Bifurcations

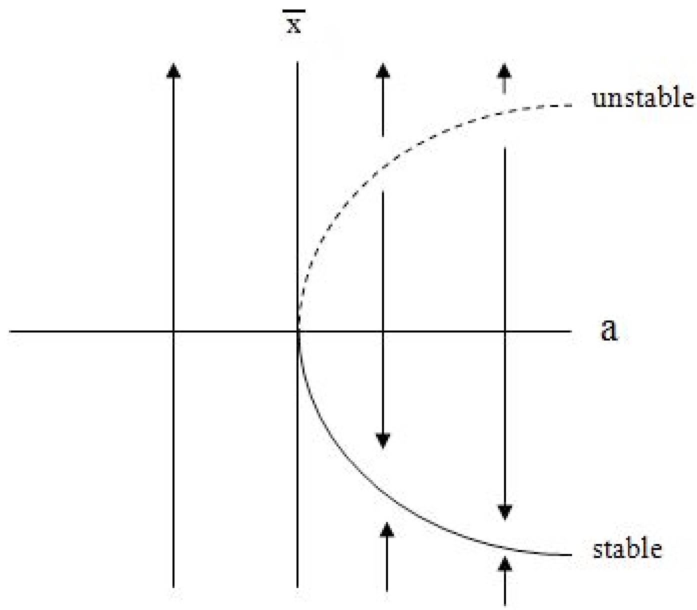
Bifurcations are one of the most important techniques in applied math. It is something that mathematicians can bring to biology that biologists know little about. Consider a family of systems $x' = F_a(x)$ where a is a real parameter. We assume F_a depends on a in a C^∞ fashion. A bifurcation occurs when solutions undergo a qualitative change as a varies.

Example 1. $x' = x^2 - a$.

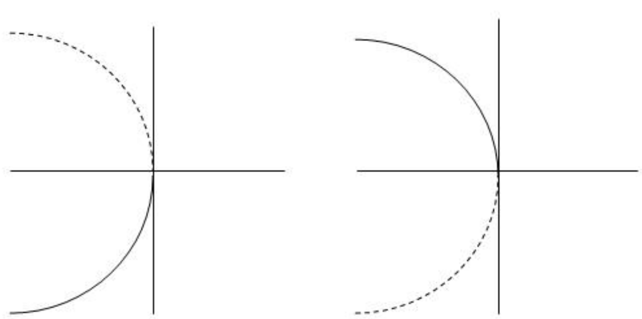
$$
\begin{array}{lrl}
\text{Equilibria:} & x^2 - a = 0 \to x = \pm\sqrt{a} & \text{if } a > 0 \\
 & x = 0 & \text{if } a = 0 \\
 & \text{No equilibria at all} & \text{if } a < 0
\end{array}
$$

Therefore, a bifurcation occurs at $a = 0$ because the fundamental nature of solutions changes there. Recall that for a 1-dimensional ODE, $\bar{x}$ is stable if $f'(\bar{x}) < 0$ and unstable if $f'(\bar{x}) > 0$.

$f'(x) = 2x$
$f'(\sqrt{a}) = 2\sqrt{a} > 0$
$f'(-\sqrt{a}) = -2\sqrt{a} < 0$
Therefore $\sqrt{a}$ is unstable whenever $a > 0$.
We illustrate this in a *bifurcation diagram*
plotting a vs $\bar{x}$



Note: stable and unstable curves always come in pairs (why?). This type of bifurcation, with no equilibria on one side and two on the other, is called a saddle-node bifurcation. (The direction of the arrows may change, as may the way the curves face.)

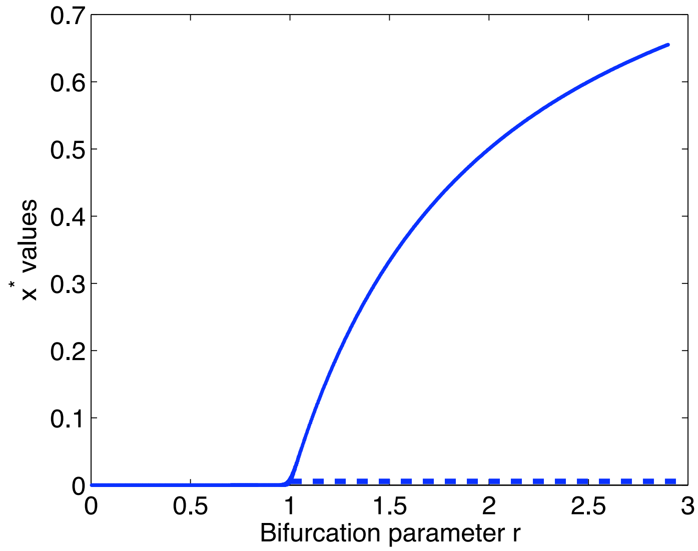


Back to our function $f(x_n) = rx_n - rx_n^2$

Differentiating:

$$
\begin{aligned}
f'(x_n) &= r - 2rx_n \\
f'(0) &= r \therefore 0 \text{ is stable if } r < 1 \text{ (and it is the only equilibrium)} \\
f'\left(\frac{r-1}{r}\right) &= r - \frac{2(r-1)r}{r} = 2 - r \text{ is stable if } |2-r| < 1 \to 1 < r < 3
\end{aligned}
$$

Bifurcation of the discrete logistic equation from a single stable fixed point to two fixed points.



What happens beyond $r = 3$?
Period 2: The system oscillates between two points ω_1 and ω_2. So $\omega_2 = f(\omega_1)$ and $\omega_1 = f(\omega_2)$. So

$$
\begin{aligned}
\omega_1 &= f(f(\omega_1)) = g(\omega_1) \\
g(\omega_1) &= f(f(\omega_1)) = rf(\omega_1)(1 - f(\omega_1)) = r[r\omega_1(1-\omega_1)][1 - r\omega_1(1-\omega_1)] \\
g(x) &= r^2 x(1-x)(1 - rx(1-x))
\end{aligned}
$$

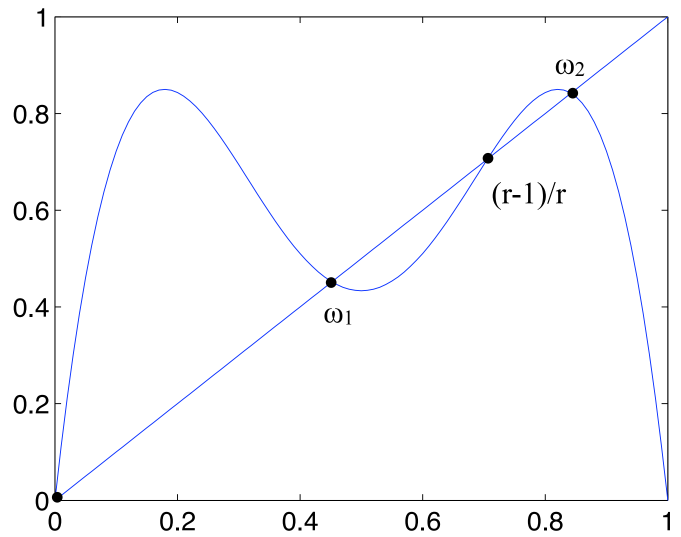
Depending on their value, the curve may intersect the line once, twice, three or four times. If $\bar{x}$ is an equilibrium of f, then it is also for g. $(g(\bar{x}) = f(f(\bar{x})) = f(\bar{x}) = \bar{x})$
Therefore two points are 0 and $\frac{r-1}{r}$.
Question: how do we find the other two?

Answer: Period 2 points of f are fixed points of g so we can look at g'.

$$
\begin{aligned}
g'(x) &= f'(f(x))f'(x) \\
g'(\omega_1) &= f'(\omega_2)f'(\omega_1) && \text{since } f(\omega_1) = \omega_2 \\
g'(\omega_2) &= f'(\omega_1)f'(\omega_2) && \text{since } f(\omega_2) = \omega_1
\end{aligned}
$$

Equilibrium values satisfy $g(x) - x = 0$

Period 2 points



$$
\begin{aligned}
g(x) - x &= r^2x(1-x)[1-rx(1-x)] - x \\
&= (r^2x - r^2x^2)(1 - rx + rx^2) - x \\
&= r^2x - r^3x^2 + r^3x^3 - r^2x^2 + r^3x^3 - r^3x^4 - x \\
0 &= -r^3x^4 + 2r^3x^3 + (-r^2 - r^3)x^2 + (r^2 - 1)x \\
0 &= r^3x^3 - 2r^3x^2 + (r^2 + r^3)x + (1 - r^2) \text{ (divide out an } x\text{)} \\
0 &= rx^3 - 2rx^2 + (1+r)x + \frac{1-r^2}{r^2} \text{ (divide by } r^2\text{)} \\
&= (x - \frac{r-1}{r})[rx^2 - (r+1)x + 1 + \frac{1}{r}] \text{ (factor out } (x - \frac{r-1}{r}) \text{ because we know this is a fixed point)} \\
x &= \frac{(r+1) \pm \sqrt{(r+1)^2 - 4(1+\frac{1}{r})r}}{2r} \\
&= \frac{(r+1) \pm \sqrt{(r+1)^2 - 4(r+1)}}{2r} \\
&= \frac{r+1 \pm \sqrt{r^2 + 2r + 1 - 4r - 4}}{2r} \\
&= \frac{r+1 \pm \sqrt{r^2 - 2r - 3}}{2r} \\
&= \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}
\end{aligned}
$$

These roots are only real when $r < -1$ or $r > 3$. Therefore, period 2 points only exist beyond $r = 3$. Stability?

$$|g'(\omega_1)| = |f'(\omega_2)f'(\omega_1)| = |r(1-2\omega_2)r(1-2\omega_1)| = r^2(1-2(\omega_1+\omega_2)+4\omega_1\omega_2)$$

Note that, from g, $(x-\omega_1)(x-\omega_2) = x^2 - \frac{r+1}{r}x + \frac{r+1}{r^2} = x^2 - (\omega_1+\omega_2)x + \omega_1\omega_2$

Therefore, $\omega_1+\omega_2 = \frac{r+1}{r}$ and $\omega_1\omega_2 = \frac{r+1}{r^2}$

Therefore, $|g'(\omega_1)| = |r^2(1-2\frac{r+1}{r}+4\frac{r+1}{r^2})|$

$$= |r^2 - 2r(r+1) + 4(r+1)| = |4+2r-r^2|$$

$$|g'(\omega_1)| = 1 \to g'(\omega_1) = \pm 1$$

$$g'(\omega_1) = 1 \to 4+2r-r^2 = 1 \to r^2-2r-3 = 0 \to (r-3)(r+1) = 0$$

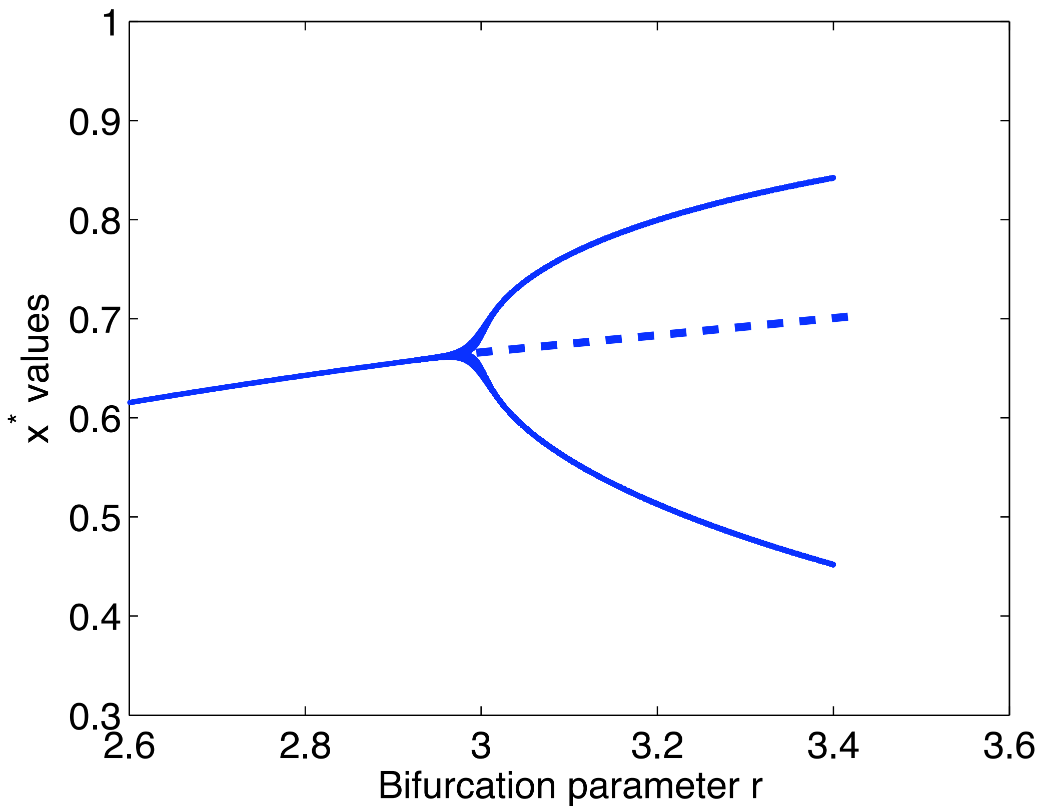
$r = 3, -1$ as before

$$g'(\omega_1) = -1 \to 4+2r-r^2 = -1 \to r^2-2r-5 = 0 \to r = \frac{2\pm\sqrt{24}}{2} = 1\pm\sqrt{6}$$

$r = 1+\sqrt{6} = 3.45$ is the next bifurcation point

That is, the period 2 orbit is stable for $3 < r < 3.45$

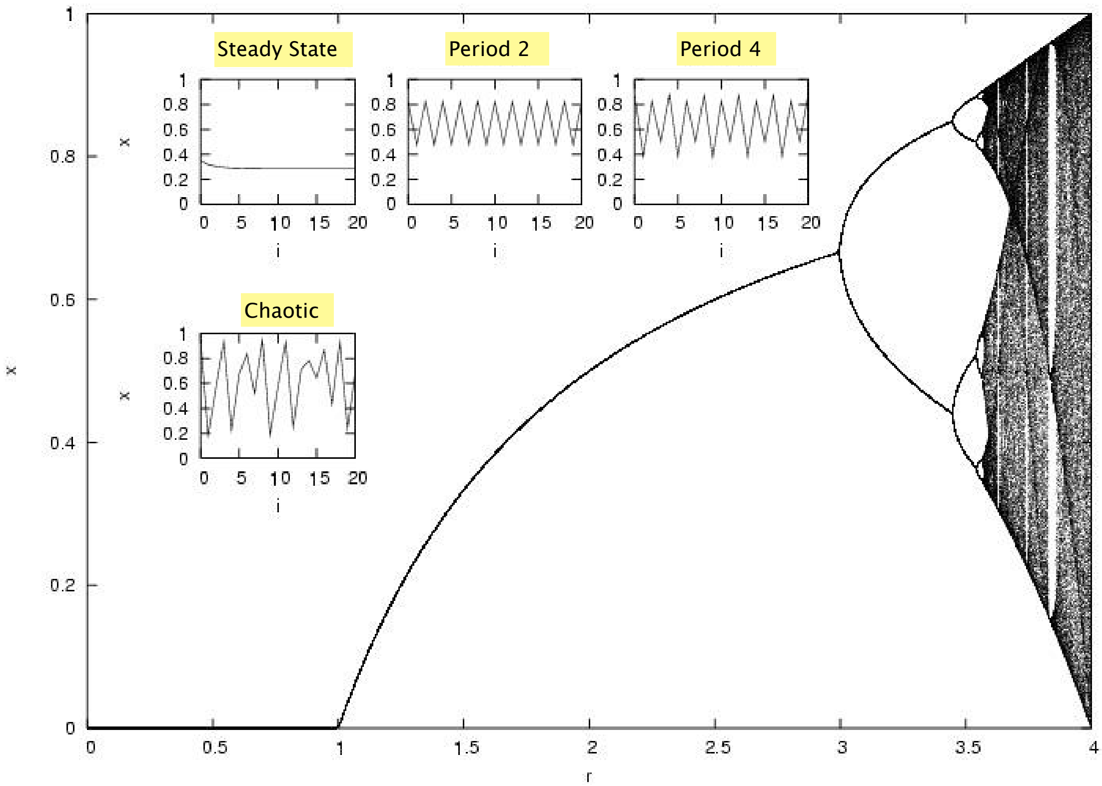
Bifurcation of the discrete logistic equation from a stable fixed point to a period 2 orbit.

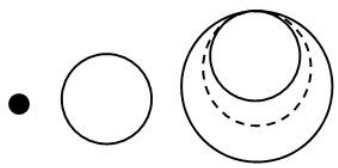


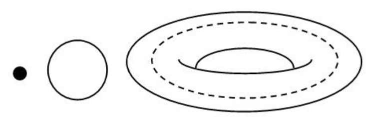
For $r > 3.45$, the period 2 orbit is unstable and a period 4 orbit appears and is stable for a while. Then a period 8 orbit appears and so on. We get chaos, and then period 3,6,12,... then period 5,10,20,... This is called the period-doubling route to chaos.

Period doubling happens in continuous dynamical systems too, although we need at least three dimensions. Another type of bifurcation in 3D is into a torus.

Full chaos bifurcation diagram







Another example is a TV camera filming its own output, like in the original Doctor Who theme.