MAT3395: Introduction to Mathematical Models

Dr. Robert Smith?

1 Solving PDEs

The diffusion equation is

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2}, \tag{1}$$

with initial condition

$$U(x,0) = 2\sin\frac{\pi x}{L} - 3\sin\frac{5\pi x}{L} + 12\sin\frac{6\pi x}{L}.$$

We are looking for a solution of the form $U(x,t)$.

However, we also need an “initial” condition for $x = 0$, aka a “boundary condition” (since x is a spatial variable).

Our corridor is of length L and droplets can’t pass through the walls, so $U(0,t) = U(L,t) = 0$.

For an arbitrary function of two variables, we can’t solve this. But if U has the form $U(x,t) = X(x)T(t)$, then we can. We call this *separation of variables*. The three derivatives are thus

$$\begin{aligned}\frac{\partial U}{\partial t} &= X(x)\dot{T}(t)\\ \frac{\partial U}{\partial x} &= X'(x)T(t)\\ \frac{\partial^2 U}{\partial x^2} &= X''(x)T(t).\end{aligned}$$

Putting this into equation (1), we have

$$\begin{aligned}X(x)\dot{T}(t) &= DX''(x)T(t)\\ \frac{\dot{T}}{DT} &= \frac{X''}{X}\end{aligned}$$

$$\underset{\substack{\text{independent}\\ \text{of } x}}{\uparrow} \qquad \underset{\substack{\text{independent}\\ \text{of } t}}{\uparrow}$$

That is, since the left-hand side cannot depend on x and the right-hand side cannot depend on t, then the only possibility for both is that they must be a constant (possibly zero, but that counts). Hence

$$\frac{\dot{T}}{DT} = \frac{X''}{X} = C$$

We have no idea what C is. It could be positive, negative or zero. So let’s try all three.

Case i) $C = 0$.

This case is easy: the left-hand side implies that $\dot{T} = 0$, so that $T(t) = T_0$, a constant. Likewise, $X'' = 0$ so $X' = X_0$, which means the solution is $X = X_0 x + X_1$.

Multiplying our two sub-solutions together, the solution to the PDE is thus

$$U(x,t) = T_0(X_0 x + X_1).$$

Next we apply the boundary conditions:

$$\begin{aligned} U(0,t) &= T_0 X_1 = 0 \\ U(L,t) &= T_0(X_0 L + X_1) = T_0 X_0 L = 0. \end{aligned}$$

Since $L \neq 0$ (i.e., we actually have a corridor of some length), then either $T_0 = 0$ or $X_1 = X_0 = 0$. Either way, the solution is $U(x,t) \equiv 0$. But this is not possible with our initial conditions. Hence $C \neq 0$.

Case ii) $C > 0$

Solving the T equation, we have

$$\begin{aligned} \dot{T} &= CDT \\ \int \frac{\dot{T}}{T} dt &= CD \int dt \\ \ln \frac{T}{T_0} &= CDt \\ T &= T_0 e^{CDt}. \end{aligned}$$

Solving the X equation, we have

$$\begin{aligned} X'' &= CX \\ X'' - CX &= 0 \\ \left(\frac{d}{dx}\right)^2 X - CX &= 0 \\ \left[\frac{d}{dx} + \sqrt{C}\right]\left[\frac{d}{dx} - \sqrt{C}\right] X &= 0 \end{aligned}$$

$$\begin{aligned} \frac{dX}{dx} + \sqrt{C}X &= 0 \\ X' &= -\sqrt{C}X \\ \frac{X'}{X} &= -\sqrt{C} \\ \ln \frac{X}{X_0} &= -\sqrt{C}x \\ X &= X_0 e^{-\sqrt{C}x} \end{aligned}$$

$$\begin{aligned} \frac{dX}{dx} - \sqrt{C}X &= 0 \\ X' &= \sqrt{C}X \\ \frac{X'}{X} &= \sqrt{C} \\ \ln \frac{X}{X_0} &= \sqrt{C}x \\ X &= X_0 e^{\sqrt{C}x} \end{aligned}$$

Hence the solution is

$$X = Ae^{-\sqrt{C}x} + Be^{\sqrt{C}x}.$$

The full solution is thus

$$U(x,t) = T_0 e^{CDt}\left(Ae^{-\sqrt{C}x} + Be^{\sqrt{C}x}\right).$$

Applying the first boundary conditions, we have

$$U(0,t) = T_0 e^{CDt}(A+B) = 0.$$

If $T_0 = 0$, then $U(x,t) \equiv 0$ and we know that won't satisfy the initial condition. Hence we have $A + B = 0$.

Applying the second boundary conditon, we have

$$\begin{aligned} U(L,t) = T_0 e^{CDt}(Ae^{-\sqrt{C}L} + Be^{\sqrt{C}L}) &= 0 \\ Ae^{\sqrt{C}L} - Ae^{\sqrt{C}L} &= 0 \text{ (since } B = -A) \\ e^{-\sqrt{C}L} - e^{\sqrt{C}L} &= 0 \text{ (since if } A = 0\text{, then } U \equiv 0) \\ 1 &= e^{2\sqrt{C}L} \\ 2\sqrt{C}L &= 0, \end{aligned}$$

which implies that either $C = 0$ or $L = 0$, neither of which can be true. It follows that C cannot be positive or zero and is hence negative.