Discrete Dynamical Systems

We now look at one-dimensional maps $f : \mathbb{R} \to \mathbb{R}$. A map $y_{n+1} = f(y_n), n = 0, 1, ...$ is called a discrete dynamical system. The solution is the sequence $y_0, y_1, y_2...$ A fixed point satisfies $\bar{y} = f(\bar{y})$. A periodic orbit of order n satisfies $y_0 = f^n(y_0)$ for some $n > 0$ but $y_0 \neq f^m(y_0)$ for $0 \leq m \leq n-1$.

Example 1. The discrete logistic equation
Let's consider a disease spreading annually (for example, smallpox). We can assume:

$f(0) = 0$	if nobody is infected, no subsequent infections
$f(y_n) > 0$	no negative population, disease can't die out in finite time
f is differentiable	

Linear growth: $f(y_n) = ry_n \qquad r > 0$

However spatial considerations require something less than linear or else the disease would take over the world. Therefore we want the growth rate to be slowing down as y_n increases. ie f is concave down therefore $f''(y_n) < 0 \; \forall y_n > 0$.

By Taylor's theorem,

$$
\begin{aligned}
f(y_n) &= f(0) + f'(0)y_n + \frac{f''(0)}{2!}y_n^2 + 0(y_n^3) \\
f(y_n) &> 0 \text{ when } y_n > 0 \text{ so } f'(0) > 0 \to f'(0) = r \\
f''(0) &= -2b \\
\therefore f(y_n) &\approx 0 + ry_n + \frac{-2b}{2!}y_n^2 = ry_n - by_n^2 \\
ry_n &\text{ is the linear growth term} \\
by_n^2 &\text{ is the competition term}
\end{aligned}
$$

As the disease spreads, infected invididuals compete for the same limited number of susceptibles.
We plot this by putting y_n on the x-axis and y_{n+1} on the y-axis.

We can rescale:

$$x_n = \frac{b}{r}y_n \to y_n = \frac{r}{b}x_n$$

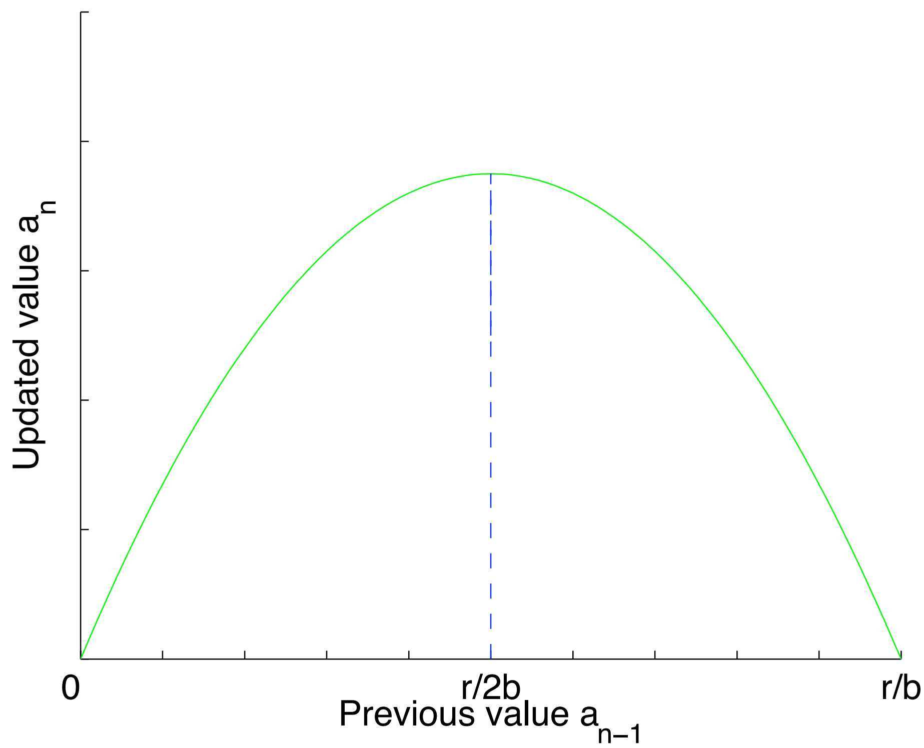
$$\frac{r}{b}x_{n+1} = r\frac{r}{b}x_n - b\frac{r^2}{b^2}x_n^2 \to x_{n+1} = rx_n - rx_n^2 = rx_n(1-x_n)$$

We follow the progression by determining $x_0, x_1, x_2, ...$ But since the old y-axis value always becomes the new x-axis value, there's an easier way to do this, called cobwebbing. Immediately, any point where the curve and the line $x_{n+1} = x_n$ meet is a fixed point.

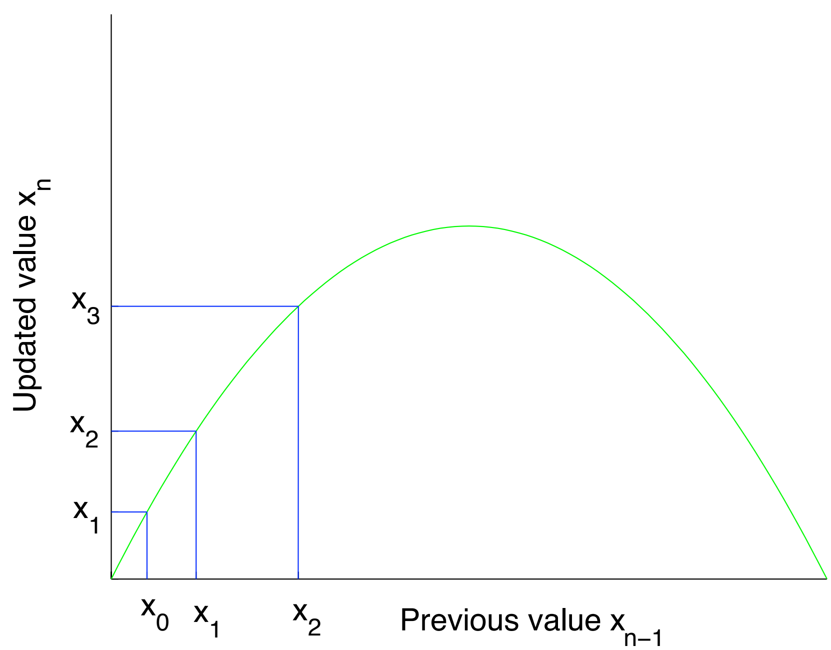
$$\therefore x = rx - rx^2 \to rx^2 + (1-r)x = 0 \to x[rx + 1 - r] = 0 \to x = 0, \frac{r-1}{r}$$

What happens at $r = 1$?
If $0 < r < 1$ then $x = 0$ is stable ($r = 0.5$)
If $r > 1$ then $x = \frac{r-1}{r}$ may be stable ($r = 2$)
If $r > 1$ then there may be an unstable equilibrium and a periodic orbit ($r = 3.2$)
If $r > 1$ then there may be chaos ($r = 4$).

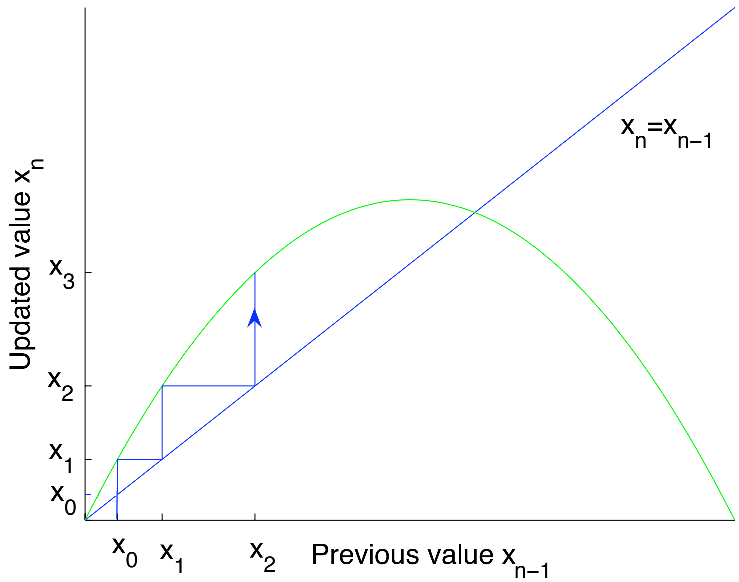
The unscaled discrete logistic function



The updating function



Suppose $\bar{x}$ is an equilibrium and x_0 is close to $\bar{x}$. ie $x_0 = \bar{x} + \epsilon$, where ϵ is small but could be positive or



negative.

$$x_1 = f(x_0) = f(\bar{x} + \epsilon) = f(\bar{x}) + f'(\bar{x})\epsilon + 0(\epsilon^2) \approx \bar{x} + f'(\bar{x})\epsilon$$

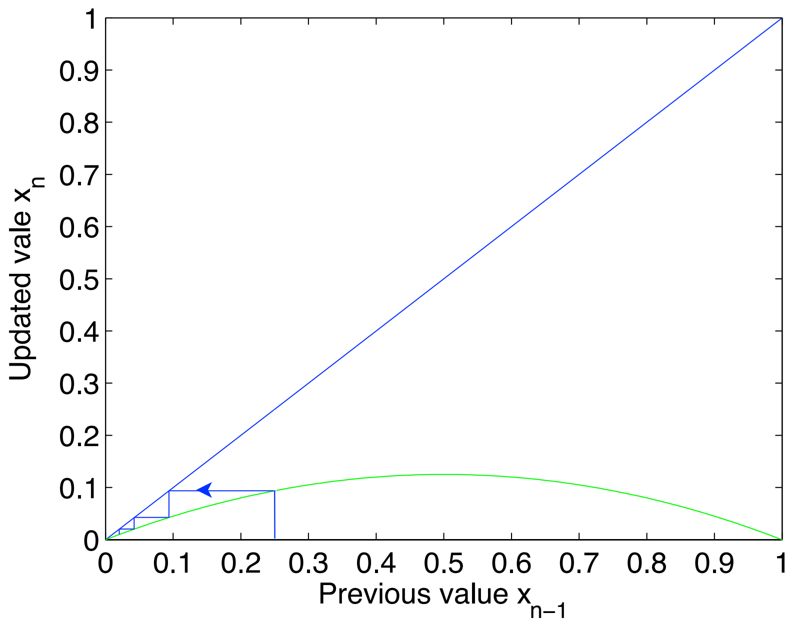
If $f'(\bar{x}) > 0$ then x_1 and x_0 lie on the same side of $\bar{x}$.

If $f'(\bar{x}) < 0$ then x_1 and x_0 lie on opposite sides of $\bar{x}$

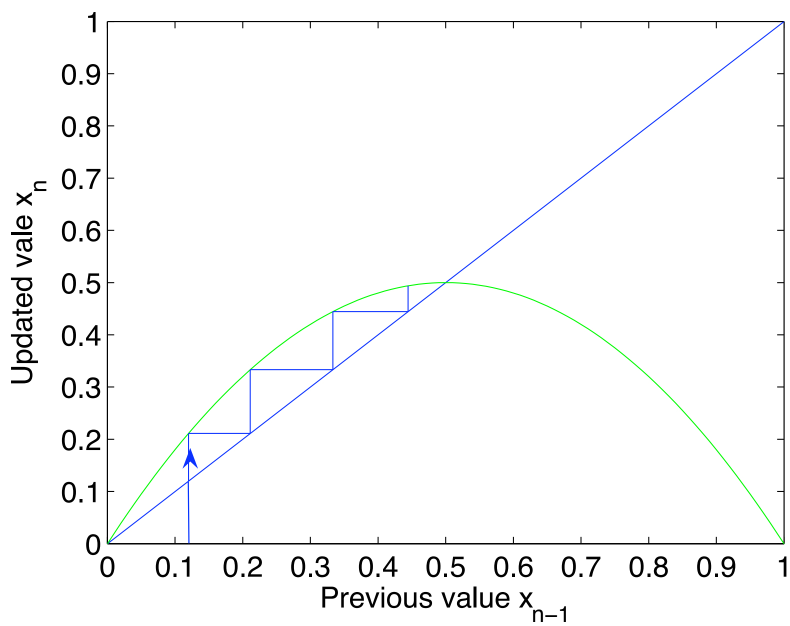
If $|f'(\bar{x})| > 1$ then x_1 is further from $\bar{x}$ than $x_0 \rightarrow \bar{x}$ is unstable.

If$|f'(\bar{x})| < 1$ then x_1 is closer to $\bar{x}$ than $x_0 \rightarrow \bar{x}$ is stable.

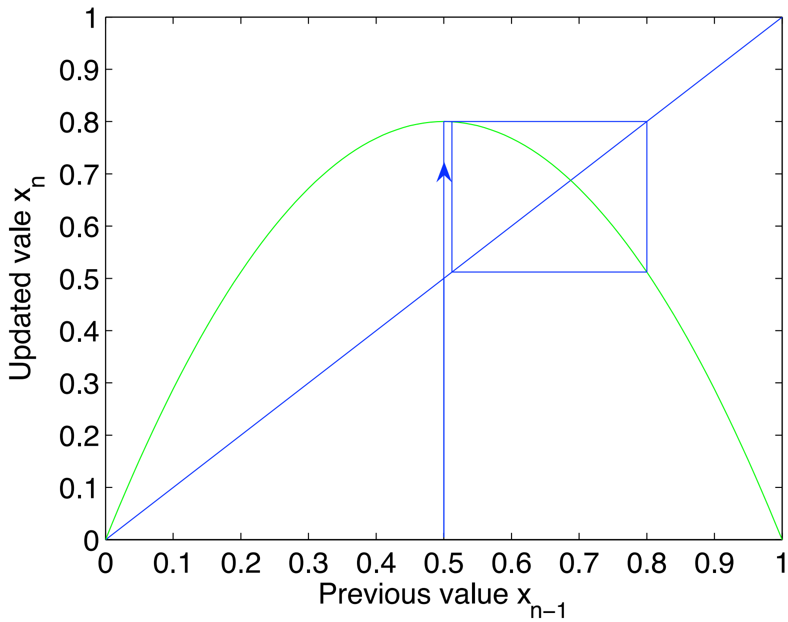
The discrete logistic equation for $r = 0.5$ (zero is stable)



The discrete logistic equation for $r = 2$ (zero is unstable, the other equilibrium is stable)



The discrete logistic equation for $r = 3.2$ (both equilibria are unstable and a stable periodic orbit arises)



The discrete logistic equation for $r = 4$ (chaos)