1 Systems With Impulsive Effect

1.1 Impulsive semidynamical systems

Definition 1. *A triple* $(X, \pi, \mathbb{R}_+)$ *is a* semidynamical system *if* X *is a metric space,* $\mathbb{R}_+$ *the set of all nonnegative reals and* $\pi : X \times \mathbb{R}_+ \to X$ *is a continuous function such that*

i) $\pi(x, 0) = x$ *for all* $x \in X$*, and*

ii) $\pi(\pi(x, t), s) = \pi(x, t + s)$ *for all* $x \in X$ *and* $t, s \in \mathbb{R}_+$*.*

Notation. $(X, \pi, \mathbb{R}_+)$ is sometimes denoted (X, π). The triple $(X, \pi, \mathbb{R})$ is said to be a *dynamical system*.

For all $x \in X$, define $\pi_x : \mathbb{R}_+ \to X$ by $\pi_x(t) = \pi(x, t)$. π_x is continuous for all x. We call π_x the *trajectory* of x. The set

$$C^+(x) \quad = \quad \{\pi(x, t) : t \in \mathbb{R}_+\}$$

is called the *positive orbit* of x. Note that $x \in C^+(x)$. We also have

$$C^+(x, r) \quad = \quad \{\pi(x, t) : 0 \leq t \leq r\}.$$

For any $M \subseteq X$, we define the following sets: for $t \in \mathbb{R}_+$,

$$G(x, t) \quad = \quad \{y \in X : \pi(y, t) = x\},$$

is the attainable set of x at $t \in \mathbb{R}_+$,

$$\begin{aligned} G(x) &= \bigcup_{t \in \mathbb{R}_+} G(x, t), \\ M^-(x) &= G(x) \cap M \backslash \{x\}, \end{aligned}$$

and

$$M^+(x) \quad = \quad C^+(x) \cap M \backslash \{x\}.$$

We then set $M(x) = M^+(x) \cup M^-(x)$. Note that $x \notin M(x)$.

Definition 2. *An* impulsive semidynamical system $(X, \pi; M, A)$ *consists of a semidynamical system* (X, π)*, a nonempty closed subset* M *of* X *and a continuous function* $A : M \to X$ *such that*

i) No point $x \in X$ *is a limit point of* $M(x)$*, and*

ii) $\{t \in \mathbb{R}_+ : G(x,t) \cap M \neq \emptyset\}$ *is a closed subset of* $\mathbb{R}_+$.

Notation. We denote the image of M under the operator A by $N = A(M)$ and, for all $x \in M$, $A(x) = x^+$.

Lemma 1. *Let* $(X, \pi; M, A)$ *be an impulsive semidynamical system. Then for any* $x \in X$*, there exists* $r, s \in \mathbb{R}_+ \cup \{\infty\}$ *such that* $0 < r,\ s \leq \infty$ *and, for* $0 < t < s$ *and* $0 < t < r$*,*

a) $\pi(x,t) \notin M$ *and if* $M^+(x) \neq \emptyset$*, then* $\pi(x,s) \in M$

b) $G(x,t) \cap M = \emptyset$ *and if* $M^-(x) \neq \emptyset$*, then* $G(x,r) \cap M \neq \emptyset$.

Notation. We call s the *time without impulse of* x. We define $\Phi : X \to \mathbb{R}_+ \backslash \{0\}$ such that $\Phi(x)$ is the time without impulse of x. If $\{x_n\}$ is the set of impulse points, then $\{s_n\}$ are the corresponding times without impulse. We can think of a given s_n as the time taken from the trajectory starting at x_n until x_{n+1} (the next impulse point). Naturally, if there is no further impulse point, then $s_{n+1} = \infty$.

Definition 3. Let $(X, \pi; M, A)$ be an impulsive semidynamical system and $x \in X$. The *(impulsive) trajectory* of x is a function $\tilde{\pi}_x$ defined on a subset $[0, s)$, $s \in (0, \infty]$ as follows:

Let $x = x_0$. If $M^+(x_0) = \emptyset$, then $\tilde{\pi}_x(t) = \pi_x(t)$ for all $t \in \mathbb{R}_+$. If $M^+(x_0) \neq \emptyset$, then by Lemma 1, there exists $s_0 \in \mathbb{R}_+ \backslash \{0\}$ such that $\pi(x_0, s_0) = x_1 \in M$ and $\pi(x_0, t) \notin M$ for all $0 < t < s_0$. We define $\tilde{\pi}_x$ on $[0, s_0]$ by

$$\tilde{\pi}_x(t) = \pi(x_0, t) \qquad 0 \leq t \leq s_0$$

We then continue this process, starting at x_1^+ (which is not equal to x_1 in general). That is, if $M^+(x_1^+) = \emptyset$ then we define $\tilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$ for all $t > s_0$ and $s = \infty$. If $M^+(x_1^+) \neq \emptyset$, then by Lemma 1 there exists $s_1 \in$

$\mathbb{R}_+\backslash\{0\}$ such that $\pi(x_1^+, s_1) = x_2 \in M$ and $\pi(x_1^+, t) \notin M$ for all $0 < t < s_1$. We define $\tilde{\pi}_x$ on $(s_0, s_0 + s_1]$ by

$$\tilde{\pi}_x(t) \;=\; \pi(x_1^+, t - s_0) \qquad\qquad s_0 < t \leq s_0 + s_1.$$

If $M^+(x_2^+) \neq \emptyset$, then by Lemma 1 there exists $s_2 \in \mathbb{R}_+\backslash\{0\}$ such that $\pi(x_2^+, s_2) = x_3 \in M$ and $\pi(x_2^+, t) \notin M$ for all $0 < t < s_2$. We define $\tilde{\pi}_x$ on $(s_0 + s_1, s_0 + s_1 + s_2]$ by

$$\tilde{\pi}_x(t) \;=\; \pi(x_2^+, t - s_0 - s_1) \qquad\qquad s_0 + s_1 < t \leq s_0 + s_1 + s_2.$$

If $M^+(x_n^+) = \emptyset$ for some n, then the process halts. On the other hand, if $M^+(x_n^+) \neq \emptyset$ for all $n = 1, 2, \ldots$ then the process continues indefinitely, with

$$\tilde{\pi}_x(t) \;=\; \pi(x_n^+, t - \sum_{i=0}^{n-1} s_i), \qquad\qquad \sum_{i=0}^{n-1} s_i < t \leq \sum_{i=0}^{n} s_i$$

for each $n \geq 1$.

Thus the process gives rise to either a finite or infinite sequence $\{x_n\}$ of points of X such that with each x_n there is associated a positive real number s_n (or ∞) and, for $s_n < \infty$, an impulse x_{n+1}, where $\pi(x_n^+, s_n) = x_{n+1}$.

The interval of definition of $\tilde{\pi}_x$ is $[0, s] = [0, \sum_{i=0}^{\infty} s_i]$.

This completes the definition of the trajectory of $\tilde{\pi}_x$.

Notation. We call $\{x_n\}$ the sequence of impulse points of x.

Definition 4. *A trajectory $\tilde{\pi}_x$ is* periodic *of period r and order k if there exists $m \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$ such that k is the smallest integer satisfying $x_m^+ = x_{m+k}^+$ and*

$$r \;=\; \sum_{i=m}^{m+k-1} s_i.$$

Remark. A periodic trajectory with no impulse points can be considered to be an impulsive trajectory with one moment of impulse, such that the trajectory is continuous at the impulse point. Thus a periodic trajectory with no impulse points is a first order periodic orbit and the period is the time taken to travel from the impulse point back to itself; hence the period in this case corresponds to the definition of period in the non-impulsive case.

Note that the trajectory $\tilde{\pi}_x$ is continuous if either $M^+(x) = \emptyset$ or for each n, $x_n = x_n^+$. Otherwise, the trajectory has discontinuities at a finite or infinite number of impulse points x_n. However, at any such point $\tilde{\pi}_x$ is continuous from the left.

Trajectories of interest for impulsive semidynamical systems are those with an infinite number of discontinuities and an interval of definition of $\mathbb{R}_+$. We call these *infinite trajectories*.

Example. Consider the autonomous system

$$x' = x \qquad y' = \alpha y, \qquad\qquad \alpha > 0,$$

the sets $M = \{(x, y) \in \mathbb{R}_+^2 : y = \frac{1}{x+1}\}$, $N = \{(x, y) \in \mathbb{R}_+^2 : x + y = 1\}$, and an operator $A : M \to N$ that associates with each point P on M the point P^+ on N which is on the ray OP. A is a continuous, bijective mapping.

We shall consider only those trajectories with initial points in the first quadrant. Note that this quadrant is invariant. We assume initial points are not on M, by convention.

Trajectories with initial points in the region $y > \frac{1}{1+x}$ do not undergo any impulsive effect. Trajectories with initial points on the x-axis also do not undergo impulsive effect, since M does not intersect the x-axis. Trajectories with initial points on the y-axis undergo impulsive effect once, at (0,1), but motion is continuous, since this is a fixed point of the operator A. Both axes are invariant.

For $0 < \alpha < 1$, trajectories with initial points in the region $y < \frac{1}{1+x}$ undergo impulsive effect an infinite number of times. $(x_n^+, y_n^+) \to (1, 0)$, $s = \infty$ and $\tilde{\pi}_{(1,0)} = \pi_{(1,0)}$.

Let $\alpha > 1$. Trajectories with initial points in the region $y < \frac{1}{1+x}$ are subject to impulsive effect an infinite number of times and tend towards the point (0,1), which is a fixed point of the impulsive effect.

When $\alpha = 1$, all trajectories with initial points in the region $y < \frac{1}{1+x}$ eventually become periodic, with order 1. Motion between N and M is performed along rays $y = cx$.

1.2 Existence, uniqueness and continuability of solutions

Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that for each $k \in \mathbb{Z}$ the functions $\tau_k : \Omega \to \mathbb{R}$ are continuous in Ω and satisfy

$$\tau_k(x) < \tau_{k+1}(x), \text{ with } \lim_{k\to\pm\infty} \tau_k(x) = \infty$$

for $x \in \Omega$. Let $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $I_k : \Omega \to \mathbb{R}^n$, $(t_0, x_0) \in \mathbb{R} \times \Omega$ and $\alpha < \beta$.

Consider the impulsive differential system

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), & t &\neq \tau_k(x), \\ \Delta x &= I_k(x), & t &= \tau_k(x), \end{aligned} \tag{1}$$

with initial condition

$$x(t_0^+) = x_0. \tag{2}$$

By definition, $\Delta x \equiv x^+ - x$, so $I_k(x) = x + A_k(x)$.

Definition 5. *The function $\varphi : \langle \alpha, \beta \rangle \to \mathbb{R}^n$ is a* solution *of (1) if*

1. $(t, \varphi(t)) \in \mathbb{R} \times \Omega$ *for* $t \in \langle \alpha, \beta \rangle$,
2. $\varphi(t)$ *is differentiable, with*
$$\frac{d\varphi}{dt}(t) = f(t, \varphi(t))$$
for $t \in (\alpha, \beta)$, $t \neq \tau_k(\varphi(t))$, *and*
3. $\varphi(t)$ *is continuous from the left in* $\langle \alpha, \beta \rangle$ *and if* $t \in \langle \alpha, \beta \rangle$, $t = \tau_k(\varphi(t))$ *and* $t \neq \beta$, *then* $\varphi(t^+) = \varphi(t) + I_k(\varphi(t))$ *and, for each* $j \in \mathbb{Z}$ *and some* $\delta > 0$, $s \neq \tau_j(\varphi(s))$ *for* $t < s < t + \delta$.

Definition 6. *A* solution of the initial value problem (1)-(2) *is a function φ which is defined in an interval of the form (t_0, β), is a solution of (1) and satisfies (2).*

1.3 Definitions of stability

The discontinuous nature of solutions of systems with impulsive effect means that we must adjust our definitions of stability. In particular, stability of a given solution $x_0(t)$ cannot be determined from the trivial solutions by a change of variables.

Definition 7. *Let $x_0(t) = x(t; t_0, x_0)$ be a given solution of the initial value problem (1)-(2), existing for $t \geq t_0$. Suppose $x_0(t)$ hits the surfaces $S_k : t = t_k(x)$ at the moments t_k such that $t_k < t_{k+1}$ and $t_k \to \infty$ as $k \to \infty$. Then the solution $x_0(t)$ of (1)-(2) is*

- stable *if for each $\epsilon > 0$, $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(t_0, \epsilon, \eta) > 0$ such that $|y_0 - x_0| < \delta$ implies $|y_0(t) - x_0(t)| < \epsilon$ for $t \geq t_0$ and $|t - t_k| > \eta$, where $y_0(t) = x(t; t_0, y_0)$ is any solution of (1)-(2) existing for $t \geq t_0$;*
- uniformly stable *if it is stable and δ is independent of t_0;*
- attractive *if for each $\epsilon > 0$, $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta_0 = \delta_0(t_0) > 0$ and a $T = T(t_0, \epsilon, \eta) > 0$ such that $|y_0 - x_0| < \delta_0$ implies $|y_0(t) - x_0(t)| < \epsilon$ for $t \geq t_0 + T$ and $|t - t_k| > \eta$;*
- uniformly attractive *if it is attractive and δ_0 and T are independent of t_0;*
- asymptotically stable *if it is stable and attractive; and*
- uniformly asymptotically stable *if it is uniformly stable and uniformly attractive*

Remarks. The standard definitions are modified so that we can choose initial points suitably close together so that trajectories remain arbitrarily close for all time, except in any neighbourhood of the impulse points, no matter how small.

If $\tau_k(x)$ is independent of x then for every solution the impulse effect occurs at the same time, so the notions of stability coincide with the standard definition.

The system (1)-(2) only possesses the trivial solution if $f(t, 0) \equiv 0$ and $I_k(0) = 0$ for all k.

If there are only a finite number of impulse points, then the usual definitions of stability can be applied to the trajectories after the last impulse point.

If there are an infinite number of impulse points then we do not want the points to accumulate at some finite value, such that $t_k \to r < \infty$. This accounts for our requiring that $t_k \to \infty$ as $k \to \infty$.

1.4 Autonomous systems with impulsive effect

Autonomous systems with impulsive effect are written in the form

$$\begin{aligned} \frac{dx}{dt} &= g(x) & x &\notin M \\ \Delta x &= I(x) & x &\in M. \end{aligned} \tag{3}$$

At an instant $t = t_k$ when $x(t)$ encounters the set M, it is instantaneously transferred to the point $x(t_k) + I(x(t_k))$ of the set N.

The set M is sometimes given in the form $\phi(x) = 0$.

The system (3) has the property of autonomy, so that $x(t; t_0, x_0) = x(t - t_0; 0, x_0)$. Note that systems of the form (1) do not possess this property, even if $f(t, x) = g(x)$ and $I_k(x) = I(x)$.

Example. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x & t_k &\neq k \\ \Delta x &= -\frac{x}{2} + 1 & t_k &= k \\ x(0.5) &= 2. \end{aligned}$$

Solutions are given by

$$x(t) = \begin{cases} 2e^{t-0.5} & 0.5 \leq t \leq 1 \\ x_k^+ e^{t-k} & k < t \leq k+1, \qquad k \geq 1. \end{cases}$$

Thus $x(1) = 2e^{0.5}$, so

$$\begin{aligned} x(2) &= x(1)^+ e \\ &= (e^{0.5} + 1)e. \end{aligned}$$

Hence $x(2; 0.5, 2) = e^{1.5} + e$.

Conversely, consider the initial condition

$$x(0) = 2.$$

Then $x(1) = 2e$, so

$$\begin{aligned} x(1.5) &= x(1)^+ e^{0.5} \\ &= (e+1)e^{0.5}. \end{aligned}$$

Thus $x(1.5; 0, 2) = e^{1.5} + e^{0.5}$, so $x(2; 0.5, 2) \neq x(1.5; 0, 2)$. Hence the system does not have the property of autonomy.

2 Floquet Theory for Impulsive Differential Equations

2.1 Introduction

The Floquet theory for ordinary differential equations has analogues in impulsive differential equations. We outline the basic theory for stability of periodic solutions. We also provide the proofs of some of the basic theorems. These proofs are straightforward, but were not included in the literature, so we have included them here for completeness.

For two-dimensional systems there is a detailed, but relatively straightforward formula for calculation of the second multiplier for a periodic orbit. The theory here is developed in Bainov and Simeonov [1], [2]. This allows the theory of Floquet multipliers to be applied to two dimensional systems, or systems that can be reduced to two dimensional systems, with ease.

2.2 Floquet theory

Consider the linear T-periodic system with fixed moments of impulsive effect

$$\begin{aligned} \frac{dx}{dt} &= P(t)x & t &\neq t_k \\ \Delta x &= B_k x & t &= t_k, \end{aligned} \tag{4}$$

subject to the following assumptions:

H1 The matrix $P(\cdot) : \mathbb{R} \to \mathbb{C}^{n\times n}$ is piecewise continuous and $P(t+T) = P(t)$ for $t \in \mathbb{R}$.

H2 $t_k < t_{k+1}$ for $k \in \mathbb{Z}$, $B_k \in \mathbb{C}^{n\times n}$ and $\det(I + B_k) \neq 0$, where I is the $n \times n$ identity matrix.

H3 There exists an integer $q > 0$ such that $B_{k+q} = B_k$, $t_{k+q} = t_k + T$ for $k \in \mathbb{Z}$.

Definition 8. *Let $x_1(t), ..., x_n(t)$ be solutions to (4) defined on the interval $(0, \infty)$. Let $X(t) = \{x_1(t), ..., x_n(t)\}$ be a matrix valued function whose columns are these solutions. $x_1(t), ..., x_n(t)$ are linearly independent if and only if* $\det X(0^+) \neq 0$. *In this case, we say that $X(t)$ is a* fundamental matrix *of solutions of (4).*

Lemma 2. *Suppose H1-H3 hold and $\lim_{k\to\infty} t_k = \infty$. Let $X(t)$ be a fundamental matrix of solutions of (4) in $\mathbb{R}_+$. Then*

1. *For any constant matrix $\bar{M} \in \mathbb{C}^{n\times n}$, $X(t)\bar{M}$ is also a solution of (4).*

2. *If $Y : \mathbb{R} \to \mathbb{C}^{n\times n}$ is a solution of (4), there exists a unique matrix $\bar{M}$ such that $Y(t) = X(t)\bar{M}$. Furthermore, if $Y(t)$ is also a fundamental matrix of solutions, then* $\det \bar{M} \neq 0$.

Proof. 1. $X(t)\bar{M}$ satisfies

$$\begin{aligned} \frac{d}{dt}\left(X(t)\bar{M}\right) &= \frac{dX(t)}{dt}\bar{M} \\ &= P(t)X(t)\bar{M} \end{aligned}$$

for $t \neq t_k$, and

$$\begin{aligned} \Delta\left(X(t_k)\bar{M}\right) &= X(t_k^+)\bar{M} - X(t_k)\bar{M} \\ &= \left[X(t_k^+) - X(t_k)\right]\bar{M} \\ &= \left[\Delta X(t_k)\right]\bar{M} \\ &= B_k X(t_k)\bar{M}. \end{aligned}$$

2. Since $X(t)$ is a fundamental matrix, it is invertible for each t. Let $\bar{M} = X(0^+)^{-1}Y(0^+)$ and let $Z(t) \equiv Y(t) - X(t)\bar{M}$. Then $Z(0^+) = 0$ and

$$\begin{aligned} \frac{dZ(t)}{dt} &= \frac{dY(t)}{dt} - \frac{dX(t)}{dt}\bar{M} \\ &= P(t)Y(t) - P(t)X(t)\bar{M} \\ &= P(t)Z(t) \\ \Delta Z &= Y(t_k)^+ - X(t_k)^+\bar{M} \; - \; \left[Y(t_k) - X(t_k)\bar{M}\right] \\ &= \Delta Y(t_k) - \Delta X(t_k)\bar{M} \\ &= B_k Y(t_k) - B_k X(t_k)\bar{M} \\ &= B_k Z(t_k), \end{aligned}$$

so $Z(t) \equiv 0$ is the unique solution satisfying $Z(0^+) = 0$. Hence $Y(t) = X(t)\bar{M}$

If Y is fundamental, then

$$\begin{aligned} \det \bar{M} &= \frac{1}{\det X(0^+)}\det Y(0^+) \\ &\neq 0. \end{aligned}$$

$\square$

Theorem 1. *Suppose conditions H1-H3 hold. Then each fundamental matrix of (4) can be represented in the form*

$$X(t) = \varphi(t)e^{\Lambda t} \qquad t \in \mathbb{R}$$

for a non-singular, T-periodic matrix $\varphi(\cdot) \in PC^1(\mathbb{R}, \mathbb{C}^{n\times n})$ and a constant matrix $\Lambda \in \mathbb{C}^{n\times n}$.

Proof. Let $X(t)$ be a fundamental matrix for (4) and define $Y(t) = X(t+T)$. Then using H1, we have

$$\begin{aligned} \frac{dy_j(t)}{dt} &= \frac{dx_j(t+T)}{dt} \\ &= P(t+T)x_j(t+T) \\ &= P(t)y_j(t) \end{aligned}$$

for $t \neq t_k$, and using H3,

$$\begin{aligned} \Delta y_j(t_k) &= \Delta x_j(t_k+T) \\ &= \Delta x_j(t_{k+q}) \\ &= B_{k+q}x_j(t_{k+q}) \\ &= B_k x_j(t_k+T) \\ &= B_k y_j(t_k) \end{aligned}$$

for each j. $\det Y(0^+) = \det X(T^+) \neq 0$, since $x_1(t), ..., x_n(t)$ are linearly independent in the interval $(0, \infty)$ and are hence independent in the interval (T, ∞). Thus $Y(t)$ is also a fundamental matrix.

By the lemma, there exists a unique matrix $\bar{M} \in \mathbb{C}^{n\times n}$ such that

$$X(t+T) = X(t)\bar{M}$$

for all $t \in \mathbb{R}$. Set

$$\begin{aligned} \Lambda &= \frac{1}{T}\ln \bar{M} \\ \varphi(t) &= X(t)e^{-\Lambda t}. \end{aligned}$$

Hence $\varphi(t)$ is non-singular and belongs to the class $PC^1(\mathbb{R}, \mathbb{C}^{n\times n})$. Furthermore,

$$\begin{aligned}\varphi(t+T) &= X(t+T)e^{-\Lambda T}e^{-\Lambda t} \\ &= X(t)\bar{M}e^{-\Lambda T}e^{-\Lambda t} \\ &= X(t)e^{-\Lambda t} \\ &= \varphi(t)\end{aligned}$$

since $\bar{M} = e^{\Lambda T}$, by definition of Λ. Hence φ is T-periodic.

□

To the fundamental matrix $X(t)$ there corresponds a unique matrix $\bar{M}$ such that $X(t+T) = \bar{M}X(t)$ for all $t \in \mathbb{R}$. The eigenvalues $\mu_1, \ldots, \mu_n$ of $\bar{M}$ are called *Floquet multipliers* of (4). The eigenvalues $\lambda_1, \ldots, \lambda_n$ of Λ are called the *characteristic exponents* of (4).

Corollary 1. *Let conditions H1-H3 hold. Then $\mu \in \mathbb{C}$ is a Floquet multiplier of (4) if and only if there exists a non-trivial solution $\gamma(t)$ such that $\gamma(t+T) = \mu\gamma(t)$ for all $t \in \mathbb{R}$.*

The following theorem is from Bainov and Simeonov [2].

Theorem 2. *Suppose conditions H1-H3 hold. Then (4) is*

1. *stable if and only if all multipliers μ_j satisfy $|\mu_j| \leq 1$; and for those multipliers for which $|\mu_j| = 1$, the corresponding characteristic exponent (which has zero real part) is a simple zero of the characteristic polynomial of Λ,*

2. *asymptotically stable if and only if all multipliers satisfy $|\mu_j| < 1$, and*

3. *unstable if $|\mu_j| > 1$ for some j.*

2.3 Orbital stability in $\mathbb{R}^2$

Consider the two dimensional autonomous system

$$\begin{aligned}\frac{dx}{dt} &= P(x,y), & \frac{dy}{dt} &= Q(x,y) & (x,y) &\notin M \\ \Delta x &= a(x,y), & \Delta y &= b(x,y) & (x,y) &\in M\end{aligned} \tag{5}$$

where $t \in \mathbb{R}$, and $M \subset \mathbb{R}^2$ is the set defined by the equation $\phi(x, y) = 0$.

Let $\gamma(t)$, $t \in \mathbb{R}$ be a solution of (5), with instants of impulsive effect t_k, such that

$$0 < t_1 < t_2 < ...; \qquad \lim_{k\to\infty} t_k = \infty$$

and let $L_+ = \{u \in \mathbb{R}^2 : u = \gamma(t), t \in \mathbb{R}_+\}$. Denote by $J^+(t_0, z_0)$ the maximal interval of the form (t_0, ω) in which the solution $z(t; t_0, z_0)$ of (5) is defined.

For $y \in \mathbb{R}^2$, let $d(y, L^+) = \min_{u\in L^+} |y - u|$ and $B_\eta(\gamma(t_1))$ be the ball of radius η centred at $\gamma(t_1)$.

Definition 9. *The solution $z = \gamma(t)$ of (5) is called*

1. orbitally stable *if for all $\epsilon > 0$, $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta > 0$ such that $d(z_0, L^+) < \delta$ and $z_0 \notin \bar{B}_\eta(\gamma(t_k)) \cup \bar{B}_\eta(\gamma(t_k^+))$ implies $d(z(t), L^+) < \epsilon$ for $t \in J^+(t_0, z_0)$ and $|t_0 - t_k| > \eta$, where $z(t) = z(t; t_0, z_0)$ is any solution of (5) for which $z(t_0^+; t_0, z_0) = z_0$.*

2. orbitally attractive *if for all $\epsilon > 0$, $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta > 0$ and $T > 0$ such that $t_0 + T \in J^+(t_0, z_0)$ and $d(z_0, L^+) < \delta$ and $z_0 \notin \bar{B}_\eta(\gamma(t_k)) \cup \bar{B}_\eta(\gamma(t_k^+))$ implies $d(z(t), L^+) < \epsilon$ for $t \geq t_0 + T$, $t \in J^+(t_0, z_0)$ and $|t_0 - t_k| > \eta$, where $z(t) = z(t; t_0, z_0)$ is any solution of (5) for which $z(t_0^+; t_0, z_0) = z_0$.*

3. orbitally asymptotically stable *if it is orbitally stable and orbitally attractive.*

Definition 10. *The solution $z = \gamma(t)$ of (5) has the property of* asymptotic phase *if for all $\epsilon > 0$, $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta > 0$, $c > 0$ and $T > |c|$ such that $t_0 + T \in J^+(t_0, z_0)$ and $|z_0 - \gamma(t_0)| < \delta$ implies $|z(t + c) - \gamma(t)| < \epsilon$ for $t \geq t_0 + T$, $t \in J^+(t_0, z_0)$ and $|t_0 - t_k| > \eta$, where $z(t + c) = z(t; t_0 - c, z_0)$ is any solution of (5) for which $z(t_0^+; t_0, z_0) = z_0$.*

Suppose (5) has a T-periodic solution

$$\vec{p}(t) \;=\; \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix},$$

with

$$\left|\frac{d\xi}{dt}\right| + \left|\frac{d\eta}{dt}\right| \;\neq\; 0.$$

Assume further that the periodic solution $\vec{p}(t)$ has q instants of impulsive effect in the interval $(0, T)$. Since we have a periodic orbit, one multiplier is equal to 1. The other is calculated according to the formula

$$\mu_2 = \prod_{k=1}^{q} \Delta_k \exp\left[\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t))\right) dt\right], \qquad (6)$$

where

$$\Delta_k = \frac{P_+\left(\frac{\partial b}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial b}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}\right) + Q_+\left(\frac{\partial a}{\partial x}\frac{\partial \phi}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right)}{P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y}}.$$

P, Q, $\frac{\partial a}{\partial x}$, $\frac{\partial b}{\partial x}$, $\frac{\partial a}{\partial y}$, $\frac{\partial b}{\partial y}$, $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ are computed at the point $(\xi(t_k), \eta(t_k))$ and $P_+ = P(\xi(t_k^+), \eta(t_k^+))$, $Q_+ = Q(\xi(t_k^+), \eta(t_k^+))$.

We then have the following theorem, from Bainov and Simeonov [2] which is an analogue of the Poincaré criterion.

Theorem 3. *The solution $\vec{p}(t)$ of (5) is orbitally asymptotically stable and has the property of asymptotic phase if the multiplier μ_2 calculated by (6) satisfies the condition $|\mu_2| < 1$.*

The Floquet theory for impulsive dynamical systems in $\mathbb{R}^n$, $n \geq 3$ is also developed in Bainov and Simeonov [1], [2], but calculation of the multipliers is much more difficult.

In practice, the theory is only useful in low dimensional systems. If we are in $\mathbb{R}^2$ or the system can be reduced to a two dimensional system, then we can apply the results in this section.

References

[1] D.D. Bainov and P.S. Simeonov, *Systems with Impulsive Effect*, Ellis Horwood Ltd, Chichester [1989].

[2] D.D. Bainov and P.S. Simeonov, *Impulsive differential equations: periodic solutions and applications*, Longman Scientific and Technical, Burnt Mill [1993].