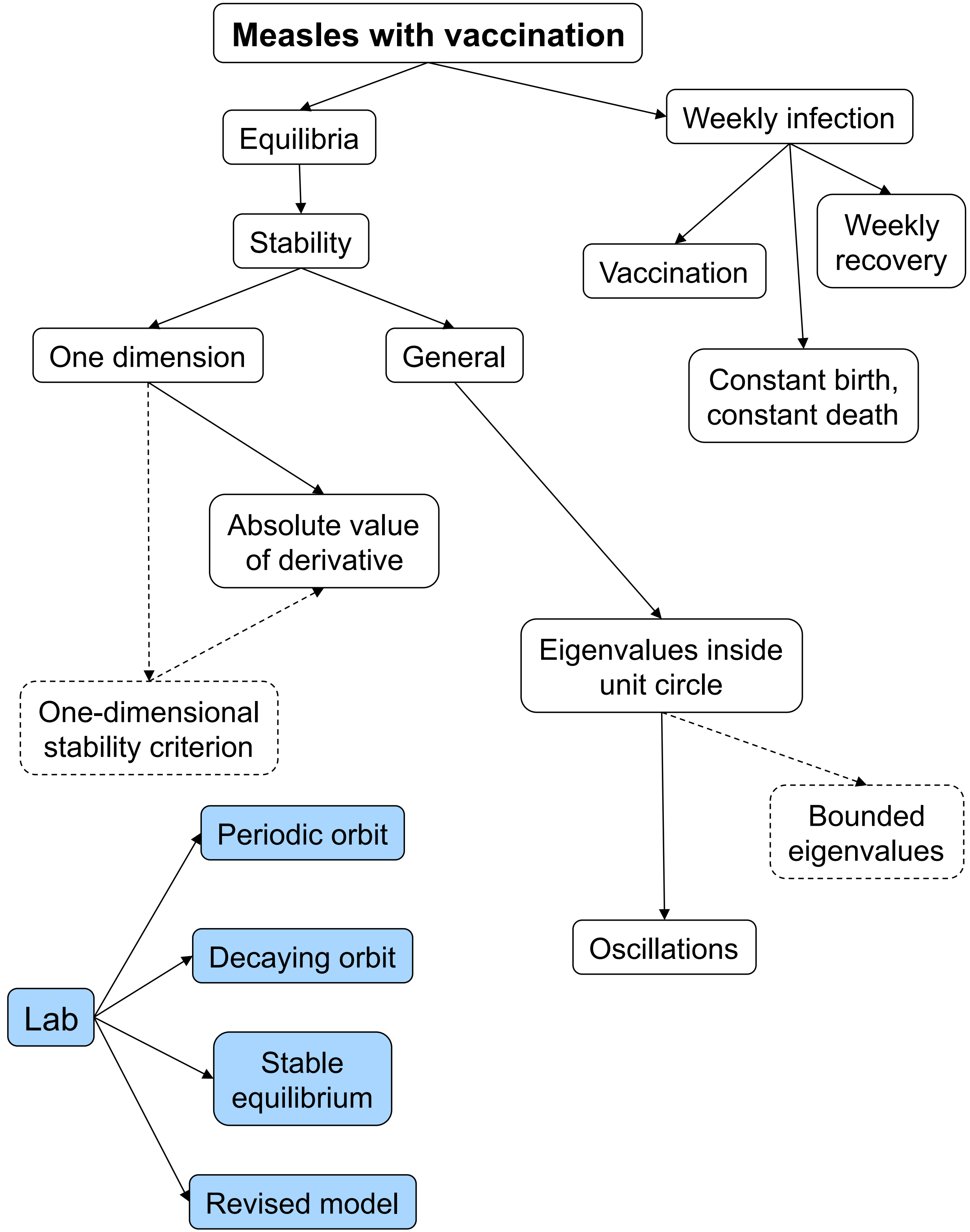
Measles with vaccination

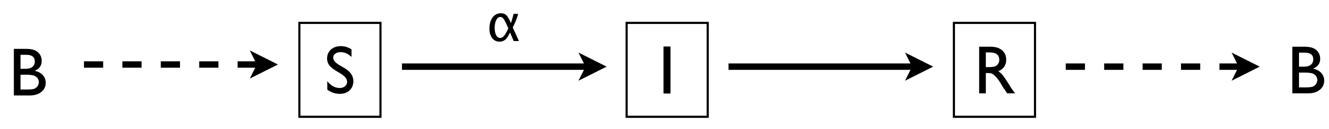
- Higher-order discrete-time dynamical systems
- Driven by an underlying timestep
- In this case, measles infection lasts a week
- We also add in fixed vaccination rates.


Measles with vaccination
Equilibria
Weekly infection
Stability
Vaccination
Weekly recovery
One dimension
General
Constant birth, constant death
Absolute value of derivative
Eigenvalues inside unit circle
One-dimensional stability criterion
Bounded eigenvalues
Periodic orbit
Oscillations
Decaying orbit
Lab
Stable equilibrium
Revised model

Assumptions

- Assume the births and deaths are equal each week
  - not true in general, but okay on a weekly timescale
- Individuals recover from measles within a week
  - so there are only new measles cases each week
- Newborns are born susceptible
- Only recovered individuals die
  - since everyone catches it or is vaccinated.

Difference equations



- We use difference equations rather than ODEs, since time is discrete
- Instead of derivative, we update each class at each timestep
- Many updates don't change the state of that variable
  - if there were susceptibles last week, there will be susceptibles this week.

The basic model

$$S_{t+1} = S_t - \alpha I_t S_t + B$$
$$I_{t+1} = \alpha I_t S_t$$
$$R_{t+1} = R_t + I_t - B$$

↑ Anyone infected last week is now recovered

- We're ignoring death due to the disease
  - reasonable in the Global North, less so elsewhere
- Birth and death are constant, not proportional to population size
- The infectious period = the time step.

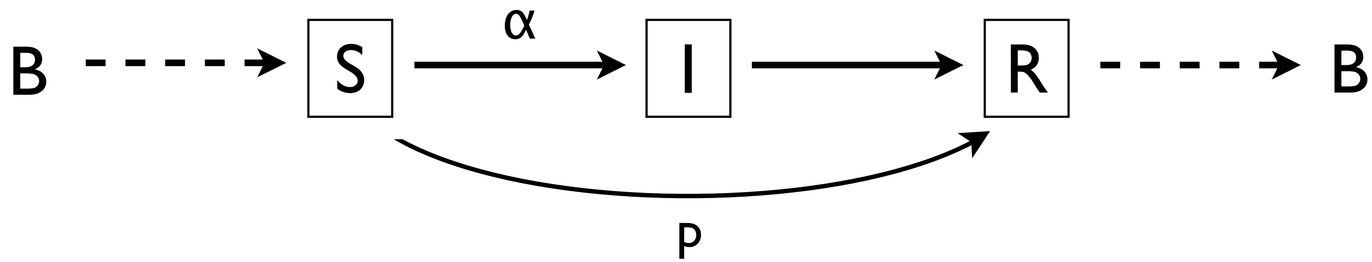
Equilibria

$$S_{t+1} = S_t - \alpha I_t S_t + B$$
$$I_{t+1} = \alpha I_t S_t$$
$$R_{t+1} = R_t + I_t - B$$

- Equilibria occur when there's no change in time
- For discrete systems, that means the timestep becomes irrelevant
- i.e. $S_{t+1}=S_t=S$, $I_{t+1}=I_t=I$ and $R_{t+1}=R_t=R$
- Note: if I=0, then there's no solution if B>0...

  ...and no info if B=0
- The only equilibrium is $(S,I,R)=(1/\alpha,B,R)$, where R is arbitrary.

Adding vaccination



$$S_{t+1} = (1-p)S_t - \alpha I_t S_t + B$$
$$I_{t+1} = \alpha I_t S_t$$
$$R_{t+1} = R_t + I_t - B + pS_t$$

- Vaccination takes some susceptibles directly to the recovered class
- A proportion p are vaccinated each week.

Total population

- Set $N_t = S_t + I_t + R_t$
- Adding the equations together, we have

$$N_{t+1} = S_t + I_t + R_t = N_t$$

- We've thus deduced that the population size remains constant over time
  - not true in general
  - we could thus write $N_t = N$.

Decoupling

- The third equation decouples from the model
  - since S_{t+1} and I_{t+1} do not depend on R_t
- We can thus look at the susceptible and infected classes only

$$S_{t+1} = (1-p)S_t - \alpha I_t S_t + B$$
$$I_{t+1} = \alpha I_t S_t.$$

Finding equilibria

- Equilibria must satisfy $S_{t+1}=S_t$ and $I_{t+1}=I_t$
- Plugging into the model, we have

$$S = (1-p)S - \alpha I S + B$$
$$I = \alpha I S$$

- From the second equation, either I=0 or S=1/α
- We thus have two cases
  - let’s do them both.

Equilibria

$$S = (1-p)S - \alpha I S + B$$
$$I = \alpha I S$$

I=0

S=S-pS+B

pS=B

S=B/p;

S=1/α

1/α=(1-p)/α-I+B

I=B-p/α

(only exists if B-p/α>0);

- Hence the equilibria are

$$(S, I) = \left(\frac{B}{p}, 0\right), \left(\frac{1}{\alpha}, B - \frac{p}{\alpha}\right).$$

Existence of equilibria

- The endemic equilibrium only exists if $B - p/\alpha > 0$
  - if the vaccination rate is high or the transmission low, it won't exist
- If there's no vaccination ($p=0$), there's no disease-free equilibrium
  - just as there wasn't in the basic model
- If there's no disease, the system would blow up
  - the death assumption assumes that everyone passes through to the recovered stage.

Jacobian

- The Jacobian is given by

$$J_p = \begin{bmatrix} 1 - p - \alpha I_t & -\alpha S_t \\ \alpha I_t & \alpha S_t \end{bmatrix} \qquad p > 0$$

$$J_0 = \begin{bmatrix} 1 - \alpha I_t & -\alpha S_t \\ \alpha I_t & \alpha S_t \end{bmatrix} \qquad p = 0$$

- The linearisation is thus

$$\begin{pmatrix} S_{t+1} \\ I_{t+1} \end{pmatrix} = J_i \begin{pmatrix} S_t \\ I_t \end{pmatrix} \text{ for } i = 0, p$$

- Let's first figure out how stability works in one-dimensional discrete-time systems.

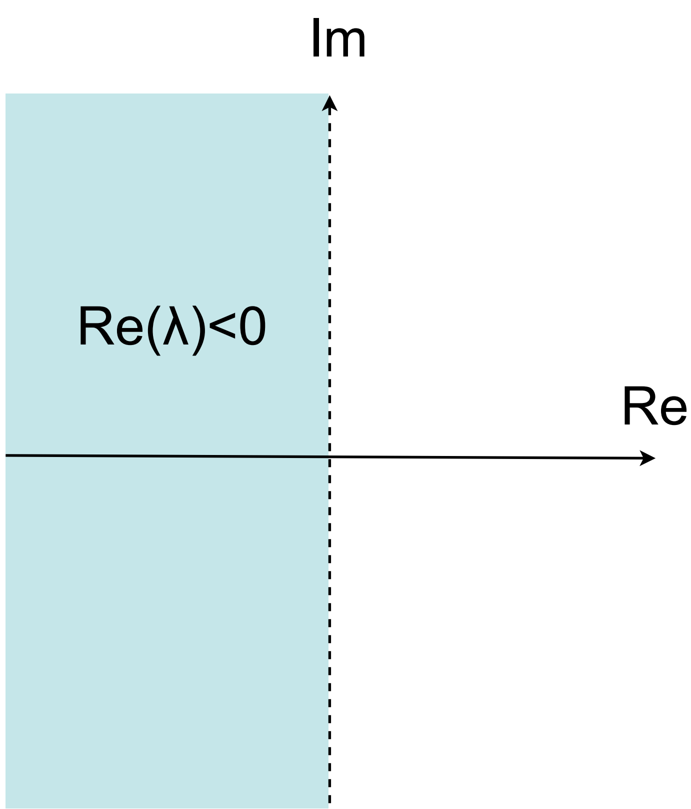
One-dimensional stability

- An equilibrium x_0 of $x_{t+1}=f(x_t)$ is (locally) stable if $|f'(x_0)|<1$ and unstable if $|f'(x_0)|>1$
  - See Appendix J for details
- In one dimension, $f'(x_0)$ is just the (single) eigenvalue of the (1×1) Jacobian matrix
  - we'll generalise this shortly
- If $|f'(x_0)|=1$, the results aren't predictable

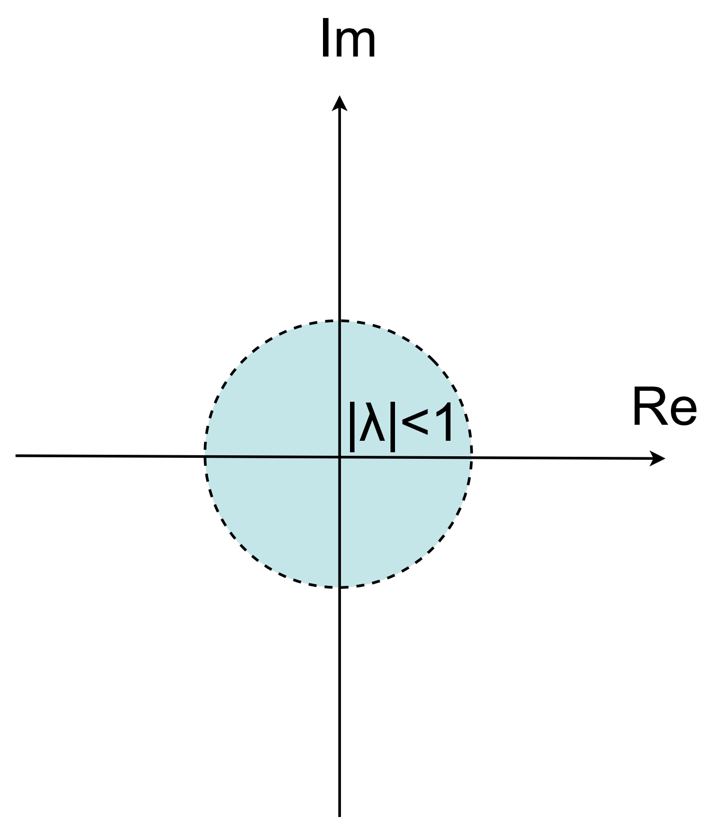
  ...but they weren't predictable for $\lambda_{max}=0$ either
- In general, $f'(x_0)$ can be complex
  - so we have stability when the eigenvalues are within the unit circle.

Eigenvalue comparison

- ODEs



- Discrete-time systems



Stability in two dimensions

- First let's look at p=0
- The Jacobian at the endemic equilibrium is

$$J_0\left(\frac{1}{\alpha}, B\right) = \begin{bmatrix} 1-\alpha B & -1 \\ \alpha B & 1 \end{bmatrix}$$

(Remember that this is the only equilibrium in the case p=0)

- Let's find the eigenvalues.

Finding eigenvalues

$$\begin{aligned}
\det\left(J_0\left(\frac{1}{\alpha}, B\right) - \lambda I\right) &= \det\begin{bmatrix} 1-\alpha B-\lambda & -1 \\ \alpha B & 1-\lambda \end{bmatrix} \\
&= (1-\alpha B-\lambda)(1-\lambda) + \alpha B \\
&= \lambda^2 - (2-\alpha B)\lambda + 1 \\
\lambda_{1,2} &= \frac{2-\alpha B \pm \sqrt{(2-\alpha B)^2 - 4}}{2} \\
&= \frac{2-\alpha B \pm \sqrt{\alpha B(\alpha B - 4)}}{2}
\end{aligned}$$

- There are two cases here:
  - $\alpha B > 4$
  - $\alpha B < 4$.

αB>4

$$\begin{aligned}\lambda_2 &= \frac{2 - \alpha B - \sqrt{\alpha B(\alpha B - 4)}}{2} \\ &< \frac{2 - 4 - \sqrt{\alpha B(\alpha B - 4)}}{2} \\ &= -1 - \frac{\sqrt{\alpha B(\alpha B - 4)}}{2}\end{aligned}$$

- Since the part under the square root is positive, it follows that $|\lambda_2|>1$
- Hence the equilibrium is unstable
  - the behaviour of λ_1 is irrelevant.

αB<4

- In this case, the roots are complex conjugates
- But this is no problem
- We can write $\lambda_{1,2} = \frac{2 - \alpha B \pm \sqrt{\alpha B(4 - \alpha B)}i}{2}$
- Then

$$
\begin{aligned}
|\lambda_{1,2}| &= \sqrt{\left(\frac{2 - \alpha B}{2}\right)^2 + \frac{\alpha B(4 - \alpha B)}{4}} \\
&= \sqrt{\frac{4 - 4\alpha B + (\alpha B)^2 + 4\alpha B - (\alpha B)^2}{4}} \\
&= 1.
\end{aligned}
$$

Knife-edge stability

- Since $|\lambda_{1,2}|=1$, we cannot say much about its stability
- It might be stable
- Or it might not be
- We would need more sophisticated methods to determine this
  - beyond the scope of this course
- For now, all we can really say is that the equilibrium is not asymptotically stable.

The case p≠0

- The Jacobian at the DFE is

$$J_p\left(\frac{B}{p},0\right)=\begin{bmatrix}1-p & -\alpha B/p\\ 0 & \alpha B/p\end{bmatrix}$$

- This is upper triangular, so the eigenvalues are $\lambda_{3,4}$=1-p, αB/p
- Since 0<p<1, $|\lambda_3|$<1
- Hence the DFE is stable if

$$|\lambda_4|=\left|\frac{\alpha B}{p}\right|<1$$

$$\text{i.e., if } \alpha B<p$$

- We can thus define R_0= αB/p.

Endemic equilibrium

- Recall that the endemic equilibrium is $\left(\frac{1}{\alpha}, B - \frac{p}{\alpha}\right)$
- This only exists if $R_0>1$; i.e., if $\alpha B>p$
- The Jacobian at this equilibrium is

$$J_p\left(\frac{1}{\alpha}, B - \frac{p}{\alpha}\right) = \begin{bmatrix} 1 - p - \alpha\left(B - \frac{p}{\alpha}\right) & -\alpha\left(\frac{1}{\alpha}\right) \\ \alpha\left(B - \frac{p}{\alpha}\right) & \alpha\left(\frac{1}{\alpha}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \alpha B & -1 \\ \alpha B - p & 1 \end{bmatrix}$$

- This is not upper triangular, so we need to calculate the eigenvalues.

Two eigenvalues

$$\det(J_p - \lambda I) = \det \begin{bmatrix} 1 - \alpha B - \lambda & -1 \\ \alpha B - p & 1 - \lambda \end{bmatrix}$$
$$= (1 - \alpha B - \lambda)(1 - \lambda) + \alpha B - p$$
$$= \lambda^2 - (2 - \alpha B) + 1 - p$$
$$\lambda_{5,6} = \frac{2 - \alpha B \pm \sqrt{\Delta}}{2}$$

where

$$\Delta = (2 - \alpha B)^2 - 4(1 - p)$$
$$= \alpha^2 B^2 - 4\alpha B + 4p$$

- Once again, we have two cases.

Case (i): $\Delta \geq 0$

- The roots are real
  - also recall that $\alpha B > p$
- First we have

$$\begin{aligned}\lambda_5 &= \frac{2 - \alpha B + \sqrt{(\alpha B)^2 - 4\alpha B + 4p}}{2} \\ &< \frac{2 - \alpha B + \sqrt{(\alpha B)^2}}{2} \\ &= 1\end{aligned}$$

- We can also show $\lambda_5 > -1$
  - see Appendix K
- Hence $|\lambda_5| < 1$.

Examining λ_6

- Instead of proving anything, let's test two values
- If $\alpha B=1$, then

$$\lambda_6 = \frac{2-1-\sqrt{1-4(1)+4p}}{2}$$
$$= \frac{1-\sqrt{-3+4p}}{2}$$

- For the range of p that gives real roots (eg p slightly smaller than 1), we have $|\lambda_6|<1$
- Hence the endemic equilibrium is stable in this example.

Another example

- If $\alpha B=4$, then

$$\begin{aligned}\lambda_6 &= \frac{2-4-\sqrt{16-4(4)+4p}}{2}\\ &= \frac{-2-\sqrt{4p}}{2}\\ &= -1-2\sqrt{p} < -1\end{aligned}$$

- Hence $|\lambda_6|>1$ in this example
- Since λ_6 can be both inside and outside the unit circle, we can conclude that the equilibrium is sometimes stable and sometimes unstable.

Case (ii): $\Delta \leq 0$

- We have complex roots, so

$$
\begin{aligned}
|\lambda_{5,6}| &= \sqrt{\left(\frac{2-\alpha B}{2}\right)^2 + \frac{-\Delta}{4}} \\
&= \sqrt{\frac{4-4\alpha B+\alpha^2 B^2-\alpha^2 B^2+4\alpha B-4p}{4}} \\
&= \sqrt{\frac{4-4p}{4}} \\
&= \sqrt{1-p} \\
&< 1
\end{aligned}
$$

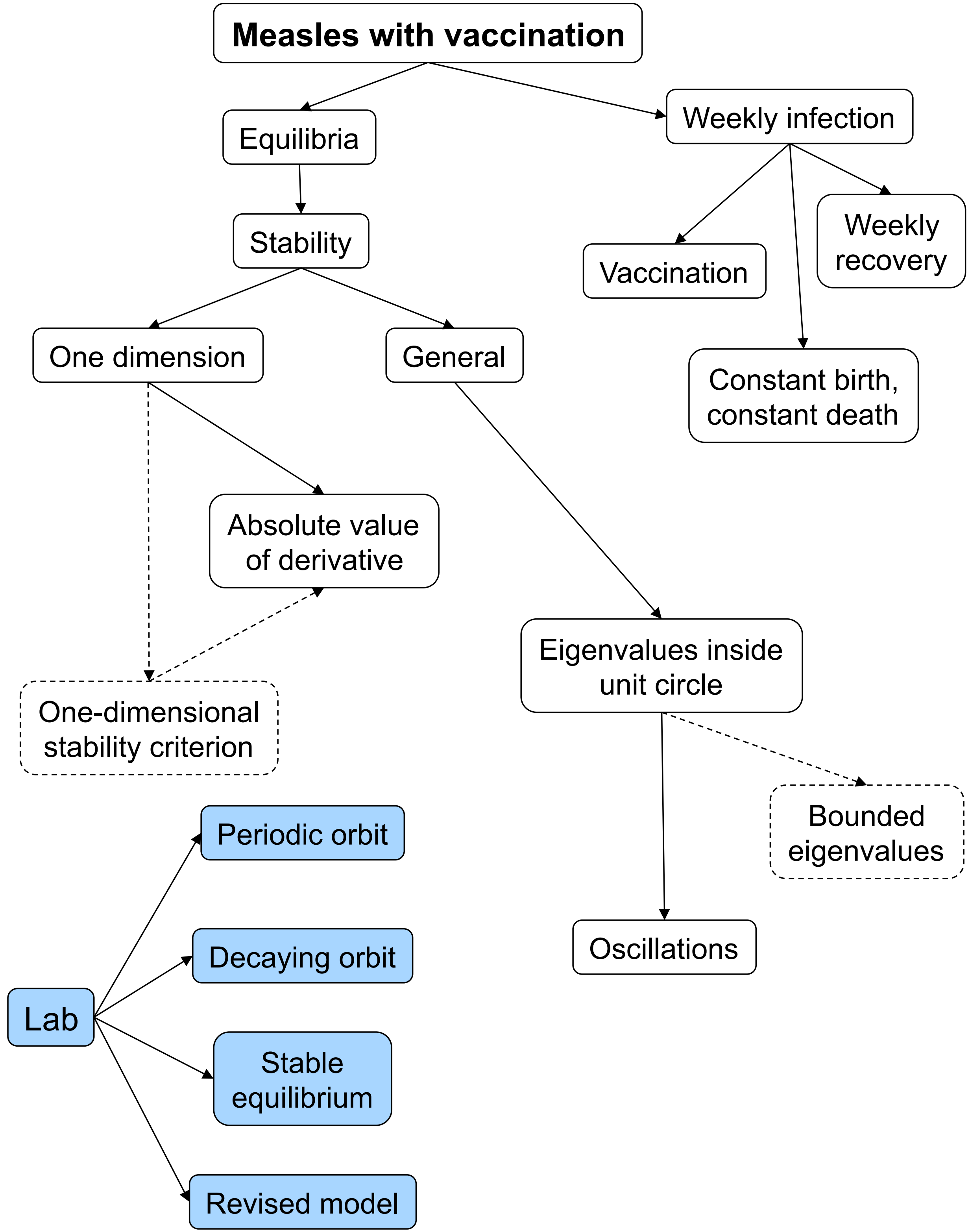
- Hence the endemic equilibrium is stable when $R_0 > 1$ and complex roots arise.

Complex roots

- Just as in the ODE case, complex roots imply oscillations
- Since we have stability, these are damped oscillations
- Hence, possible behaviours are
  - damped oscillations (complex roots) and a stable endemic equilibrium
  - potentially stable or unstable endemic equilibrium without oscillations
  - unstable DFE
  - a DFE that is not asymptotically stable.

Lab work

- In the lab, we'll plot solutions for different cases
- We'll choose parameter values and plot both time series and phase portraits
  - we'll numerically deal with the case $p=0$ where we couldn't determine stability
- We'll also adjust the model to account for more realistic birth and death rates
  - as well as a non-weekly recovery rate.



Measles with vaccination
Equilibria
Weekly infection
Stability
Vaccination
Weekly recovery
One dimension
General
Constant birth, constant death
Absolute value of derivative
Eigenvalues inside unit circle
One-dimensional stability criterion
Bounded eigenvalues
Periodic orbit
Oscillations
Decaying orbit
Lab
Stable equilibrium
Revised model