{1}{\psi}\left[1 - \frac{\mu^4(\alpha+\mu_C)}{\beta_N\beta_M\pi_M\pi_W\alpha}\right]$$

such that if $\epsilon < \epsilon^*$, then even 100% childhood vaccination coverage will not eradicate targeted types of the disease.

From (1), when $p = 1$, we have

$$\epsilon \;=\; \epsilon^* + \frac{f(\bar{\epsilon}\bar{p})}{\psi\mu}\left[(1-\psi) - \frac{\mu^4(\alpha+\mu_C)}{\beta_N\beta_M\pi_M\pi_W\alpha}\right]\;.$$

Define

$$\theta \;\equiv\; \frac{\psi\mu(\epsilon^* - \epsilon)}{\frac{\mu^4(\alpha+\mu_C)}{\beta_N\beta_M\pi_M\pi_W\alpha} - (1-\psi)}\;.$$

For $\psi > \psi^*$, and $\epsilon < \epsilon^*$, $\theta > 0$. It follows that the minimum level of adult vaccination required for eradication of targeted types, $\bar{p}^*$, satisfies

$$\bar{p}^* \;=\; \frac{\theta(1+\gamma)}{\bar{\epsilon}(c+\theta)}\;.$$

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Since $\theta > 0$, it follows that $\bar{p}^* > 0$. If the immunogenicities ϵ and $\bar{\epsilon}$ are not too small, then $\bar{p}^* < 1$ (since γ is small). Note that we are not assuming that childhood immunogenicity is necessarily the same as adult immunogenicity. Thus, if childhood immunogenicity is below ϵ^* (but not so small that the vaccine is nonfunctional), then there is a minimum level of adult vaccination coverage that must be achieved for eradication of targeted types.

To examine sensitivity of results on parameter variation, we used the output parameter as the proportion of adults who should be vaccinated in order to eradicate targeted types. Thus, rearranging equation (1), we have

$$\bar{p} = \frac{\delta(1+\gamma)}{\bar{\epsilon}(c+\delta)},$$

where

$$\delta = \frac{\beta_M \beta_N \pi_M \pi_W \alpha}{(1-\psi)\beta_M \beta_N \pi_M \pi_W \alpha - \mu^4(\alpha+\mu_C)} \left(\epsilon\mu\psi p + \frac{\mu^5(\alpha+\mu_C)}{\beta_M \beta_N \pi_M \pi_W \alpha} - \mu \right).$$