MAT4996 Assignment 4

1. An epidemic of a communicable disease that does not cause death but from which infectives do not recover may be modelled by the pair of differential equations

$$\begin{aligned} S' &= -\beta SI \\ I' &= \beta SI \end{aligned}$$

   Show that, in a population of fixed size K, such a disease will eventually spread to the entire population.

2. If a fraction λ of the population susceptible to a disease that provides immunity against reinfection moves out of the region of an epidemic, the situation may be modelled by the system

$$\begin{aligned} S' &= -\beta SI - \lambda S \\ I' &= \beta SI - \alpha I\,. \end{aligned}$$

   Show that both S and I approach zero as $t \to \infty$.

3. Consider a disease where, upon infection, susceptibles S first move to an exposed, noninfectious class E. Exposed individuals move to the infectious class I at constant rate κ. Infected individuals recover at constant rate α and thus move the recovered class R. All individuals have background death rate μ. You may assume a constant birth rate π of susceptibles.

   (a) Draw the diagram of the model.
   (b) Write down the differential equations.
   (c) Calculate the disease-free equilibrium.
   (d) Use the next-generation method to calculate $R_{0,N}$.
   (e) Show that the disease-free equilibrium is (locally asymptotically) stable for $R_{0,N} < 1$ and unstable for $R_{0,N} > 1$. (Hint: Use the Jacobian method.)

4. Consider a disease spread by carriers who transmit the disease without exhibiting symptoms themselves. Let $C(t)$ be the number of carriers and suppose that carriers are identified and isolated from contact with others at a constant per capita rate α, so that $C' = -\alpha C$. The rate at which susceptibles become infected is proportional to the number of carriers and to the number of susceptibles, so that $S' = -\beta SC$. Let C_0 and S_0 be the number of carriers and susceptibles, respectively, at time $t = 0$.

   (a) Determine the number of carriers at time t from the first equation.
   (b) Substitute the solution to part (a) into the second equation and determine the number of susceptibles at time t.
   (c) Find

$$\lim_{t\to\infty} S(t)\,,$$

   the number of members of the population who escape the disease.