

Modelling a Ponzi Scheme

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Introduction

Ponzi schemes are fraudulent investment enterprises. Investors are often promised large returns in a short amount of time. Unbeknownst to the investors, there is no legitimate money-making scheme: the returns are imaginary. Consider the following example.

The scheme promoter promises people a 20% yield on their investment every year. A person who has invested \$100 will receive \$20 every year, assuming that he does not reinvest. Since the returns are fictitious, the promoter will use the initial \$100 capital investment to pay out subsequent returns. Doing so will sustain the scheme for five years. After that, the promoter must attract more investors to finance the promised returns. The cycle is never-ending and it is impossible to pay back all the investors if they decide to pull out their initial investments.

History

The term “Ponzi scheme” was coined after Charles Ponzi, who was the promoter of the first of such schemes to be widely exposed. In the early 1900s, people used international postal reply coupons to pay for postage. For example, a person would buy a coupon in Italy and send it to someone in the US. The recipient would be able to exchange the coupon for a stamp and not have to pay for postage himself. If the values of the two commodities were different, there was a potential profit.

World War I brought about inflation in Europe. Consequently, the cost of postage in Italy decreased relative to prices in the US. Ponzi claimed that he could buy coupons in Italy, exchange them for stamps in the US, and sell them at a higher price. Arbitrage was not illegal. He claimed that profits would be over 400 per cent. Ponzi enticed his friends and associates to invest money in his scheme, promising them a 50 per cent return in 45 days, or an astounding “double your money” in just 90 days!

In reality, Ponzi was not buying and reselling stamps. He was taking the new investment money and using it to pay out the returns to older investors. Clearly there is a big problem if there is not enough money coming in because it will become impossible to pay out the ‘imaginary’ returns. Ponzi partly avoided this pitfall because many of his investors kept reinvesting their money. For a while, he was assured an influx of funds. He was also prompt to pay out to the few people who did take their money out. This instilled confidence in his investors that their outlays were fairly liquid.

Ponzi himself became a millionaire in just six months by taking money from the scheme. He obviously did not invest his own money. Eventually people started examining and questioning his scheme. They noticed that Ponzi himself wasn't investing in the scheme, despite the high rates of return. Furthermore, there were too few coupons circulating to validate Ponzi's claims. The Post Office reported a small number of coupons that were actually being bought. It was a mere fraction of the amount Ponzi would have had to buy to bring in the profits that he was allegedly making.

The investigation of Ponzi's 'business' caused panic. Eventually, Ponzi was arrested and charged with fraud. But not after losing millions of investor's dollars during the 8 months that the scheme lasted.

How Ponzi Schemes Work

Ponzi scheme promoters promise extraordinary returns on investment. They create a seemingly complicated investment scheme using advanced financial jargon so that the average person does not question its validity. Historically, these promoters have been skilled orators and able to instill confidence and convince investors.

What the investor does not know is that his returns are not actually coming from profits. Whenever a new investor enters, the promoter uses that money to pay out returns to existing investors. The Ponzi scheme only makes money when new investors enter. However, the promoter will have to pay back this investment amount AND the promised returns on investment. Unless there is an unlimited supply of people to enter the scheme, the promoter will not be able to pay back all of the investors. Especially troubling is when too many investors decide to take out their money at once. This often happens when the economy takes a turn for the worse and will cause the scheme to collapse.

Why Ponzi Schemes Are Unsustainable

The reason that Ponzi schemes do not work is that they simply redistribute money rather than create new money. In essence, money is taken from one person and given to another. Ponzi schemes are unsustainable in that eventually one of three things happen:

- 1) The promoter of the scheme will take the invested money and disappear.
- 2) If there is a panic and people start withdrawing their money too quickly, the scheme will collapse. There will eventually be too little money to pay back the promised returns to everyone.
- 3) If the promoter isn't able to validate his claims (for example, Ponzi's resale profits from stamps), authorities become suspicious and will expose the fraudulent scheme.

Model Formulation

The goal of this project is to construct a mathematical model of a simple Ponzi scheme. This model will track the flow of investment capital, the “returns” on investment, and their redistribution. The purpose is to study the stability and longevity of a Ponzi scheme. It is of most interest to determine what strategies or conditions allow Ponzi scheme promoters to profit or avoid collapse over long periods of time, and under what conditions they easily collapse.

Perhaps the most intuitive way to represent a simple Ponzi scheme is using a discrete model of difference equations. The time periods considered are the instances when transactions occur, and depending on the structure of the actual scheme, these may take place monthly, quarterly, annually or based on some other interval.

In this model there exists an initial pool of investor capital $\mathbf{Y}_0 = \mathbf{I}$. As new investors are recruited into the scheme, capital trickles into the promoter’s fund at a fixed ratio \mathbf{r}_1 . Each month, a new quantity of capital is invested into the scheme and is held in a holding state \mathbf{H}_n , since it will not accrue interest until returns are distributed the next month. Thus, at every time period (month) in the model, interest is paid on the net amount of earning capital that was in the fund the previous month, which includes the investments from the previous month’s holding and the net reinvestment capital, i.e. the total earning capital is $\mathbf{H}_{n-1} + \mathbf{E}_{n-1}$. This amount of money will accrue interest or “returns” at the monthly rate i . Some of the returns paid out each month are “reinvested” or kept in the fund so that the interest is compounded, while the rest (proportion \mathbf{r}_3) may be pocketed by some investors. Alternatively, some investors may pull out their total earnings (interest and principal) and leave the scheme entirely (proportion \mathbf{r}_2). The resulting amount left in the fund is the current month’s reinvestment capital, \mathbf{E}_n .

$$\begin{aligned}
 Y_n &= (1 - r_1)Y_{n-1} \\
 H_n &= r_1 Y_{n-1} \\
 E_n &= (1 - r_2)(H_{n-1} + E_{n-1}) + (1 - r_3 - r_2)(H_{n-1} + E_{n-1})i \\
 M_n &= M_{n-1} + r_2(H_{n-1} + E_{n-1}) + (r_2 + r_3)(H_{n-1} + E_{n-1})i \\
 R_n &= R_{n-1} + i(H_{n-1} + E_{n-1})
 \end{aligned}$$

Where \mathbf{Y} is the untapped investment capital and \mathbf{H} consists of investments currently in holding. \mathbf{R} is the cumulative sum of money paid out to investors as returns with interest rate i . \mathbf{M} is the cumulative sum of capital that has been removed from the fund (interest pocketed and principal pulled out). \mathbf{E} is the net capital reinvested for the following month.

The real money passing through the promoter’s hands is the balance between revenue and costs: the revenue the promoter makes is the new investments in holding plus the capital that was reinvested. However, the costs are the returns on the earning capital that are withdrawn and the earning capital that is removed. This gives the following equation describing the actual funds in

the Ponzi scheme, where the funds at time n is the capital currently in holding minus the returns paid out plus the net capital reinvested:

$$P_n = H_n - R_n + E_n$$

The initial condition are:

$$Y_0 = I, H_0 = 0, E_0 = 0, M_0 = 0, R_0 = 0, P_0 = 0$$

The model shown assumes that the holding period lasts one month, but this time could be extended to increase profits. This can be incorporated into the model by considering money remaining in the holding compartment for several months in succession, so that an investment made on month n does not mature and accrue interest until month $n+k$. It is assumed that the capital in holding cannot be removed until it matures.

$$\begin{aligned} Y_n &= (1 - r_1)Y_{n-1} \\ H_n &= r_1 Y_{n-1} \\ E_n &= 0, M_n = 0, R_n = 0, P_n = 0 \\ P_n &= P_{n-1} + H_n \end{aligned} \quad \text{for } 1 \leq n \leq k$$

and...

$$\begin{aligned} Y_n &= (1 - r_1)Y_{n-1} \\ H_n &= r_1 Y_{n-1} \\ E_n &= (1 - r_2)(H_{n=k} + E_{n-1}) + (1 - r_3 - r_2)(H_{n=k} + E_{n-1})i \\ M_n &= M_{n-1} + r_2(H_{n=k} + E_{n-1}) + (r_2 + r_3)(H_{n=k} + E_{n-1})i \\ R_n &= R_{n-1} + i(H_{n=k} + E_{n-1}) \\ P_n &= \sum_{j=0}^{k-1} H_{n-j} + E_n - R_n \end{aligned} \quad \text{for } n > k$$

In other words, earning capital is calculated based on the value of H_{n-k} rather than H_{n-1} . The total revenue includes all the investments in holding over the past k months.

Results and Discussion

1. Finite Economy

When the initial pool of capital is fixed $\mathbf{Y}_0 = \mathbf{I}$, it implies that there is a finite amount of investment dollars available to be acquired and that these resources get depleted in a geometric fashion over time. This amount of money can be scaled to 1 for simplicity. The figure below shows a typical time course of the cash flow. The dark blue curve represents the remaining untapped capital \mathbf{Y}_n , the green curve the newest investments placed in holding \mathbf{H}_n , the black curve the cumulative returns paid to investors, the cyan curve the cumulative money removed from the scheme \mathbf{M}_n , and the red curve is the net profit of the Ponzi promoter \mathbf{P}_n .

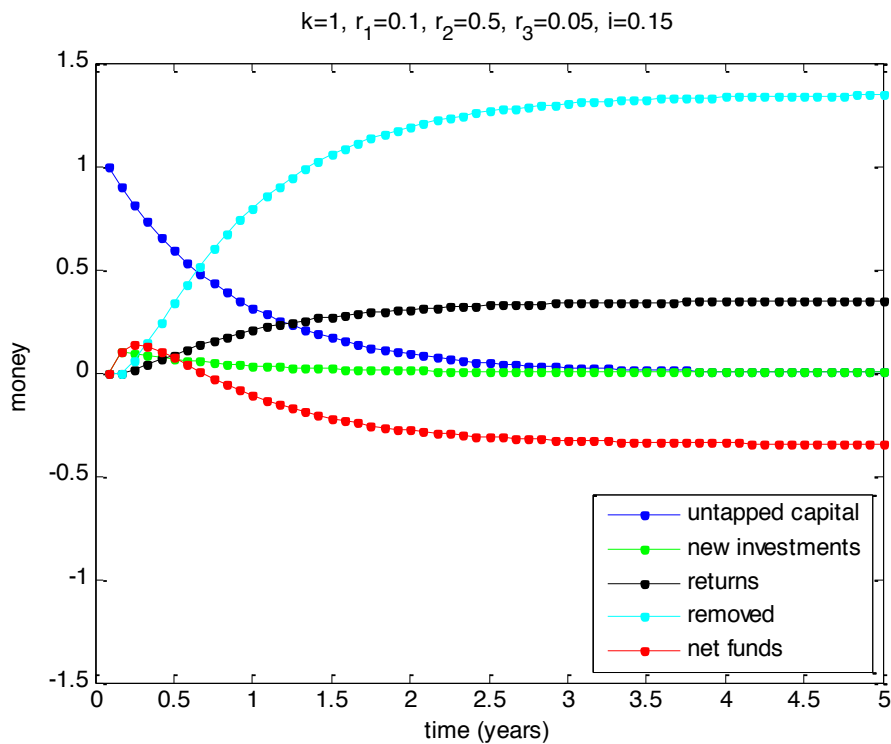


Figure 1: Time course for a Ponzi scheme

Here it is assumed that half of all earning capital is pulled out after interest is credited each month, that investors collect their interest on an additional 5% of the earning capital, and that the rest keep all their money in the scheme. Note that the promoter (red) begins to make a profit, which eventually peaks, but with time will eventually run at a loss before year's end. If the scheme lasts indefinitely, the system will reach an equilibrium ($\mathbf{Y}_\infty = \mathbf{0}, \mathbf{H}_\infty = \mathbf{0}, \mathbf{E}_\infty = \mathbf{0}, \mathbf{M}_\infty = \mathbf{I} + \mathbf{R}_\infty$), and the funds will reach an equilibrium value, equivalent to the total amount of fake income generated by the promoter to pay off his investors ($\mathbf{P}_\infty = -\mathbf{R}_\infty$).

Note that the returns paid are artificial: they are not based on actual returns on real investments. Thus, while there are sufficient funds in the scheme, the promoter is merely redistributing the wealth of new investors to earning investors. Once the funds run out, the promoter will run a deficit and accumulate large debt, although realistically, the fraud would likely be exposed before the system reaches the equilibrium predicted here. Thus, for such a scheme to be profitable, the promoter will have to abandon and pull out the funds before collapse.

1.1 Interest rate

The monthly interest rate offered to clients is crucial in determining the longevity of a Ponzi scheme as well as the extent of the potential losses.

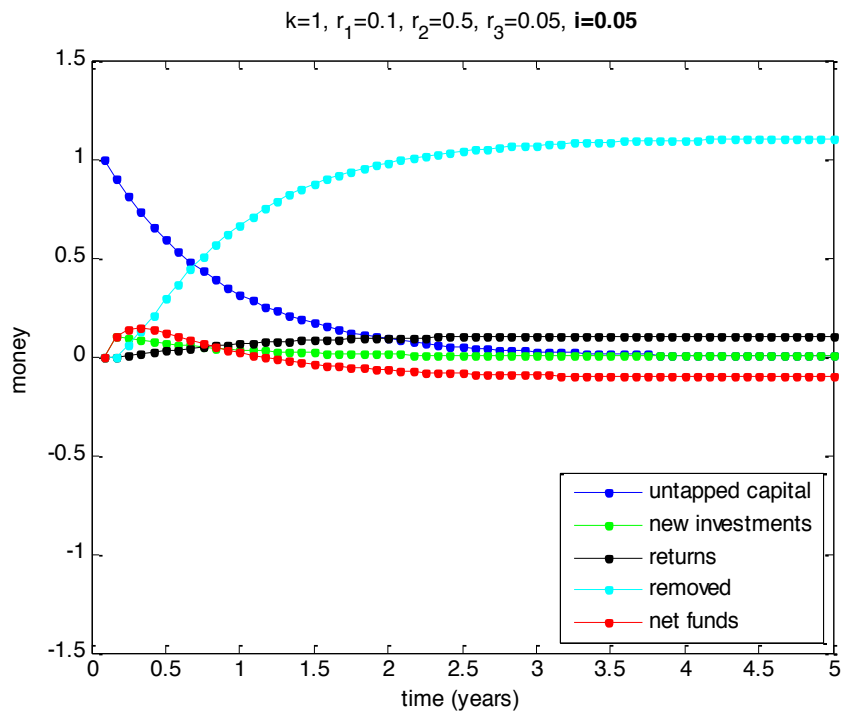


Figure 2: Interest rate of 5%

Lowering the interest rate from 15% to 5% increases the potential longevity of the scheme to just over one year before bankruptcy (Figure 2). Since the returns paid are small, there is less of a burden in paying investors when incoming capital starts to dwindle, and hence the losses are not as steep. Correspondingly, raising the interest rate from 15% to 25% reduces longevity and leads to a lower equilibrium fund balance. In reality, a high interest rate is an important incentive to attract new investors, but the trade off is that it will be disadvantageous when incoming supplies of money eventually start to fall because the costs of paying back earning investors will be much greater.

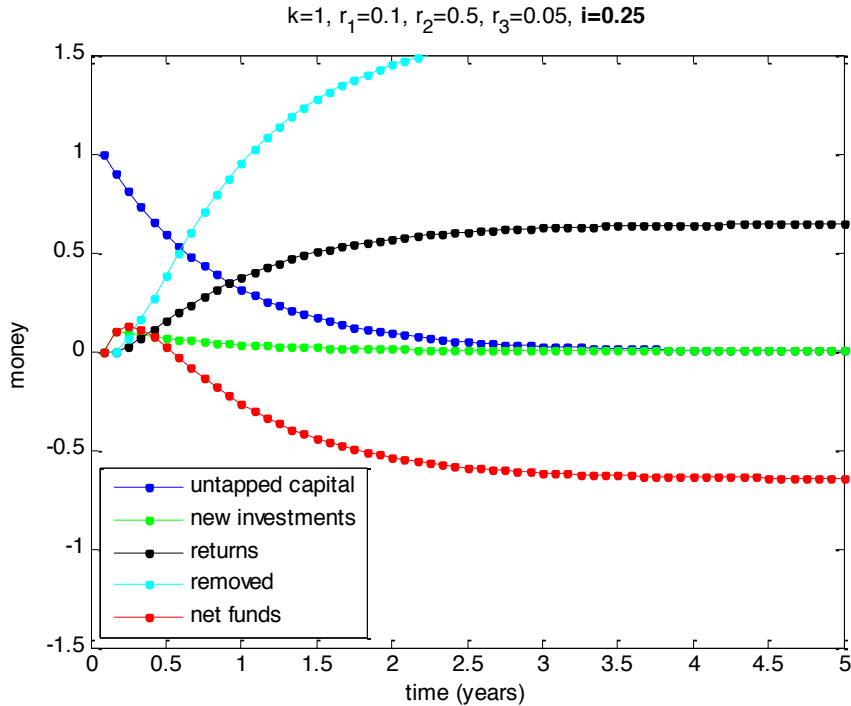


Figure 3: Interest rate of 25%

Notice that changing the interest rate does not affect peak profit if the proportional rates of capital influx (r_1) and reinvestment/removal (r_2, r_3) are the same, as they are in the above simulations. However, in reality, different interest rates would be expected to yield different investment behaviour by clients.

1.2 Reinvestment

It would seem crucial for the success of a Ponzi scheme that investors plow back their earnings into the scheme. However, each month, it is inevitable that some investors in need of cash will pocket the interest earned on their investment or pull out their entire investment altogether. This is modelled as a loss of some fraction of the interest distributed per month (r_3) as well as the loss of some fraction of the total earning capital pool (r_2). As shown in figures 4 and 5, if r_2 is decreased from 50% to 25% and 10%, profit reaches a higher maximum, but will decline more rapidly when income is from new investors limited. Moreover, the Ponzi scheme is able to last much longer before the promoter goes bankrupt.

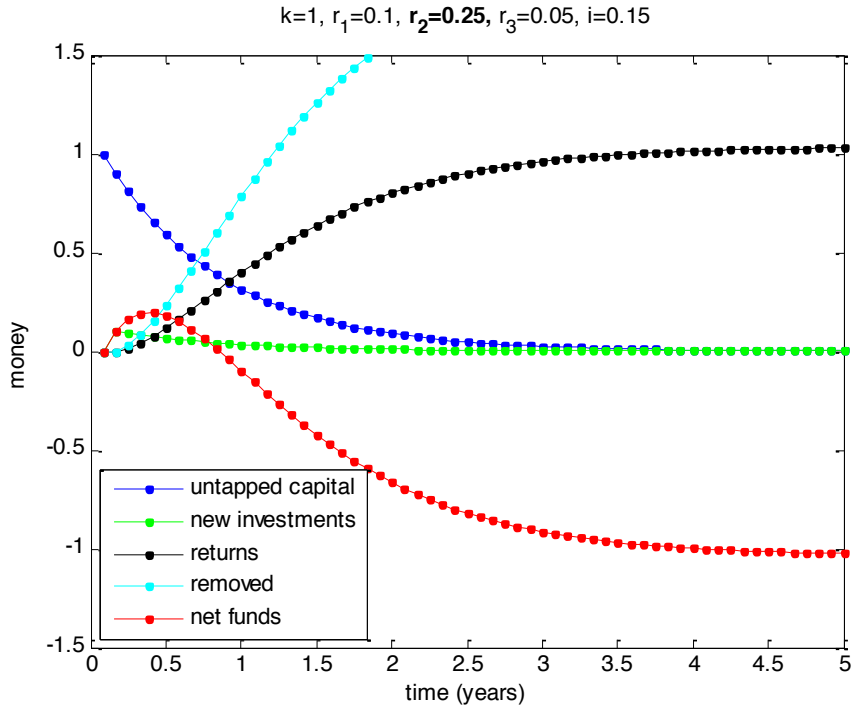


Figure 4: Pull-out rate of 25%

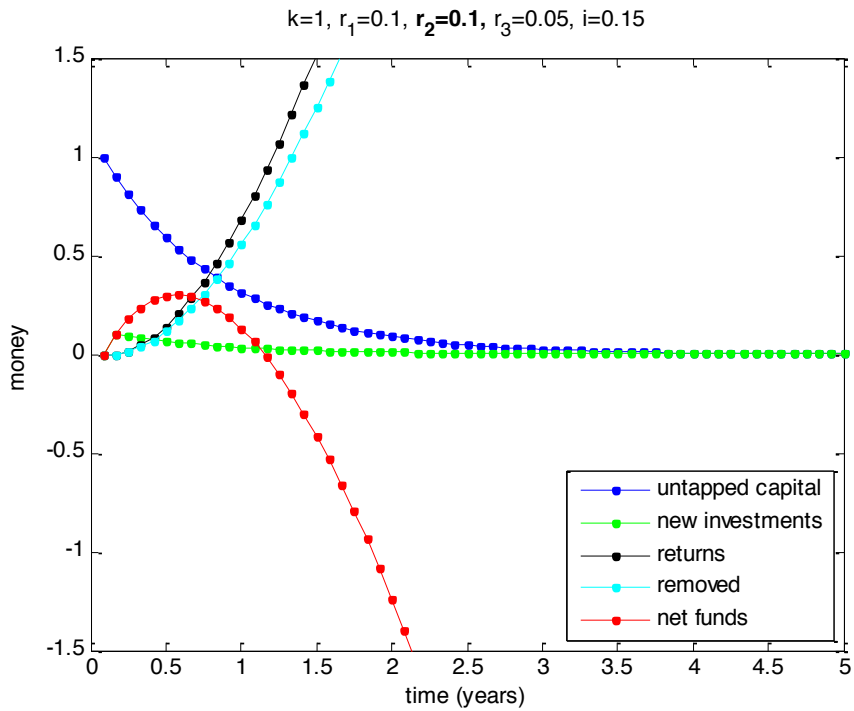


Figure 5: Pull-out rate of 10%

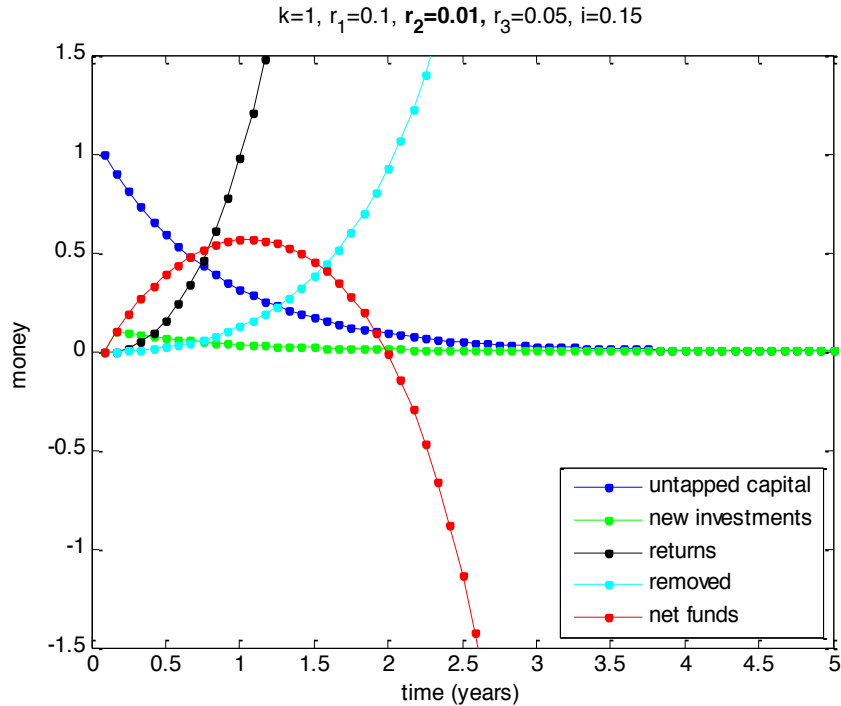


Figure 6: Pull-out rate of 1%

With a pullout rate of just 1% per month (Figure 6), a much greater deal of money gets reinvested each period. Reinvested money provides a hefty income, increasing the peak profit and longevity. However, because the promoter is compounding interest on these investments, the promoter’s virtual funds balloons rapidly. Eventually, there is not enough income available from new investors to pay off investors’ withdrawals, even if only a small proportion of earning capital is removed each month. As such, funds will eventually decline rapidly and the scheme will collapse if the promoter does not abandon in time. On the other hand, with greater peak profits, bankruptcy will take longer to occur, providing greater flexibility to the promoter.

Similarly, when the monthly withdrawal of interest is changed, it has a similar effect as complete pull-out, but obviously not as severe. If less of the interest accrued on earning capital is kept in the scheme, the earning capital pool will compound more slowly and fewer returns will need to be dispensed per month; however, this will reduce the maximum profit (Figure 7).

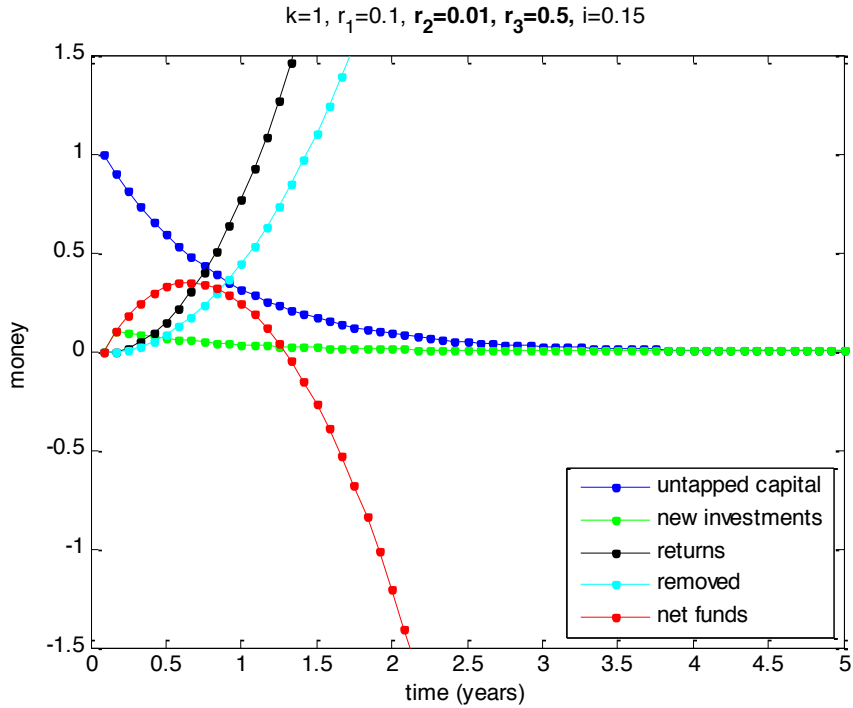


Figure 7: Interest withdrawal rate of 50%

In the ideal case where no investment capital is ever removed from the scheme (Figure 8), profits are maximized, eventually capturing all of I , and the returns compounded are purely fictitious numbers for investors to see. These usually cause public and regulatory suspicion when a Ponzi scheme has reached its height.

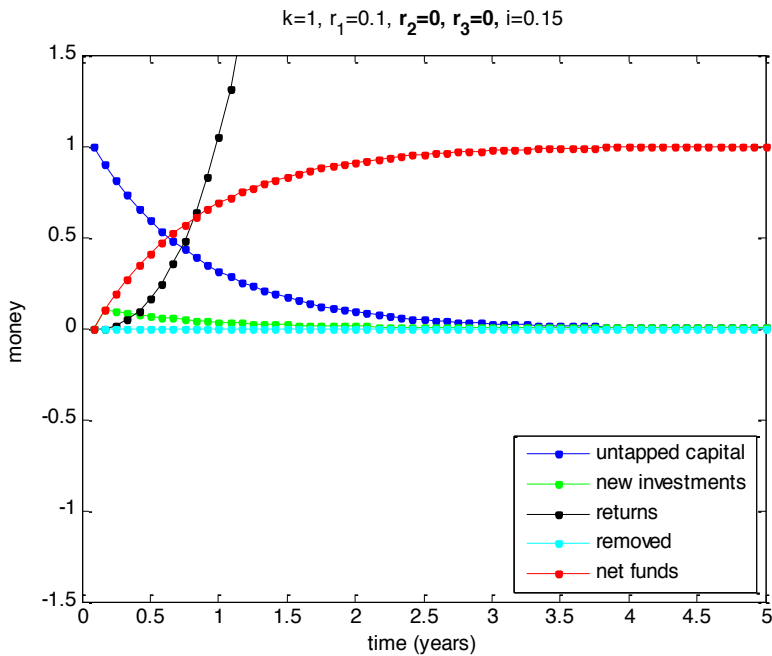


Figure 8: Complete reinvestment

1.3 Holding period

Here we explore the effects of holding new investors' money for a longer period of time in order to reduce the burden of remittances on earning capital and investor flight. Figure 9 shows the results from simulations involving identical conditions but changing the duration of the holding period.

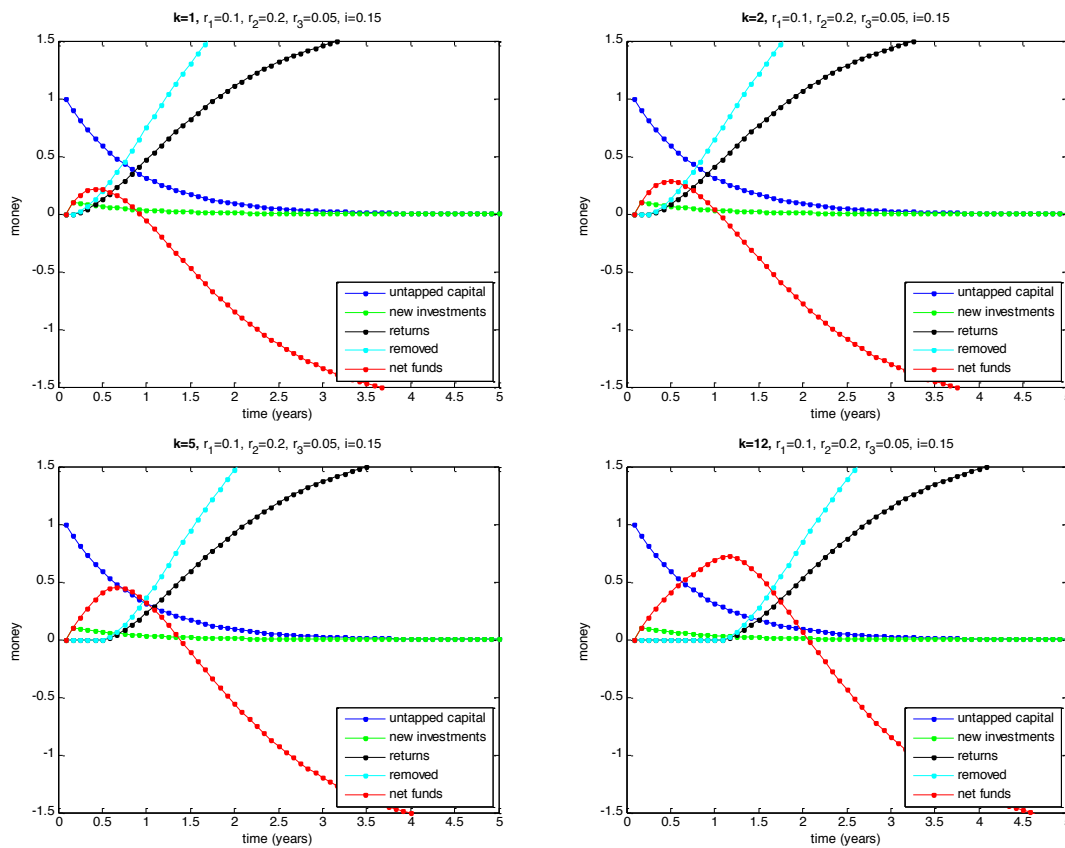


Figure 9: Varying the holding period, $k = 1, 2, 6,$ and 12 months

Increasing the holding period can make the scheme more profitable at first, since all of the funds initially are not part of earning capital and are not given interest, and investors cannot withdraw. However, once the first holding period is over, profits begin to decline with the similar speed. Realistically, long holding periods are probably not feasible, at least when they do not allow investors to sell back their shares before maturity. Alternatively, the promoter may allow an investor to pull their money out of holding before maturity without gaining any interest.

2. Volatility

It appears as though Ponzi schemes are inherently unsustainable. Therefore, a strategizing promoter would likely try to get out way before collapse. But in reality it would be very difficult to predict when profits will peak and when will be the most opportune time to quit due to volatility and uncertainty in investors' behaviour in changing economic times. It can be shown that the Ponzi scheme can be quite sensitive to volatility.

2.1 Fluctuating income and investor capriciousness

Rather than assume a diminishing income from a finite pool of money, one may consider that over time new investments per month are relatively steady. This can be modelled by setting H_n to some constant. The result is the same basic profile as seen before with a decaying income. Even with a steady income, eventually the withdrawal demands become too large to handle and net funds plummet.

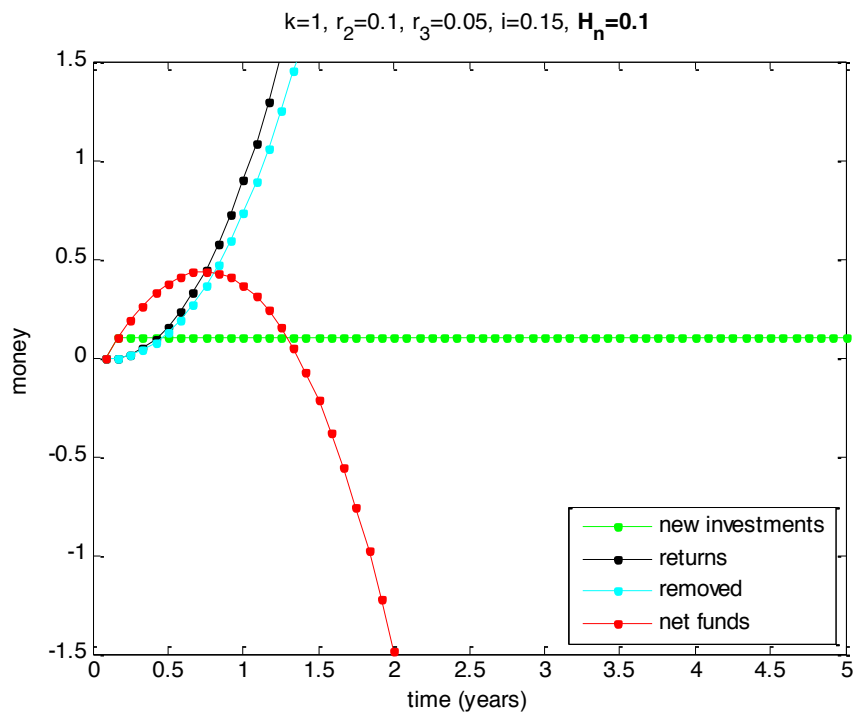


Figure 10: Ponzi scheme with steady monthly income

Random month-to-month fluctuations in incoming capital may be considered by introducing some Gaussian noise to the average monthly income from new investments.

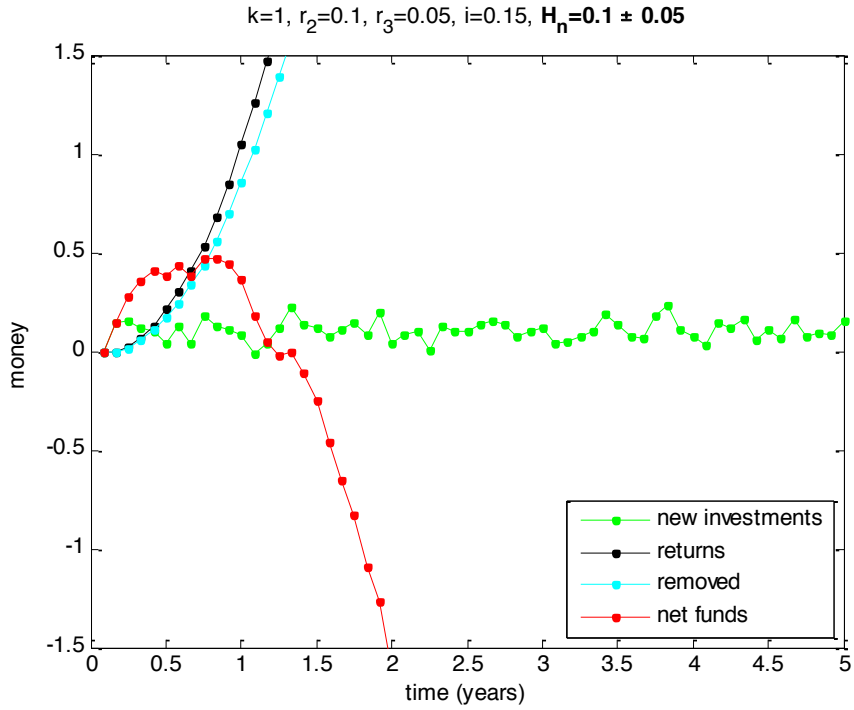


Figure 11: Noisy monthly income, s.d. = 0.05

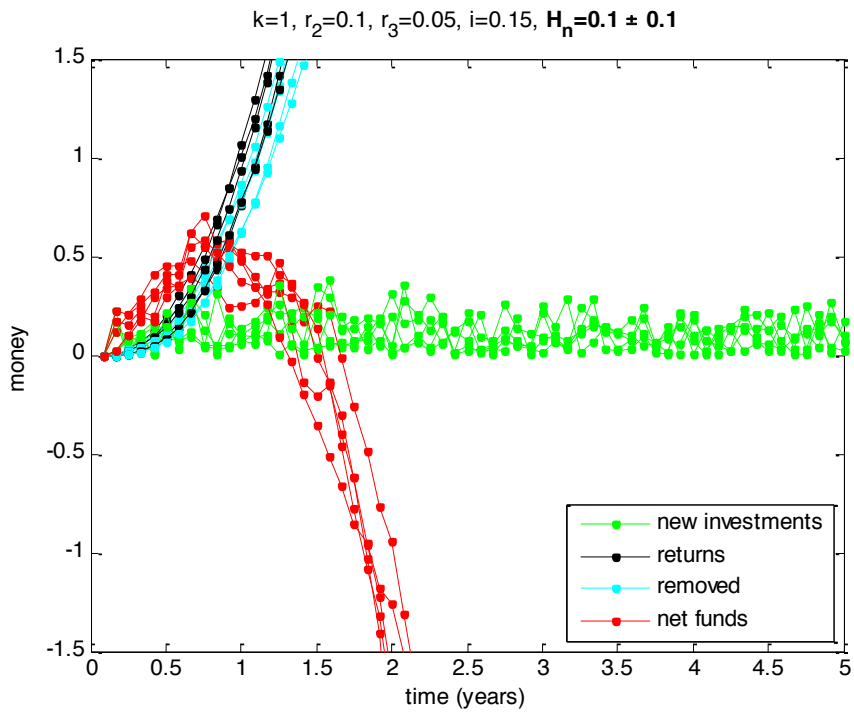


Figure 12: Noisy monthly income, s.d. = 0.1

From the above figures, it appears as though mild fluctuations and noise do not propagate too intensely, and that the basic trajectory of Ponzi scheme funds is not too strongly influenced.

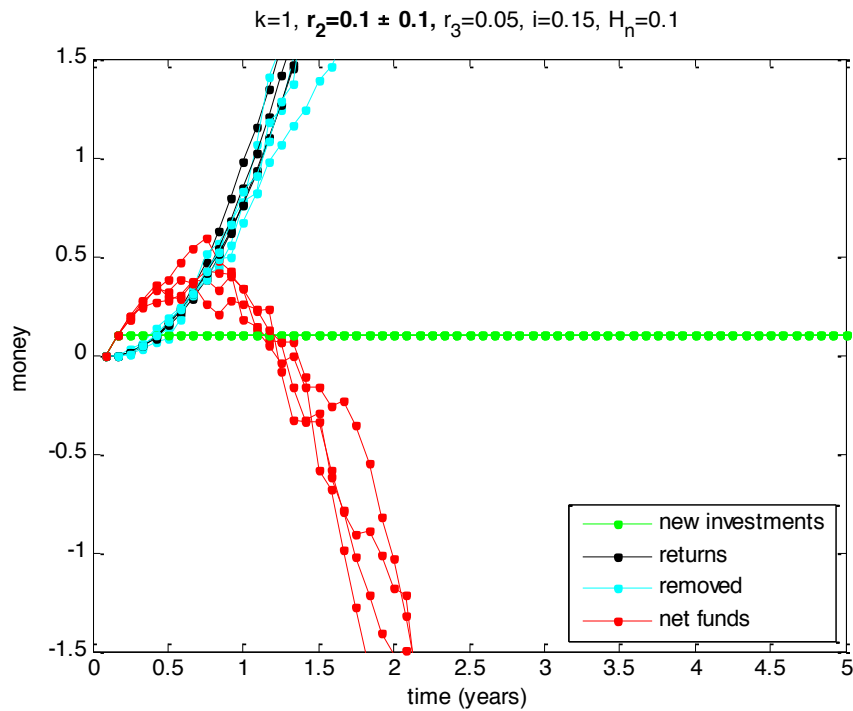


Figure 13: Noisy pullout rate

Similarly, a noisy pull-out rate will mildly influence the probable time course of the promoter's funds, but so long as the average rate is fixed, the general trend remains the same.

2.2 Sudden pullout

The real threat to a Ponzi scheme is not background variability in month-to-month behaviour, but very significant unpredictable events. A key investor may suffer losses in some other ventures and will suddenly need to take large sums of money out of his or her existing securities. Certain fears or speculation may cause a sudden exodus of investor dollars from the scheme.

Furthermore, major financial crises, like the current global economic recession, catalyze Ponzi scheme collapses by increasing such fears and liquidity crunches.

The following figures demonstrate the effect of a sudden rise in pullout rate from 10% to 90% occurring at different instances in the lifetime of a scheme.

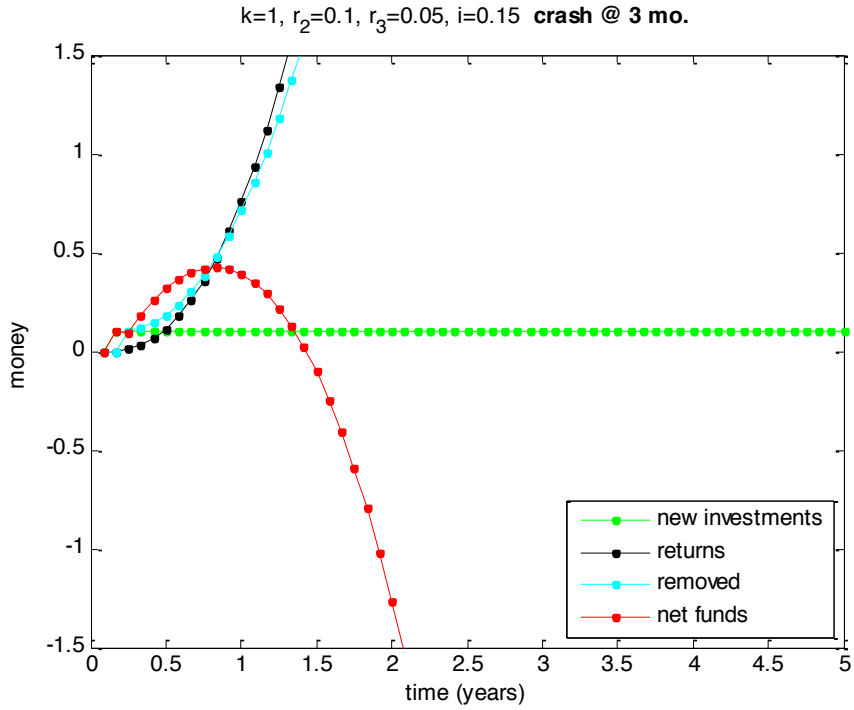


Figure 14: Sudden pullout at 3months

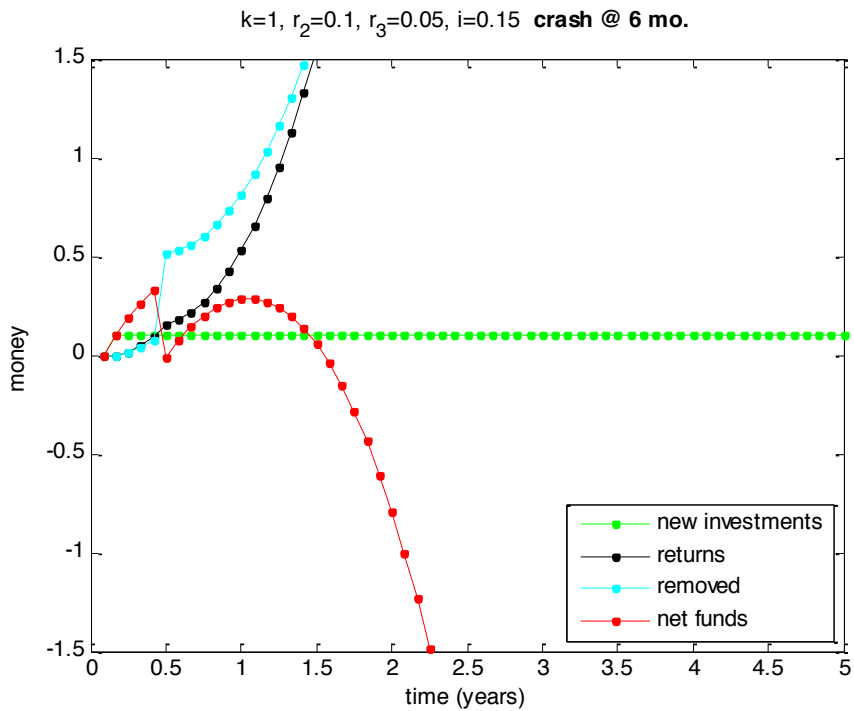


Figure 15: Sudden pullout at 6 months

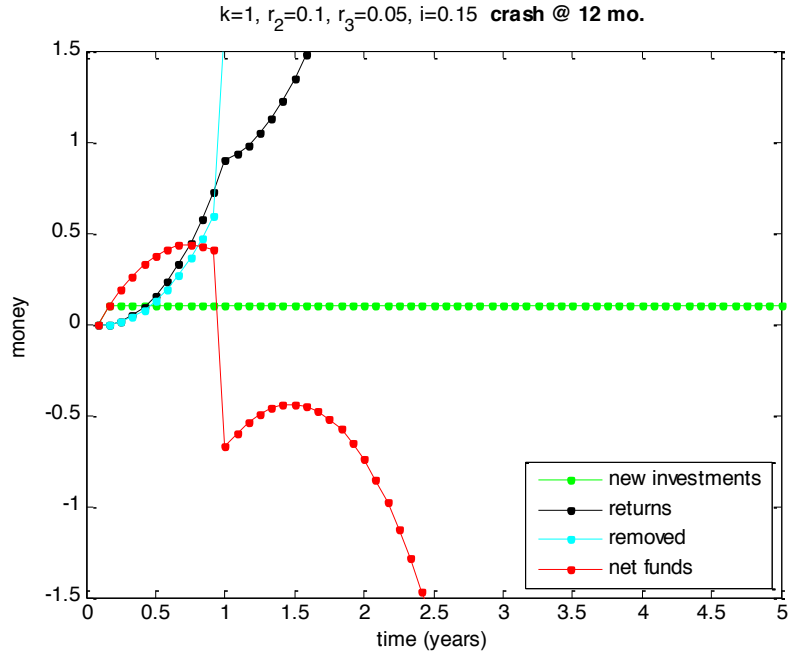


Figure 16: Sudden pullout at 12 months

If sudden pullout occurs early and transiently, the promoter may be able to recover his losses. Later on, as more of the apparent funds consist of compounded and reinvested interest, sudden pullout becomes much more dangerous. If such events are unpredictable or persistent, it makes the Ponzi scheme a very risky enterprise, as suggested in Figure 17.

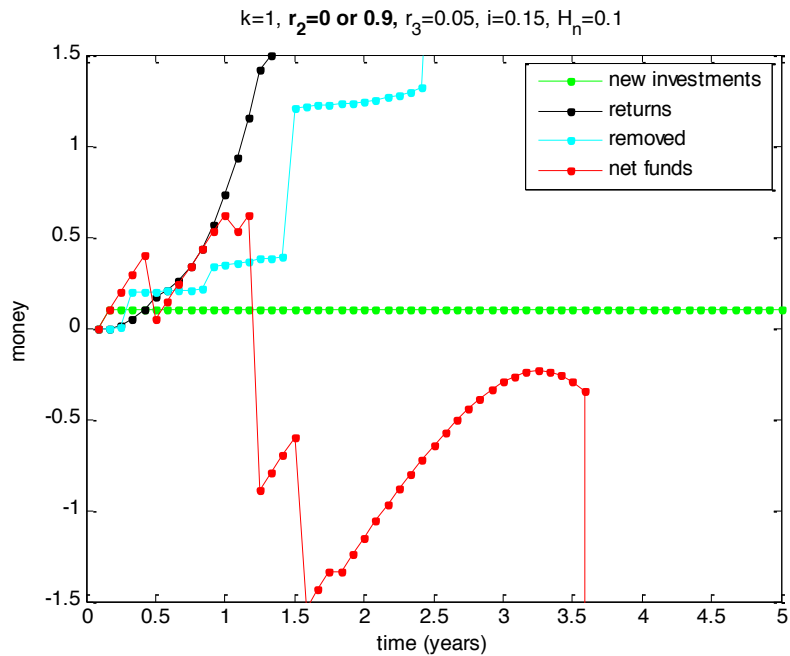


Figure 17: Unpredictable sudden withdrawals

3. Implications

The results shown highlight the two large obstacles that kill most Ponzi schemes within a year or two: unsustainable investment growth and volatility. Many schemes fail because the promoter eventually runs out of new investors and is unable to pay out promised returns. Sudden investor pullouts are also destructive to the scheme because the promoter must pay out more money than is circulating in the scheme. Essentially, a Ponzi scheme collapses if too many people leave with their investments and returns because there is only enough money circulating to finance a certain amount of withdrawals. As simulations suggest, there are big trade-offs: strategies to boost profits initially (higher interest rate, low withdrawals) cause the virtual funds of the scheme to balloon so that it becomes hypersensitive to sudden large withdrawals or limiting income from new investments.

In a volatile economy characterized by a succession of “boom” and “bust” periods, it would be nearly impossible to predict how long such a scheme could be sustained or what time would be best for the promoter to abandon. Is it therefore simply a combination of luck and stable economic times that allow a promoter to get away with fraud for a long period of time? A recently uncovered Ponzi scheme survived twenty years provides insight in how the adverse factors can be mitigated.

In December 2008, Bernard Madoff was arrested and charged with fraud after being accused of bilking up to \$50 billion from his investors. Despite eventually getting caught, it was the largest scale Ponzi scheme ever committed. Madoff circumvented the above problems by targeting only certain investors. His most interesting tactic was limiting his clientele to mainly charities. Federal law stipulates that charitable foundations must spend five percent of their funds on services: “The 5% Rule”. Consequently, Madoff was guaranteed that these investors would take out only a small share of investments every year (thus, r_2 and r_3 are kept very low). Being non-profit organizations, there was hardly any danger of experiencing a run on the bank (low volatility). By only having to pay out five percent of the principal every year, Madoff would have run out of money after only twenty years if there were no new investors. Furthermore, Madoff promised a relatively modest interest rate of 12% which was in line with returns that many stocks were promising, and not as likely to rouse suspicion. As the model predicts, a lower interest rate increases the sustainability of the scheme and reduces the impact of short term losses when incoming capital is short.

Madoff’s “charitable” works not only guaranteed stability, but instilled confidence and trust in people. Some even begged for him to take their money. By denying certain investors, Madoff created an aura of exclusivity. This enticed new investors and promoted sustainable growth – there would be no shortage of available investment capital if some years saw larger withdrawals than usual. However, even this carefully managed fraud was not immune to the powerful effect of the global recession, which prompted mass pullout and exposed its true form.

4. Limitations

Some limitations should be considered when interpreting the predictions offered by the proposed model. First, the assumption that a finite pool of capital trickles into the scheme at a decaying rate is a crude approximation to the fact that the amount of investors that can be sought to join the scheme is limited. A steady but noisy investment income may be more realistic over a short time period, but in reality the investment income is influenced by many factors, including the promoter's recruitment abilities. Early on, one would expect positive feedback from satisfied clients to attract new investors, providing an initial boost. Such effects could be explored by simulation.

The model also assumes that all of the promised returns are fictitious. Real Ponzi schemes are often hybrids where part of the returns is legitimate and the rest must be fabricated because the promoter uses exaggerated interest rates to attract investors. A hybrid model would be more shock-resistant and stable since there is more money circulating. Furthermore, it would appear more legitimate and less likely to be discovered by authorities. Perhaps adapting such a hybrid scheme to the model could prove it to be theoretically sustainable under certain conditions.

The model does not explicitly account for many other complications in the accounting of such fraudulent schemes. Because the promoter can steal some of the money, convoluting it with personal nonliquid assets like property or hiding it offshore, tracking the funds is not as straightforward as following a balance sheet. If the promoter has taken out a significant amount of money, the scheme will be less resilient to shocks and more likely to crash earlier than the model predicts. Furthermore, the aforementioned shocks have a huge impact on what happens in the model. However, they can be neither forecasted nor modelled accurately. At any given moment the scheme can deviate from the model due to unanticipated events.

The rates used in the model are arbitrary. The rate at which people leave the scheme or reinvest their money are captured as rates of cash flow and often assumed to be constant. Investors are groups of individuals, whose wealth is not equally distributed, making it difficult to generalize and represent the actions of the group as average flows of money. Furthermore, because we do not explicitly model the investor population (which would be difficult), the model does not capture some of the important negative effects of Ponzi schemes, such as the net loss the individual investors suffer when such schemes terminate. For instance, those individuals who pull out early actually do make profit (but from other people's money), while the investors who are less fickle, such as the charities Madoff targeted, will be subjected to great losses if they have large amounts of money invested when the scheme runs out of funds.

Conclusion

The model reveals some of the intricacies of a Ponzi scheme. Along with the Ponzi and Madoff case studies, it uncovers the inevitable pitfalls and helps in understanding how some fraudsters may try to avoid them. The most important features appear to be sustainable growth and volatility.

The scheme must grow in terms of attracting new investments because these are key in paying back existing investors. It is crucial to the longevity of the scheme that existing investors also reinvest their money. For sustainable growth, the promoter wants as little money withdrawn as possible. In order for sustained reinvestment to occur, the promoter must instill confidence and trust in investors.

Since there is a finite amount of investment capital, the promoter will eventually run out of money and growth will come to a halt. In Madoff's case, not all willing investors were allowed entrance into the scheme. This exclusivity controlled the amount of money coming in as well as the promised returns. Assuming that everyone and anyone is able to invest, the promoter could potentially make a lot of money in a very short amount of time. However, this would increase volatility and, as the model predicts, the large amount of promised returns is a death sentence in an unpredictable economic climate. Madoff had many willing investors thanks to his reputation. In other words, he had a lot of money that he could access whenever funds were too low. Slow sustainable growth is one reason that his scheme was able to last so many years.

Just as importantly, introducing volatility in the model can be destructive. The environment itself cannot be controlled since economic conditions are generally unpredictable. However, it is possible to control the investments and induce stability indirectly. Madoff's approach is ingenious in that he targeted investors who were not profit-seeking and thus unlikely to withdraw money depending on economic conditions. His investors were content with long-term stable growth. However, the scheme proved not to be immune to the 2008 financial crisis, which actually exposed numerous other Ponzi schemes worldwide, an effect coined "Ponzimonium".

Though Ponzi schemes are inherently self-destructive, it is possible to extend their lifetime and profitability. In the end, the scheme could be successful so long as the promoter does not run out of money in his own lifetime. Ultimately, in many cases hybrid Ponzi schemes may be the result of a strategy to rake in large starting capital to be put into real investments, with the intent to eventually recoup the costs of the returns paid to investors at a later period. Under volatile conditions, such schemes to cheat the system are likely to be uncovered early. However, they may thrive unnoticed when times are stable. The devastating consequences of frauds such as Madoff's can be minimized if such schemes are discovered early. Therefore, from the standpoint of regulatory agencies, it would be advisable to be most vigilant for suspicious investment activity *when times are good*, because when times are bad, Ponzi schemes and the like will tend to reveal themselves on their own.