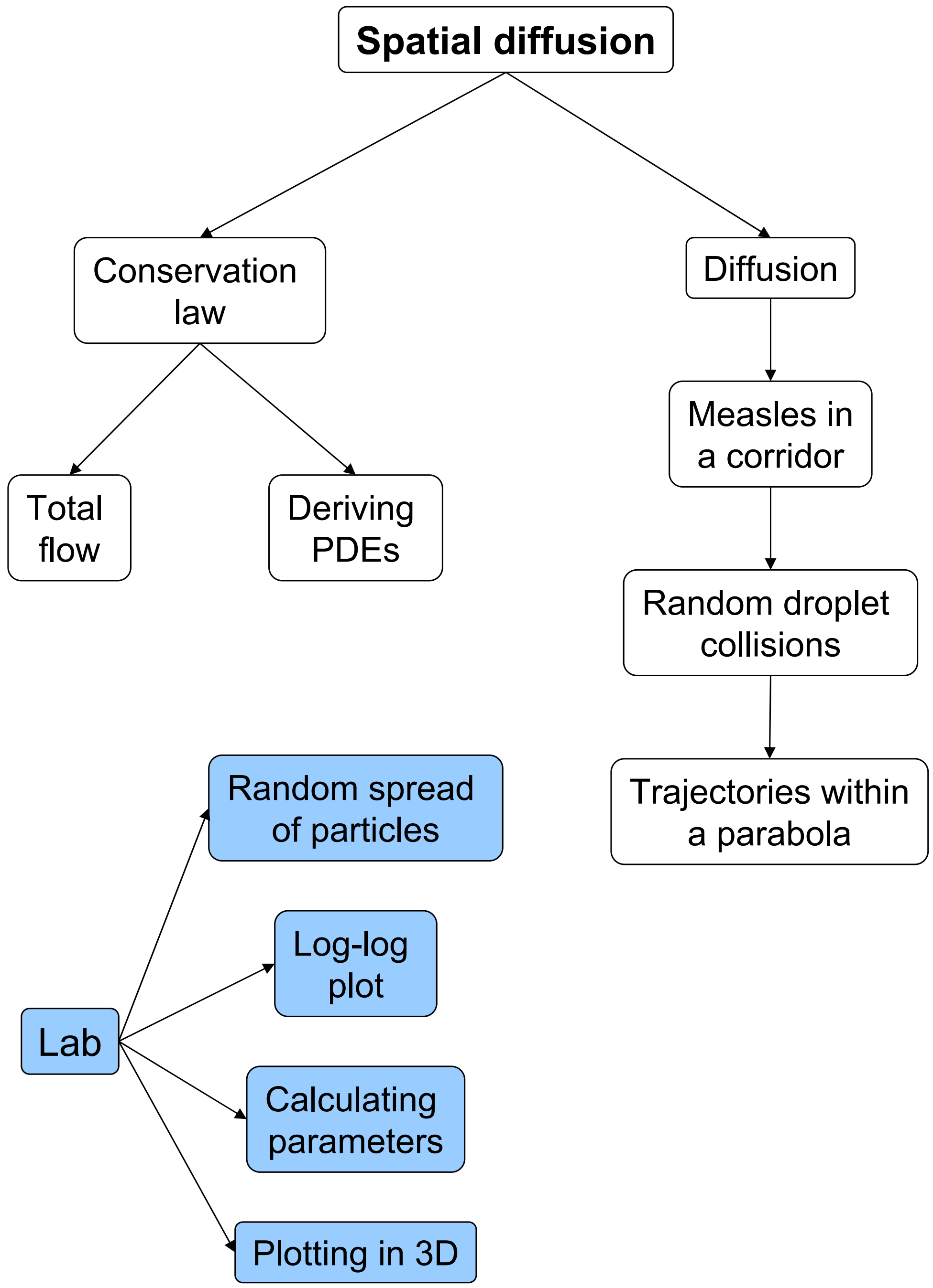
Spatial models

- Many systems change with respect to both space and time
- Instead of ODEs, we use PDEs (Partial Differential Equations)
- PDEs → differential equations with more than one independent variable.

ODEs = Ordinary Differential Equations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One spatial dimension

- We'll illustrate the methods using one spatial variable
- Two dimensional models with time
- More spatial dimensions...

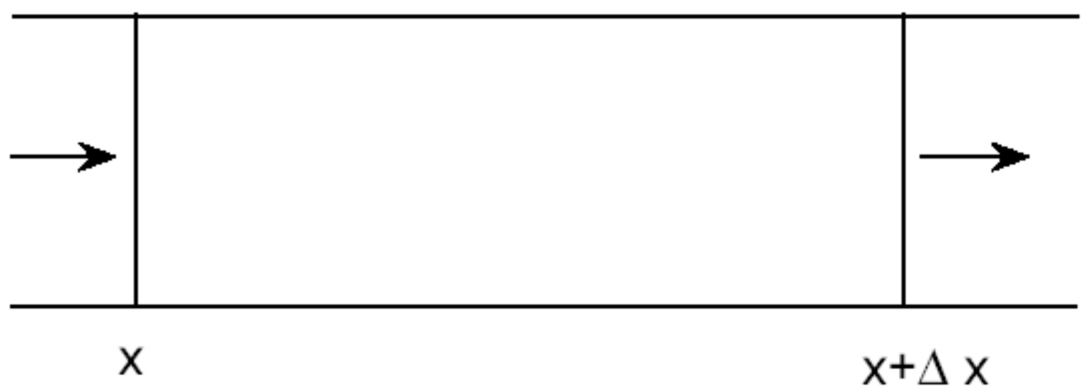
→ things get *very* complicated 😐.

The conservation law

- Mass can neither be created or destroyed
- We'll model infectious measles droplets
- We must ensure that droplets do not spontaneously appear or disappear.

A stretch of space

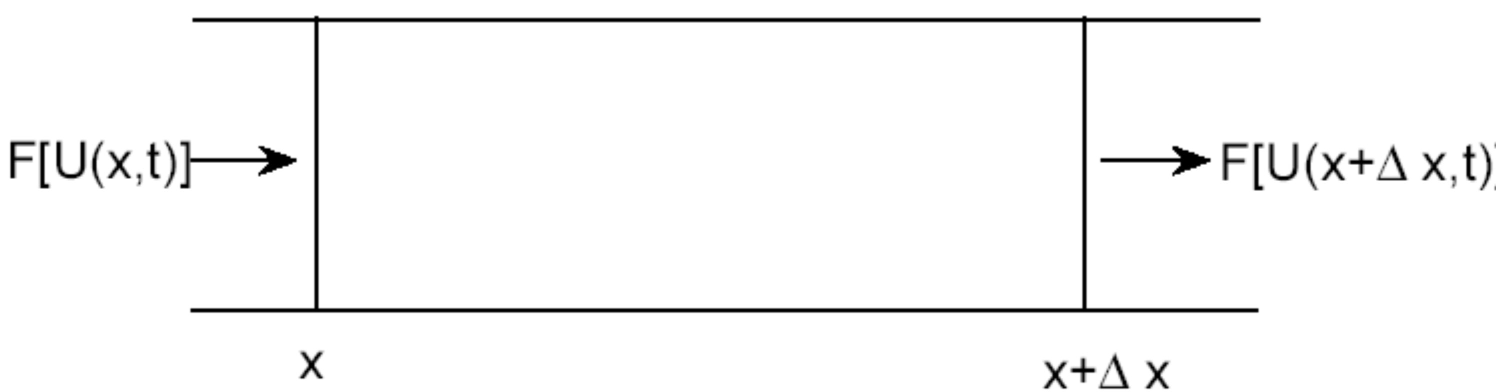
- Consider a stretch of space Δx long
- Infected measles droplets flow along this space.



x = space

The flow of droplets

- The density at a particular location and time is $U(x,t)$
- The flow of droplets F will be a function of the density
- Thus $F = F[U(x,t)]$.



F = flow of droplets x = space U = droplet density t = time

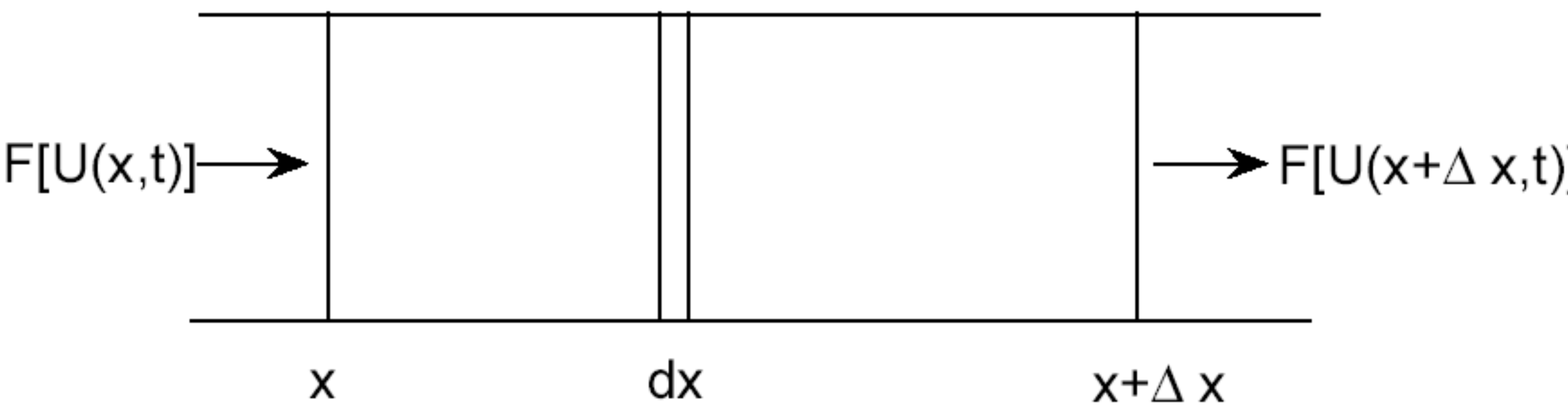
The total flow (part 1)

- Thus the total flow is $F[U(x,t)]-F[U(x+\Delta x,t)]$ (The inward flow minus the outward flow)
- Let's hold that thought for a moment.

F = flow of droplets x = space
U = droplet density t = time

Another method

- There's another way to calculate the total flow in this stretch of space
- Consider a small length of space *dx* wide.



F = flow of droplets x = space
U = droplet density t = time

The total number of droplets

- We want the total number of droplets in our stretch of space
- Sum the density of these small portions over the total interval

$$\int_{x}^{x+\Delta x} U(x,t)\,dx$$

Remember that an integral is really just a sum over lots of tiny pieces.

U = droplet density x = space t = time

The total flow (part 2)

- The total flow is the change in this number with respect to time
- Thus, the total flow is

$$\frac{d}{dt}\int_{x}^{x+\Delta x} U(x,t)\,dx$$

We want the change with respect to time only, so we have an ordinary derivative.

U = droplet density x = space t = time

Equating our two expressions

$$\frac{d}{dt}\int_{x}^{x+\Delta x} U(x,t)dx \quad = \quad F[U(x,t)] - F[U(x+\Delta x,t)]$$

$$\int_{x}^{x+\Delta x} \frac{\partial U(x,t)}{\partial t}dx \quad = \quad -\{F[U(x+\Delta x,t)] - F[U(x,t)]\}$$

The ordinary derivative moves inside, becoming partial

Swap the minus signs.

F = flow of droplets x = space
U = droplet density t = time

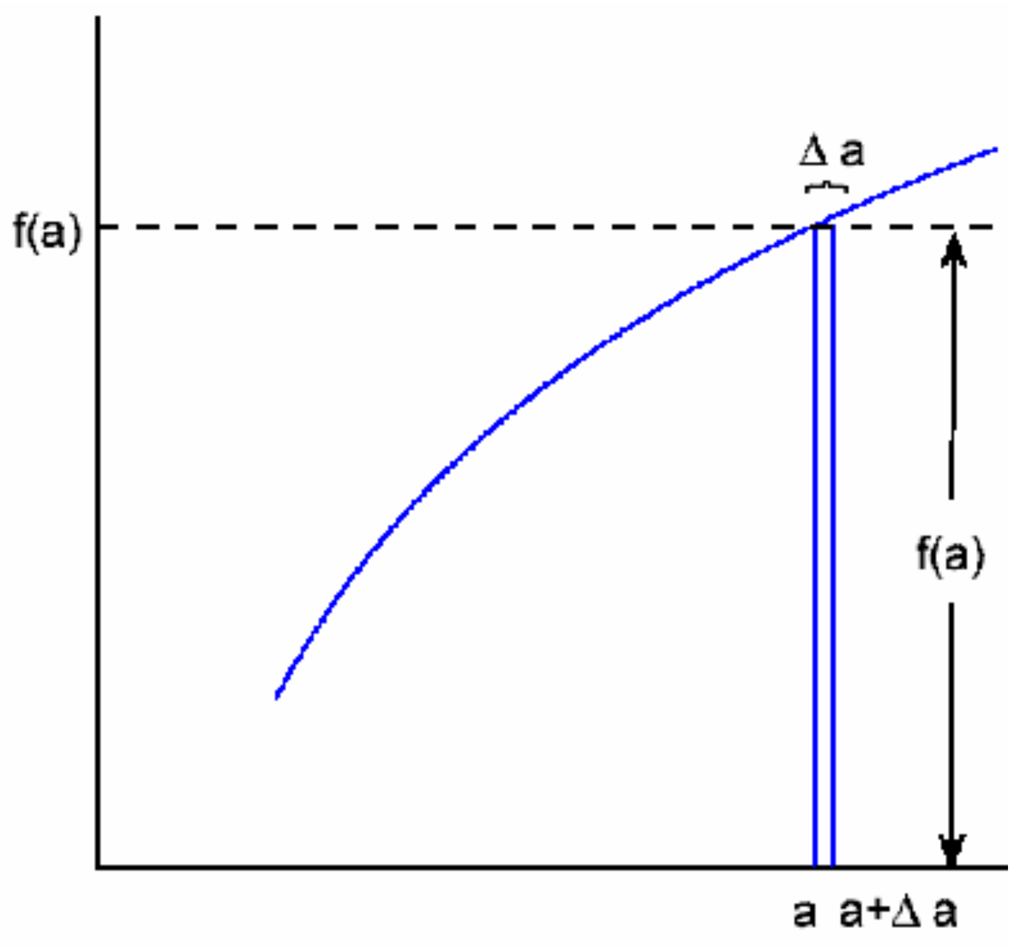
Partial derivatives

- The partial derivative $\partial g(y,z)/\partial y$ is the derivative of g with respect to y, ignoring z
- The partial derivative is used when the outcome depends on several variables
- In our case $U=U(x,t)$ and we only want the derivative with respect to t.

U = droplet density x = space t = time

An approximation

- The integral of *f(x)* from *a* to *a*+Δ*a* is the area under the curve from *a* to *a*+Δ*a*
- If Δ*a* is small, we can approximate by a rectangle.



f(x) is any function

Area of a rectangle

- The area of a rectangle is

  height × width = $f(a)\Delta a$

- Thus, the integral (an area) is

$$\int_{a}^{a+\Delta a} f(u)\,du \approx f(a)\Delta a.$$

f(x) is any function

Applying $\int_a^{a+\Delta a} f(u)du = f(a)\Delta a$

$$\int_x^{x+\Delta x} \frac{\partial U(x,t)}{\partial t} dx = -\{F[U(x+\Delta x, t)] - F[U(x,t)]\}$$

$$\frac{\partial U(x,t)}{\partial t} \Delta x \approx -\{F[U(x+\Delta x, t)] - F[U(x,t)]\}$$

$$\frac{\partial U(x,t)}{\partial t} \approx -\frac{\{F[U(x+\Delta x, t)] - F[U(x,t)]\}}{\Delta x}.$$

F = flow of droplets x = space
U = droplet density t = time

The conservation equation

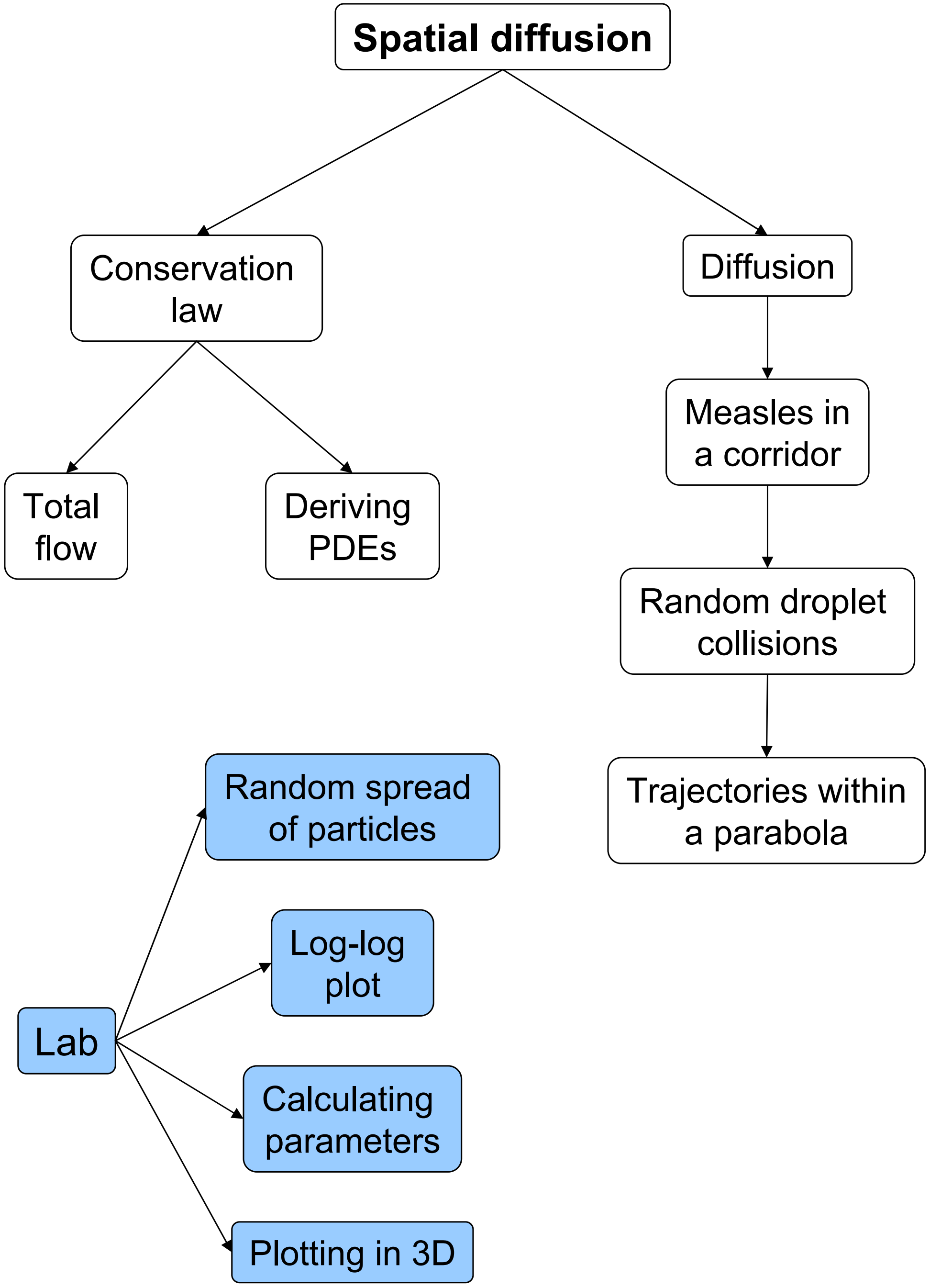
- Let $\Delta x \rightarrow 0$
- By definition, the right side is then the partial derivative with respect to x

$$\frac{\partial U(x,t)}{\partial t} \approx -\frac{\{F[U(x+\Delta x,t)] - F[U(x,t)]\}}{\Delta x}$$

$$\frac{\partial U(x,t)}{\partial t} = -\frac{\partial F[U(x,t)]}{\partial x}$$

This is the conservation equation.

F = flow of droplets x = space
U = droplet density t = time



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Diffusion

- If you drop dye into water, the particles spread out from the centre
- Diffusion is the movement from high concentration of particles to low concentrations
- It is the result of random collisions between molecules.

The flow of diffusing droplets

- For diffusion, the flow is

$$F(U) = -D\frac{\partial U}{\partial x}$$

where *D* is the diffusion constant, reflecting the viscosity of the medium

- Thus, the flow is proportional to the change in density over distance
- Density decreases with distance $\Rightarrow \partial U/\partial x < 0$.

F = flow of droplets x = space
U = droplet density t = time

From the conservation equation, $\frac{\partial U}{\partial t} = -\frac{\partial F(U)}{\partial x}$

- Differentiate: $F(U) = -D\frac{\partial U}{\partial x}$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(-D\frac{\partial U}{\partial x}\right)$$

$$= -D\frac{\partial^2 U}{\partial x^2}$$

- The diffusion equation is thus

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2}.$$

F = flow of droplets
U = droplet density
D = diffusion constant
x = space
t = time

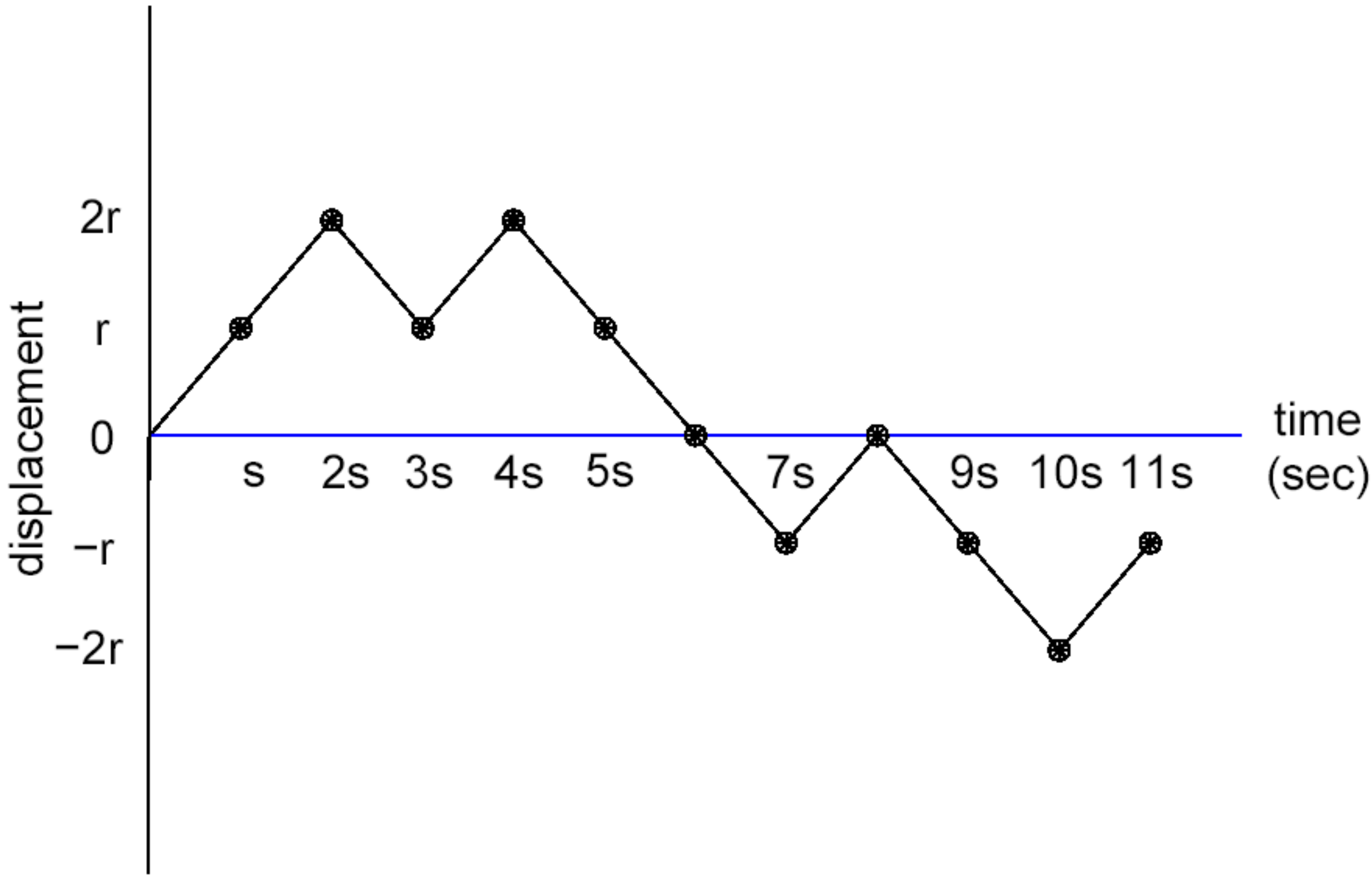
Measles in a corridor

- Harry is infected with measles
- After he sneezes in a corridor at school, the infectious droplets spread out
- Random collisions may knock them left or right.

Assumptions

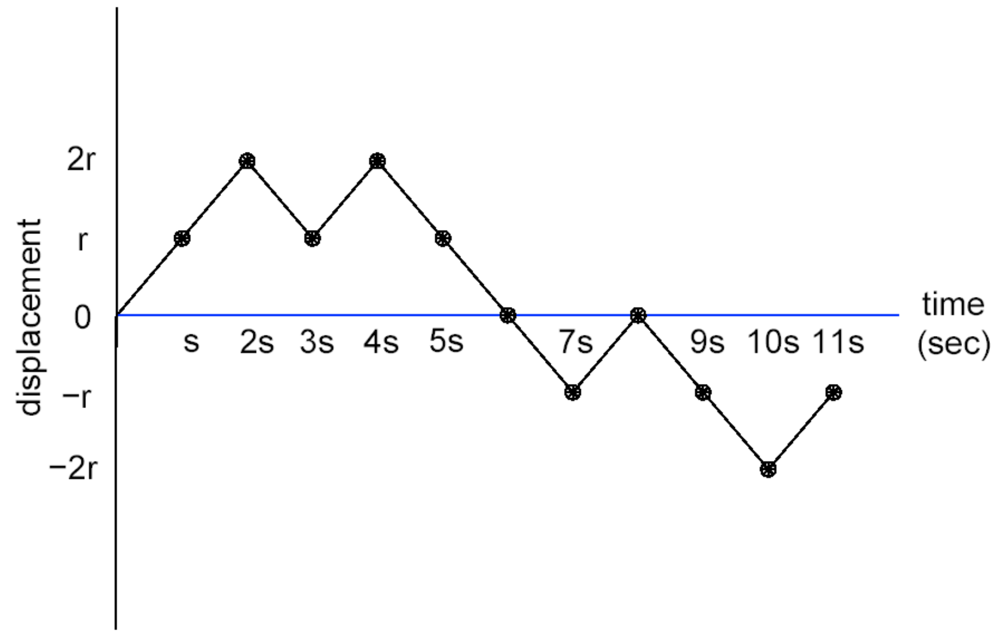
- There are *s* seconds between collisions
- Collisions displace each droplet by $\pm r$ mm
- Each collision (left or right) is independent of the previous one.

The trajectory of a random droplet



Displacement from 0

- Let r_n be the displacement from 0 at the *n*th step
- $r_1=r, r_2=r, r_3=-r$ etc in our figure.

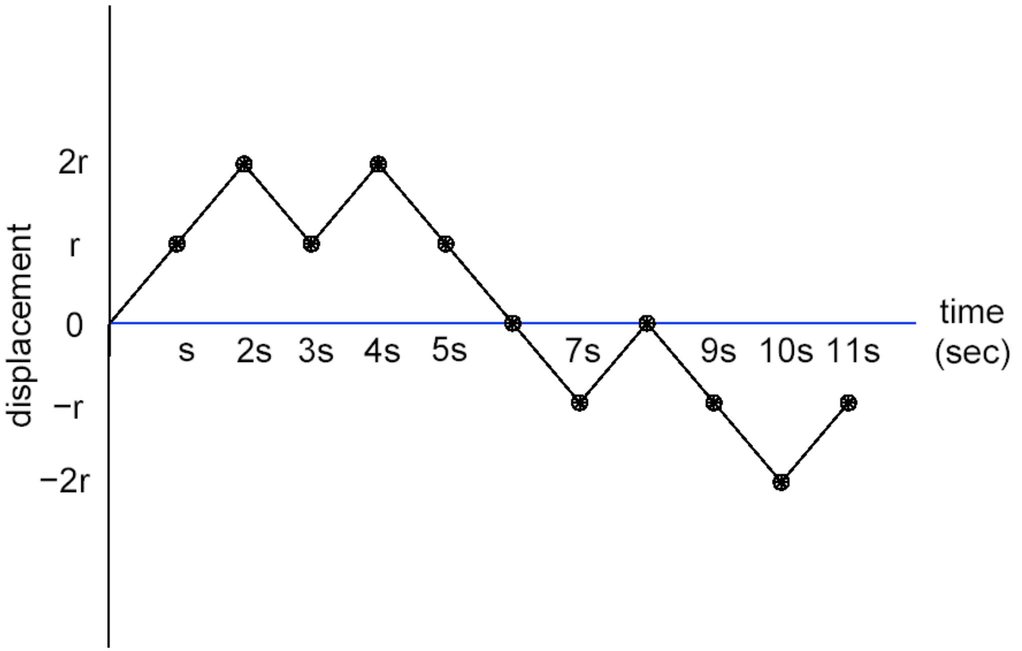


The total displacement

- The total displacement after *n* collisions is

$$y_n = r_1 + r_2 + \cdots + r_n$$

- $y_1 = r$, $y_2 = 2r$, $y_3 = r$ etc in our figure.



r_n = *displacement from 0 at nth step*

Average displacement

- On average, y_n will be zero
- Collisions will likely send the droplets left as often as right
- How can we measure distance without things cancelling each other out?

y_n = total displacement after n collisions

Mean square distance

- Answer: Sum of squares

$$y_n^2 = (r_1 + r_2 + \cdots + r_n)^2$$

$$= r_1^2 + r_2^2 + \cdots + r_n^2 + 2(r_1 r_2 + r_1 r_3 + \cdots)$$

But these terms are each equal to r^2...

...and these terms will all cancel each other out

$$\therefore \; y_n^2 = nr^2.$$

y_n = total displacement after n collisions
r_n = displacement from 0 at nth step
r = collision displacement

Mean square distance formula

- After n steps, $t = sn$ seconds have elapsed.

$$\therefore\ n = t/s$$

$$y_n{}^2 = nr^2$$

$$= tr^2/s$$

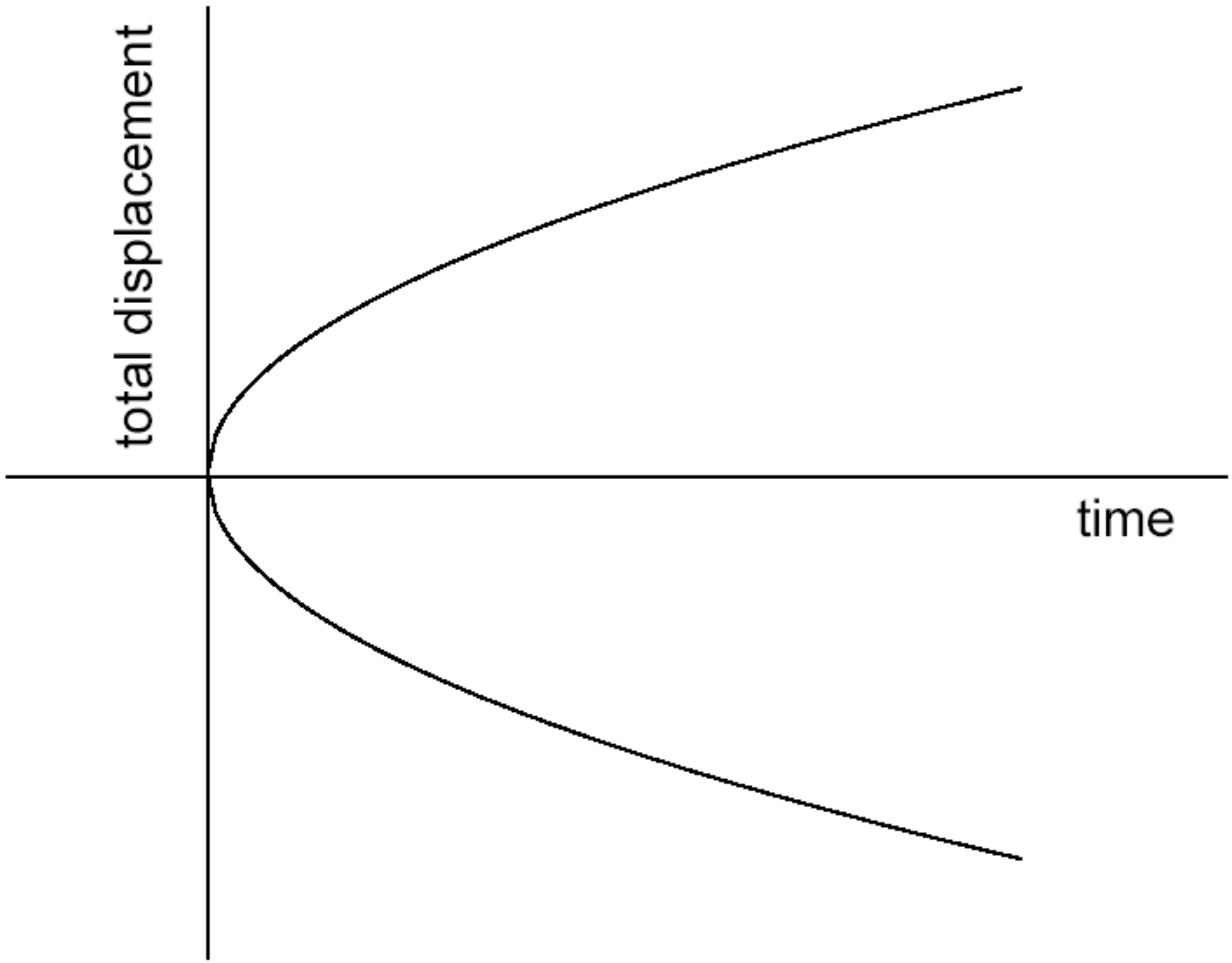
$$= C^2 t$$

C² is just some constant that we happen to know is positive

$$\therefore \quad y_n = \pm C\sqrt{t}.$$

y_n = total displacement after n collisions
r = collision displacement
s = seconds between collisions

Average trajectory over time



Extending to many droplets

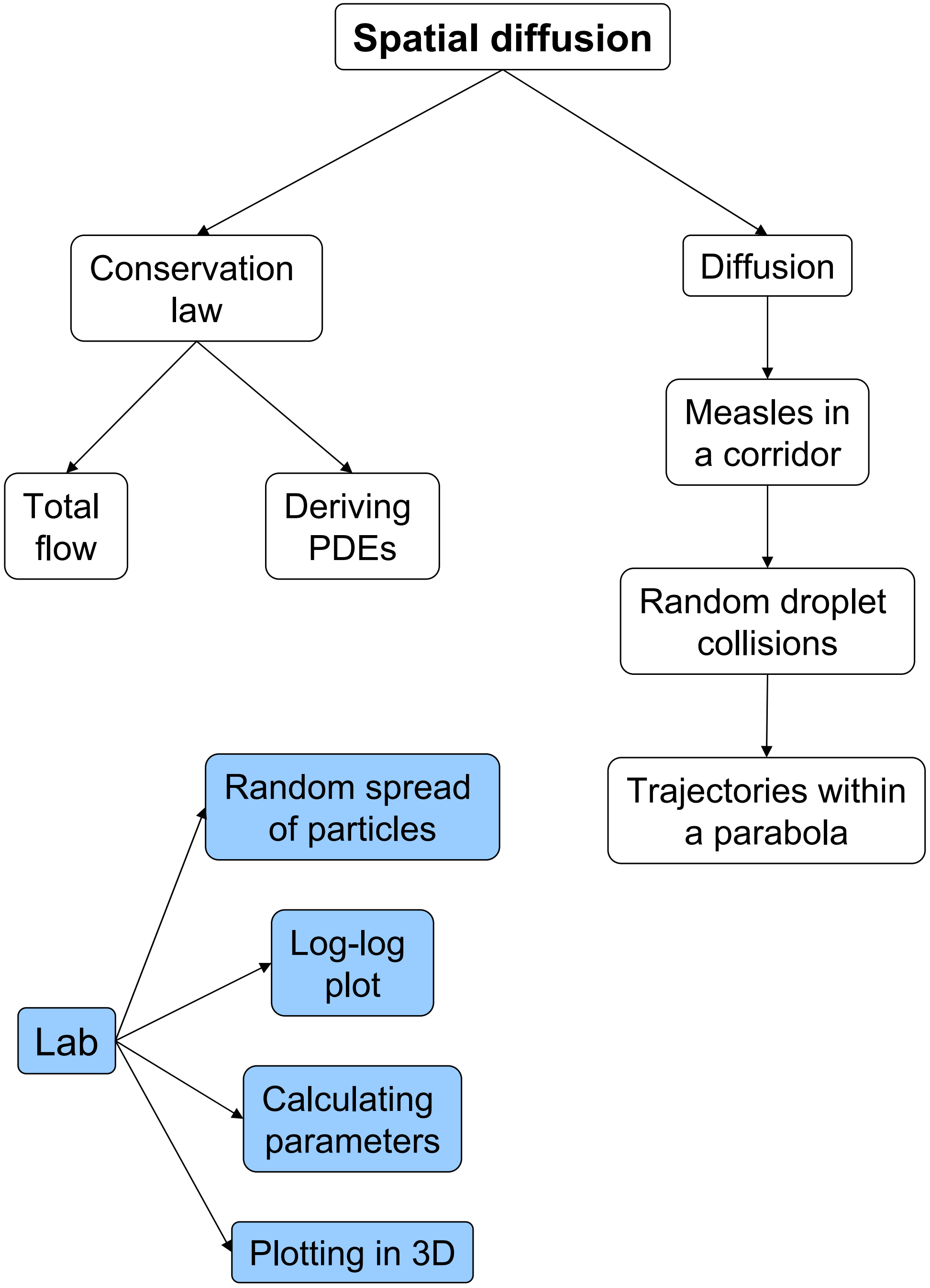
- Suppose we kept track of a whole lot of droplets
- The tendency as a whole would be to follow paths within this parabolic trajectory.

Normally distributed droplets

- Suppose the spatial spread is normally distributed at any given time
- The standard deviation at each time is governed by this parabola
- At any time, 70% of droplets should be found within this parabola.

Measles spreading over time

- Thus, we have an idea of how far measles spreads with time
- Important for determining quarantine measures
- Could extend to more dimensions, but the mathematics is harder.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Lab work

- In the lab we'll keep track of hundreds of measles particles
- We'll include randomness to simulate the real situation
- We'll also fit parameters to data to verify our theoretical approximations.

Demo

- For loops repeat statements a specific number of times
- If statements may include an else or an elseif
- Be sure to match each with a corresponding end.

```
N=3;
n=4;
a=rand(N,n)                  %creates a random matrix
for i=1:N
   for j=1:n
      if a(i,j)>0.5
         a(i,j)=1;
      else
         a(i,j)=-1;
      end
   end
end
a                            %now it's a random matrix of ±1's
pause(1)                     %pause for 1 second
for i=1:N
   for j=1:n
      y(i,j)=sum(a(i,1:j));
   end
end
y                            %See how y comes from a? Run it again.
```