Our model consists of five ordinary differential equations, which specify the rate of change over time of five categories of individuals: susceptible individuals (who are unvaccinated or vaccine failures) (X), "successfully" vaccinated individuals (V), HIV-infected individuals who were previously "successfully" vaccinated (Y_V), HIV-infected individuals who were never vaccinated or who were vaccine failures (Y_U), and AIDS cases (A). The size of the sexually active community is specified by N (where $N = X + V + Y_V + Y_U$). New susceptible individuals enter the sexually active community at rate π, and they leave at an average rate μ. A fraction p of newly susceptible individuals are vaccinated, and the vaccine "takes" (i.e., produces a protective immunological response) in a fraction ε of those vaccinated; therefore, the fraction of new susceptibles that enter the susceptible pool is $(1 - \varepsilon p)$, and the fraction that is "successfully" vaccinated is εp. Susceptible individuals become HIV-infected at rate $\lambda c X$, where λ is the average per capita risk of infection and c is the average number of new sex partners acquired per unit time. The average per capita risk of infection (λ) is calculated as the product of two factors: the per partnership transmission probability and the fraction of the community that is infected and infectious. Thus, $\lambda = [\beta_U (Y_U/N) + \beta_V (Y_V/N)]$, where β_U represents the per partnership transmission probability of Y_U individuals and β_V represents the per partnership transmission probability of Y_V individuals. Vaccine-induced immunity in the "successfully" vaccinated wanes at a rate ω; thus the average duration of vaccine-induced immunity is $1/\omega$ years, and the number of "successfully" vaccinated individuals entering the susceptible pool (per unit time) is ωV. Hence, the rate of change in the number of susceptible individuals (X) per unit time is specified by:

$$dX/dt = (1 - \varepsilon p)\pi - \mu X - \lambda c X + \omega V. \qquad (1)$$

The fraction of new susceptibles in whom the vaccine "takes" enter the "successfully" vaccinated state at rate $\varepsilon p \pi$. They may leave this state for one of three reasons: they may leave the community (at average rate μ), their vaccine-induced immunity may wane (at an average rate ω), or they acquire HIV-infection. The degree of vaccine-induced protection against HIV infection is ψ; thus, their probability of becoming HIV-infected is $\lambda\, c\,(1 - \psi)$. Hence, the rate of change in the number of "successfully" vaccinated individuals (V) per unit time is specified by:

$$dV/dt = \varepsilon p \pi - \mu\, V - \omega\, V - (1 - \psi)\, c \lambda\, V. \tag{2}$$

"Successfully" vaccinated individuals (i.e., individuals in whom the vaccine "took" and did not wane) who become HIV-infected enter the infectious class Y_V. Individuals leave this class if they leave the sexually active community (at average rate μ) or as they progress to AIDS (at average rate γ_V). Thus the rate of change in the number of infectious individuals (Y_V) per unit time is specified by:

$$dY_V/dt = \lambda\, c(1 - \psi)\, V - (\mu + \gamma_V)\, Y_V. \tag{3}$$

Unvaccinated individuals and vaccine failures (i.e., individuals in whom the vaccine either did not "take," or in whom the vaccine "took" but waned) who become HIV-infected enter the infectious class Y_U. Individuals leave this infectious class if they leave the sexually active community (at average rate μ) or if they progress to AIDS (at average rate γ_U). Thus the rate of change in the number of infectious individuals (Y_U) per unit time is specified by:

$$dY_U/dt = \lambda\, cX - (\mu + \gamma_U)\, Y_U. \tag{4}$$

The number of AIDS cases (A) per unit time increases at rate $\gamma_V Y_V + \gamma_U Y_U$ (due to the incidence of disease) and decreases as AIDS patients leave the community (at average rate μ) or die of AIDS (at rate α):

$$dA/dt = \gamma_V Y_V + \gamma_U Y_U - (\mu + \alpha)\, A. \tag{5}$$

At the disease-free equilibrium, $\lambda = 0$, since $Y_U = Y_V = 0$. Clearly $dY_V/dt = dY_U/dt = dA/dt = 0$. Thus, the remaining nonzero equations satisfy

$$\begin{aligned} dX/dt &= (1-\epsilon p)\pi - \mu X + \omega V \\ dV/dt &= \epsilon p \pi - \mu V - \omega V \,. \end{aligned}$$

From the second equation,

$$\bar{V} = \frac{\epsilon p \pi}{\mu + \omega} \,.$$

We then have

$$\begin{aligned} \bar{X} &= \frac{1}{\mu}\left[(1-\epsilon p)\pi + \frac{\epsilon p \pi \omega}{\mu+\omega}\right] \\ &= \frac{1}{\mu}\left[\frac{\pi\mu - \epsilon p \pi \mu + \pi\omega}{\mu+\omega}\right] \\ \bar{X} + \bar{V} &= \frac{1}{\mu}\left[\frac{\pi\mu - \epsilon p \pi \mu + \pi\omega + \epsilon p \pi \mu}{\mu+\omega}\right] \\ &= \frac{\pi}{\mu} \,. \end{aligned}$$

Thus, the probability of "successful" vaccination is

$$\begin{aligned} S &= \frac{\bar{V}}{\bar{X}+\bar{V}} \\ &= \frac{\epsilon p \mu}{\mu + \omega} \,. \end{aligned}$$

If there is no vaccination, then (ignoring the AIDS equation, which decouples from the others), the system

simplifies to

$$\begin{aligned} dX/dt &= (1-\epsilon p)\pi - \mu X - \Lambda c X \\ dY_U/dt &= \lambda c X - (\mu+\gamma_U) Y_U \,. \end{aligned}$$

Note that

$$\begin{aligned} \frac{\partial \lambda}{\partial X} &= \frac{N-X}{N^2} \\ \frac{\partial \lambda}{\partial Y_U} &= \beta_U \frac{N-Y_U}{N^2} \,. \end{aligned}$$

Then the Jacobian is

$$J = \begin{bmatrix} -\mu - \lambda \frac{N-X}{N^2} cX - \lambda c & -\beta_U \frac{N-Y_U}{N^2} cX \\ \lambda c + \lambda \frac{N-X}{N^2} cX & \beta_U \frac{N-Y_U}{N_2} cX - \mu - \gamma_U \end{bmatrix} .$$

At the disease-free equilibrium, $Y_U = 0$, $\lambda = 0$ and $X = N$. Thus

$$J\big|_{\text{DFE}} = \begin{bmatrix} -\mu & -\beta_U c \\ 0 & \beta_U c - \mu - \gamma_U \end{bmatrix} .$$

It follows that

$$R_0 = \frac{\beta_U c}{\mu + \gamma_U} \,.$$

For the entire system, we have

$$J\big|_{\text{DFE}} = \begin{bmatrix} -\mu & \omega & -\beta_V c\frac{X}{N} & -\beta_U c\frac{X}{N} & 0 \\ 0 & -\mu-\omega & -(1-\psi)\beta_V c\frac{V}{N} & -(1-\psi)\beta_U c\frac{V}{N} & 0 \\ 0 & 0 & (1-\psi)\beta_V c\frac{V}{N} - \mu - \gamma_V & (1-\psi)\beta_U c\frac{V}{N} & 0 \\ 0 & 0 & \beta_V c\frac{X}{N} & \beta_U c\frac{X}{N} - \mu - \gamma_V & 0 \\ 0 & 0 & \gamma_V & \gamma_U & -\mu-\alpha \end{bmatrix}.$$

The disease-free equilibrium is stable if

$$\det \begin{bmatrix} (1-\psi)\beta_V c\frac{V}{N} - \mu - \gamma_V & (1-\psi)\beta_U c\frac{V}{N} \\ \beta_V c\frac{X}{N} & \beta_U c\frac{X}{N} - \mu - \gamma_V \end{bmatrix} < 0\,.$$

Equivalently,

$$\begin{aligned} -(1-\psi)\beta_V c\frac{V}{N}(\mu+\gamma_U) - \beta_U c\frac{X}{N}(\mu+\gamma_V) & \\ +(\mu+\gamma_U)(\mu+\gamma_V) &< 0 \\ R_p \;\equiv\; \frac{\beta_U c}{\mu+\gamma_U}\cdot\frac{X}{N} + \frac{(1-\psi)\beta_V c}{\mu+\gamma_V}\cdot\frac{V}{N} &< 1\,. \end{aligned}$$

We have

$$\begin{aligned} S &= \frac{V}{X+V} = \frac{V}{N} \\ 1-S &= \frac{X+V-V}{X+V} = \frac{X}{N}\,. \end{aligned}$$

It follows that

$$R_p = S(1-\psi)R_V + (1-S)R_0\,.$$

For perversity where the infected unvaccinated change their behavior by a multiplicative factor m_U and the infected vaccinated change their behavior by a multiplicative factor m_V, we require

$$S(1-\psi)R_V m_V + (1-S)R_0 m_U > R_0$$
$$f > \frac{1-m_U + m_U S}{S(1-\psi)m_V} \equiv z.$$

Note that

$$\frac{\partial z}{\partial S} = \frac{m_U - 1}{S^2(1-\psi)m_V}.$$

Thus if $m_U > 1$ then z is increasing with respect to S, whereas if the reverse inequality holds, then z is decreasing with respect to S. Thus, the behavior changes in the unvaccinated are a critical determinant of the epidemic control strategy for vaccines that offer a low degree of protection.