MAT4996 Assignment 5

1. Consider the one-dimensional impulsive system studied in class

$$\begin{aligned} x' &= -\alpha x & t &\neq t_k \\ \Delta x &= c & t &= t_k\,, \end{aligned}$$

with $\tau = t_{k+1} - t_k$. Show that the impulsive orbit, given by

$$x(t) = \frac{ce^{-\alpha(t-t_k)}}{1-e^{-\alpha\tau}} \qquad t_k < t \leq t_{k+1}$$

with endpoints

$$\begin{aligned} x(t_k^+) &= \frac{c}{1-e^{-\alpha\tau}} \\ x(t_k^-) &= \frac{ce^{-\alpha\tau}}{1-e^{-\alpha\tau}} \end{aligned}$$

is stable.

2. Consider the spread of Methicilin-Resistant Staphylococcus Aureus (MRSA) in prisons, where new (susceptible) prisoners are transferred in when there is room, while infected prisoners are quarantined at regular intervals.

   (a) Suppose both events happen simultaneously. Then

$$\begin{aligned} S' &= -\beta SI & S &\neq C \\ I' &= \beta SI - dI & S &\neq C \\ \Delta S &= \lambda & S &= C \\ \Delta I &= -\alpha I & S &= C \end{aligned}$$

   i. Under what conditions will there be an impulsive periodic orbit?
   ii. Sketch the impulsive periodic orbit in the S-I plane.
   iii. Show that this orbit is orbitally asymptotically stable.
   iv. Find the period of the periodic orbit.

   (b) Let $n > 0$ be an integer and assume that new prisoners arrive at rate λ at each impulsive effect, while infected prisoners are quarantined only at the moments of impulsive effect τ_k whose ordinal number k is a multiple of n. That is,

$$\begin{aligned} S' &= -\beta SI & & & S &\neq C \\ I' &= \beta SI - dI & & & S &\neq C \\ \Delta S(\tau_k) &= \lambda & & & S &= C \\ \Delta I(\tau_k) &= \begin{cases} 0 & \text{if } k \text{ is not divisible by } n \\ -\alpha I(\tau_k) & \text{if } k \text{ is divisible by } n \end{cases} & & & S &= C \end{aligned}$$

   i. Under what conditions will there be an impulsive periodic orbit?
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   iii. Show that this orbit is orbitally asymptotically stable.
   iv. Find the period of the periodic orbit.