MAT4996/5187 Assignment 2

1. (Next-generation method) Consider an SEIR model

$$\begin{aligned}
\dot{S} &= \Omega - \beta S I - \mu S \\
\dot{E} &= \beta S I - (\mu + k)E \\
\dot{I} &= kE - (\gamma + \mu)I \\
\dot{R} &= \gamma I - \mu R,
\end{aligned} \tag{1}$$

   where Ω is the birth rate, μ is the per capita natural death rate, β is the efficacy of infection of susceptible individuals S, k is the rate at which a latent individual becomes infectious and γ is the per capita recovery rate.

   (a) Show that

   $$F = \begin{pmatrix} 0 & \beta\Omega/\mu \\ 0 & 0 \end{pmatrix}.$$

   Hint: only two compartments are infected.

   (b) Show that

   $$V = \begin{pmatrix} \mu + k & 0 \\ -k & \gamma + \mu \end{pmatrix}.$$

   (c) Find V^{-1}.

   (d) Finally, show that

   $$R_{0,N} = \frac{k\beta\Omega}{(\mu + k)(\mu + \gamma)\mu}. \tag{2}$$

   (Note that this is also the value of R_0 determined by the survival function method.)

2. (Jacobian) Calculate the Jacobian matrix for model (1) and find the eigenvalues for the disease-free equilibrium. Does this match the R_0 value found here? Which method do you like better?

3. (Endemic equilibrium) Consider this model (from Blower *et al.*, 1998) of herpes simplex virus:

$$\begin{aligned}
\frac{dX}{dt} &= \pi - Xc\beta\frac{H}{N} - X\mu \\
\frac{dQ}{dt} &= H(\sigma + q) - Q(\mu + r) \\
\frac{dH}{dt} &= Xc\beta\frac{H}{N} - H(\mu + \sigma + q) + rQ,
\end{aligned}$$

   where X is the susceptible population, Q represents those infected with the virus in the non-infectious latent state, H represents those infected with the virus in infectious state and $N = X + Q + H$. (Other letters are positive parameters.)

(a) Show that, at equilibrium,

$$\begin{aligned}
\bar{N} &= \frac{\pi}{\mu} \\
\bar{X} &= \frac{\pi}{\mu} - \frac{\mu+\sigma+q+r}{\mu+r}\bar{H} \\
\bar{Q} &= \frac{\sigma+q}{\mu+r}\bar{H},
\end{aligned}$$

where $\bar{H}$ is yet to be determined.

(b) Find the disease-free equilibrium.

(c) Show that, if $\bar{H} \neq 0$, then

$$\bar{H} = \frac{\pi}{\mu}\left[\frac{\mu+r}{\mu+\sigma+q+r} - \frac{\mu}{c\beta}\right].$$

(d) Show that the endemic equilibrium only exists when

$$R_{0,E} \equiv c\beta\left(\frac{r+\mu}{\mu(r+\mu+\sigma+q)}\right) > 1$$

and does not exist if the reverse inequality holds.