Dynamics of Continuous, Discrete and Impulsive Systems
Series **B:** *Applications & Algorithms* **22 (2015)** 163-197

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF GENERAL IMPULSE EXTENSION EQUATIONS WITH SPECIFICATION TO LINEAR EQUATIONS

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Abstract. Analogues of the classical existence and uniqueness of solutions are proven for impulse extension equations. An exposition on matrix solutions, their properties and Floquet's theorem for periodic linear systems is provided, including applications to stability. Where applicable, comparison is made to the analoguous results from impulsive differential equations.

Keywords. Impulsive differential equation, impulse extension, predictable set, matrix solution, fundamental matrix, periodic solution, Floquet's theorem.

AMS (MOS) subject classification: 34A36, 34A37.

1 Introduction

Impulsive differential equations have a host of applications to both biological and physical problems [8, 12, 15, 18, 19, 24, 25, 27, 28]. Classic monographs on the subject (see, for example, [13, 23]) preface the exposition of the theory by writing that it is often natural to assume that sufficiently short perturbations in the system occur instantaneously, since their length is negligible in comparison with the duration of the process. A key part of the study of these equations is the existence, uniqueness and stability of their periodic solutions.

In the past few decades, many advances have been made in the theory of impulsive differential equations (see [7, 14, 17, 22, 26, 29] among others) and impulsive semidynamical systems ([2, 3, 4, 16]). The theory has therefore undergone extensive research; as such, scientists and those in industry have many tools with which to analyze mathematical models formulated in terms of impulsive differential equations.

In modelling with impulsive differential equations, the standing hypothesis is that these models can accurately describe continuous phenomena if the the impulse effects occur during "short" periods of time in comparison the overal dynamic process, to such a degree that they can be assumed to occur instantaneously. This hypothesis led Church and Smith? [5] to pose the following question: "Is it always safe to assume that sufficiently short processes occur instantaneously?" They found that this is not always true, albeit for a limited class of equations. They propose that an answer to this question may lie in the study of certain functional differential equations, which they call "impulse extension equations".

In the present article, we extend the results of Church and Smith? as follows. Chapter 2 contains background material on impulse extension equations. Chapter 3 pertains to initial-value problems and theorems on existence and uniqueness of solutions. Chapter 4 is devoted to linear systems, where it is shown that these systems admit matrix solutions that, under fairly mild assumptions, are "invertible enough" to develop a Floquet theory for periodic systems. In Chapter 5, the Floquet theorem is proven and associated linear stability results and theorems on existence of periodic solutions are provided. Throughout, comparisons are made to analogous results for impulsive differential equations, demonstrating that the progress made here is both consistent with and generalizes results from linear impulsive differential equations. We conclude with a discussion.

2 Impulse extension equations

Consider an impulsive differential equation with impulses at fixed times

$$\begin{aligned} \frac{dx}{dt} &= f(t,x) && t \neq \tau_k \\ \Delta x &= I_k(x) && t = \tau_k. \end{aligned} \tag{1}$$

with $t \in \mathbb{R}$, phase space $\Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$, $I_k : \Omega \to \mathbb{R}^n$ and sequence of impulses τ_k for $k \in \mathbb{Z}$; for the moment, we will not state any regularity requirements (see, for example, the monographs [1, 13, 23] for typical regularity requirements). We will now construct a (functional) differential equation with continuous solutions that in some sense "approximates" the above impulsive differential equation. It will often be notationally convenient to identify an impulsive differential equation (1) with a triple (f, I_k, τ_k), where each symbol represents the function, sequence of functions and sequence appearing in (1). The only assumption that must be imposed at this point is monotonicity and unboundedness of the impulse times. That is, $\tau_k < \tau_{k+1}$ for all $k \in \mathbb{Z}$ and $\tau_k \to \infty$ as $k \to \infty$. We begin with some definitions.

Definition 2.1. *Consider an impulsive differential equation (f, I_k, τ_k) as in* (1).

- *A* step sequence over τ_k *is sequence of positive real numbers a_k such that $\tau_k + a_k < \tau_{k+1}$ for all $k \in \mathbb{Z}$. We denote $\mathcal{S}_j = \mathcal{S}_j(a_k) \equiv [\tau_j, \tau_j + a_j)$ and $\mathcal{S} = \mathcal{S}(a_k) \equiv \bigcup_{j\in\mathbb{Z}} \mathcal{S}_j$.*

- *A sequence of functions* $\varphi_k : \mathcal{S}_k \times \Omega \to \mathbb{R}^n$ *is an* impulse extension for (f, I_k, τ_k) compatible with a_k *if for all* $k \in \mathbb{Z}$ *and* $x \in \Omega$*, the function* $\varphi_k(\cdot, x)$ *is locally integrable and*

$$\int_{\mathcal{S}_k(a_k)} \varphi_k(t, x)dt = I_k(x).$$

- *Given a step sequence* a_k *and a compatible impulse extension* φ_k *for the impulsive differential equation* (f, I_k, τ_k)*, the* impulse extension equation associated to (f, I_k, τ_k) induced by (a_k, φ_k) *is the (functional) differential equation*

$$\begin{aligned} \frac{dx}{dt} &= f(t,x) && t \notin \mathcal{S} \\ \frac{dx}{dt} &= f(t,x) + \varphi_k(t, x(\tau_k)) && t \in \mathcal{S}_k. \end{aligned} \tag{2}$$

As described by Church and Smith? [5] for linear systems specifically, differential equations of this type can be seen as continuous versions of impulsive systems, where the "impulse" $I_k(x)$ is carried by the function $\varphi_k(t, x)$. This can be justified as follows. If the vector field is "turned off" artificially, so that we set $f(t, x) = 0$, then a solution for $t \in \mathcal{S}_k$ of the initial-value problem with $x(\tau_k) = x_k$ is a solution of the differential equation

$$\frac{dx}{dt} = \varphi_k(t, x_k).$$

Consequently, a unique solution exists (see Carathéodory conditions [9]) and is given by

$$x(t) = x_k + \int_{\tau_k}^{t} \varphi_k(s, x_k)ds.$$

For $t = \tau_k + a_k$, we obtain

$$x(\tau_k + a_k) = x_k + I_k(x_k) = x_k + \Delta x_k.$$

Therefore the impulse extension φ_k "applies" the effect of the impulse over a finite, nonzero length of time a_k. However, in reality, the vector field is not "off", so the other dynamics might contribute.

It is worth mentioning that the impulse extension equation (2) is not the only way to describe an "impulsive equation with non-instantaneous impulses". Hernández and O'Regan [11], for example, consider another such abstract impulsive differential equation for which the impulses are not instantaneous. Their equation is formulated in terms of Banach spaces,

$$\begin{aligned} \frac{dx}{dt} &= Ax(t) + t(t, x(t)) && t \in (s_i, t_{i+1}] \\ u(t) &= g_i(t, u(t)) && t \in (t_i, s_i], \end{aligned}$$

where X is a Banach space, $A : D(A) \subset X \to X$ generates a C_0 semigroup of bounded linear operators on X, $f : [0, a] \times X \to X$ is a suitable function, the times $t_i, s_i \in \mathbb{R}$ satisfy the inequalities $t_i < s_i < t_{i+1}$ for all $i \geq 0$, and there are functions $g_i : C((t_i, s_i] \times X; X)$ for each $i \in \mathbb{N}$. The primary difference between our approach and theirs is in how the impulse effect is specified. In their approach, solutions $u : t \mapsto u(t) \in X$ are required to satisfy the very general X-valued equation $u(t) = g(t, u(t))$ for all $t \in (t_i, s_i]$. In ours, solutions are required to satisfy a very particular functional differential equation, which serves to make the connection to (discontinuous) impulsive differential equations very explicit. Since our goal was to compare impulsive systems to suitable continuous systems that formally resemble them, our explicit approach is ideally suited. The drawback, of course, is the lack of generality. It is worth mentioning that equation (2) could be posed in a Banach-space framework, if one wished to study it in more generality (for example, if one wanted to repeat the construction for impulsive partial differential equations). See also the discussion in Section 3.3 for how equation (2) can be suitably interpreted as a delay differential equation.

3 Formulations of the initial-value problem and existence and uniqueness of solutions

One aspect that makes equation (2) interesting is that there are several inequivalent ways to pose initial-value problems for it. This will be the focus of the present section.

3.1 One-point formulation

The first formulation is one that we consider to be, in a sense, most "natural", when viewing the impulse extension equation (2) as being derived from an impulsive differential equation (1). This begins with a classification of the phase space $\mathbb{R} \times \Omega$. First, if $X \subset \mathbb{R}^n$, we denote X° its interior.

Definition 3.1. *Consider an impulse extension equation* (2) *with impulse space* $\mathcal{S} = \bigcup_k [\tau_k, \tau_k + a_k)$. *Denote*

$$\mathcal{S}^+ = \bigcup_{k \in \mathbb{Z}} (\tau_k, \tau_k + a_k] = \bigcup_{k \in \mathbb{Z}} \mathcal{S}_k^+.$$

The point $(t_0, x_0) \in \mathbb{R} \times \Omega$ *is*

- admissible for (2) *if* $(t_0, x_0) \in (\mathbb{R} \setminus \mathcal{S}^\circ) \times \Omega$,
- indeterminate for (2) *if* $(t_0, x_0) \in \mathcal{S}^+ \times \Omega$,
- strongly indeterminate for (2) *if* $(t_0, x_0) \in \mathcal{S}^\circ \times \Omega$
- k-indeterminate for (2) *for some* $k \in \mathbb{Z}$ *if* $(t_0, x_0) \in \mathcal{S}_k^+ \times \Omega$.

Definition 3.2. *A function $\phi : I \to \mathbb{R}^n$ defined on an interval $I \subset \mathbb{R}$ is a* classical solution *of* (2) *if it satisfies the following conditions.*

1. *ϕ is absolutely continuous,*
2. *$(t, \phi(t)) \in \mathbb{R} \times \Omega$,*
3. *if $I \cap \mathcal{S}_k$ is nonempty, then $\tau_k \in I$,*
4. *$\phi(t)$ is differentiable almost everywhere in I,*
5. *$\frac{d\phi}{dt}(t) = f(t, \phi(t))$ almost everywhere on $I \setminus \mathcal{S}$,*
6. *$\frac{d\phi}{dt}(t) = f(t, \phi(t)) + \varphi_k(t, \phi(\tau_k))$ almost everywhere on $I \cap \mathcal{S}_k$.*

Given an initial condition

$$x(t_0) = x_0, \tag{3}$$

with $(t_0, x_0) \in \mathbb{R} \times \Omega$, the function $\phi(t)$ is a solution of the one-point initial-value problem (2)-(3) *if, in addition, $\phi(t_0) = x_0$.*

The reason for imposing conditions 1 and 2 should be obvious. Condition 3 guarantees that, for each time t in the domain of a solution, the evaluation of its derivative according to equation (2) can always be determined. Conditions 4, 5 and 6 ensure that the function ϕ satisfies the differential equation (2) almost everywhere.

Existence and uniqueness of solutions of the one-point initial-value problem for an admissible point can be handled similarly to ordinary differential equations. The proof of the following lemma is obvious and is omitted.

Lemma 3.1. *If $f(t, x)$ is continuous and $\varphi_k(t, y)$ is continuous in t for all $k \in \mathbb{Z}$ and $y \in \Omega$, then, for every admissible point (t_0, x_0), the one-point initial-value problem* (2)-(3) *has a solution defined on an interval $I \subset \mathbb{R}$. If f is locally Lipschitz continuous in x, then there is an interval I on which there is defined a unique solution. If $t_0 \notin \{\tau_k + a_k\}_{k\in\mathbb{Z}}$, then the interval I can be chosen to contain a subinterval of the form $[t_0 - \epsilon, t_0 + \epsilon]$ for some $\epsilon > 0$.*

Forward continuation of solutions can be accomplished by similar means as with ordinary differential equations; as the proof is nearly identical (at each step, either $t \in \mathcal{S}$ or $t \notin \mathcal{S}$, so typical ODE arguments related to forward continuation apply), we omit it.

Theorem 3.1. *Suppose $f(t, x)$ is locally Lipschitz continuous in x, continuous in t and $\varphi_k(t, x)$ is continuous in t for all $k \in \mathbb{Z}$ and $x \in \Omega$. Let (t_0, x_0) be admissible for* (2)*. If $\phi : I \to \Omega$ is any solution of the initial-value problem* (2)-(3) *defined on an interval I containing t_0, then there is a unique forward continuation of ϕ to a maximal forward interval of existence I^+. Moreover, if $\phi^+ : I^+ \to \Omega$ is solution of the initial-value problem and I^+ is the maximal forward interval of existence, then $(t, \phi^+(t))$ approaches the boundary of $\mathbb{R} \times \Omega$ as t approaches* $\sup I^+$.

For an indeterminate point (t_0, x_0), there may be no solution $x(t)$ of the impulse extension equation that satisfies $x(t_0) = x_0$. We refer to such an initial-value problem as an *indeterminate initial-value problem.* The following sufficient condition for existence of a solution for the indeterminate initial-value problem arises when one attempts to emulate the Peano existence proof from ordinary differential equations in the impulse extension equations case. We state it without proof, as the proof is nearly identical to the ordinary differential equations case (see [9] for such a proof).

Theorem 3.2. *Suppose $f(t,x)$ and $\varphi_k(t,x)$ are continuous. Let $(t_0, x_0) \in \mathbb{R} \times \Omega$ be k-indeterminate for the impulse extension equation* (2)*. For $\alpha, \beta > 0$, define the set*

$$U(\alpha, \beta) = (\tau_k, t_0 + \alpha) \times B_\beta(x_0)$$

where $B_\beta(x_0)$ is the open ball of radius β centered at x_0. Suppose there exist $\alpha, \beta > 0$ such that $B_\beta(x_0) \subset \Omega$ and

$$\max\{\alpha, t_0 - \tau_k\} \cdot \left(\sup_{U(\alpha,\beta)} |f(t,x)| + \sup_{U(\alpha,\beta)} |\varphi_k(t,x)| \right) \leq \beta. \tag{4}$$

Then there exists a classical solution $x(t)$ of (2) *that is defined on $[\tau_k, t_0 + \alpha]$ and satisfies $x(t_0) = x_0$. That is, a solution of the one-point initial-value problem $x(t_0) = x_0$ exists.*

The "usual" proof fails because the maximum term on the left of (4) cannot necessarily be made sufficiently small by an appropriate choice of α, and because $U(\alpha, \beta)$ does not become arbitrarily small when we take α and β small. As an example of this failure in action, consider the following simple (linear) example.

Example 3.1.

$$\begin{aligned} \frac{dx}{dt} &= 0 & t &\notin \mathcal{S} \\ \frac{dx}{dt} &= -x(3k) & t &\in [3k, 3k+2). \end{aligned} \tag{5}$$

Let us attempt to solve the initial-value problem $x(1) = x_1$ for arbitrary $x_1 \in \mathbb{R}$. By definition of a classical solution, this amounts to finding a solution $x(t)$ that satisfies $x(0) = x_0$ for some $x_0 \in \mathbb{R}$ and $x(1) = x_1$. It is easy to check that for, $x_0 \in \mathbb{R}$, the solution of the initial-value problem $x(0) = x_0$ is

$$x(t; x_0) = \begin{cases} (1-t)x_0 & t \in [0,2) \\ -x_0 & t \in [2,3) \\ -(4-t)x_0 & t \in [3,5) \\ x_0 & t \in [5,6), \end{cases}$$

extended periodically with period 6. *However, note that* $x(1; x_0) = 0$ *for all* $x_0 \in \mathbb{R}$. *Consequently, there is no solution of the initial-value problem* $x(1) = x_1$ *for* $x_1 \neq 0$. *There are however infinitely many solutions to the initial-value problem* $x(1) = 0$.

Consistent with this is the form that equation (4) *takes. We find that, for this equation, for* $t_0 = 1$ *and* $x_1 \in \mathbb{R}$, *we require*

$$\max\{\alpha, 1\} \cdot (|x_1| + \beta) \leq \beta.$$

However, this obviously fails for all $x_1 \neq 0$. *Conversely, when* $x_1 = 0$, *the inequality holds for all* $\alpha \leq 1$, *so Theorem 3.2 implies the existence of a solution* $x(t)$ *of the impulse extension equation satisfying* $x(1) = 0$. *The results of the theorem are therefore consistent with the above analysis.*

Under this formulation of the initial-value problem, we take a single point as the initial condition. The result is that some initial-value problems are well-posed, while others are not. Specifically, a one-point initial-value problem with an admissible initial condition is well-posed under fairly typical regularity requirements, while those with indeterminite initial conditions are not.

3.2 Two-point formulation

By doubling the dimension of the ambient space Ω, the impulse extension equation (2) can be converted into an impulsive differential equation, some of whose solutions will in fact be solutions of one-point initial-value problems.

Consider the impulse extension equation (2). If $X \subset \mathbb{R}$, we denote by $\mathbb{1}_X(t)$ the indicator function on the set X. Define the function $\Phi : \mathbb{R} \times \Omega \to \Omega$ by

$$\Phi(t, y) = \sum_{k=-\infty}^{\infty} \mathbb{1}_{\mathcal{S}_k}(t) \cdot \varphi_k(t, y).$$

Consider now the impulsive differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) + \Phi(t, y) && t \neq \tau_k \\ \frac{dy}{dt} &= 0 && t \neq \tau_k \\ \Delta x &= 0 && t = \tau_k \\ \Delta y &= x - y && t = \tau_k, \end{aligned} \tag{6}$$

where $(x, y) \in \Omega \times \Omega$.

We will discuss the structure of equation (6). If $t \notin \mathcal{S}$, then $\Phi(t, y) = 0$, so the solutions are govered by the ordinary differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) && t \notin \mathcal{S} \\ \frac{dy}{dt} &= 0 && t \notin \mathcal{S}. \end{aligned}$$

If $t = \tau_k$, then any solution $(x(t), y(t))$ satisfies $x(\tau_k^+) = x(\tau_k)$ and $y(\tau_k^+) = x(\tau_k)$. Consequently, for $t \in \mathcal{S}_k^\circ$, since $\Phi(t, y) = \varphi_k(t, y)$, solutions satisfy the ordinary differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(t,x) + \varphi_k(t, x(\tau_k)) && t \in \mathcal{S}_k \\ \frac{dy}{dt} &= 0 && t \in \mathcal{S}_k. \end{aligned}$$

The above two differential equations are essentially the impulse extension equation (2) except with an extra component for the $\dot{y}$ equation. An initial-value problem for this impulsive differential equation with initial data $t_0 \in \mathbb{R}$ and $(x_0, y_0) \in \Omega \times \Omega$ then requires finding a solution $(x(t), y(t))$ of (6) that satisfies $x(t_0) = x_0$ and $y(t_0) = y_0$. That is, we now require two points in the ambient space Ω.

It can be verified that if $x(t)$ is a classical solution (as in Definition 3.2) of the impulse extension equation (2) defined on some interval $\mathcal{I} \subset \mathbb{R}$, then the function $(x(t), y(t))$ with

$$y(t) = \sum_{\tau_k \in \mathcal{I}} \mathbb{1}_{[\tau_k, \tau_{k+1})}(t) \cdot x(\tau_k),$$

is a solution of (6).[1]

Therefore all classical solutions of (2) can be seen as solutions of the impulsive system (6). However, the converse is not true. Specifically, the following proposition is easily verified.

Proposition 3.1. *Suppose $Z(t) = (x(t), y(t))$ is a solution of* (6) *defined on some $\mathcal{I} \subset \mathbb{R}$. Then $x(t)$ is a classical solution of* (2) *if and only if* $\inf \mathcal{I} \notin \mathcal{S}^\circ$ *and, for all $\tau_k \in \mathcal{I}$, there exists $v_k \in \Omega$ such that $Z(\tau_k^+) = (v_k, v_k)$.*

In this way, we see that a classical solution $x(t)$ of the initial-value problem (2)-(3) with k-indeterminate initial data (t_0, x_0) is in fact a solution (possibly with domain restricted) of the parameterized family of boundary value problems (6)-(7)

$$\begin{aligned} Z(t_0) &= (x_0, v) \\ Z(\tau_k^+) &= (v, v) \end{aligned} \tag{7}$$

with parameter $v \in \Omega$, where we have used the notation appearing in Proposition 3.1.

The advantage of this formulation is that it allows techniques of impulsive differential equations and non-autonomous dynamical systems to be used, since then, under suitable regularity, solutions of an initial-value problem with initial time t_0 will be defined for all time $t \geq t_0$. Unfortunately, the set

[1] As a note of possible clarification, if $\inf \mathcal{I} = \tau_k - \alpha$ and $\tau_k - \alpha > \tau_{k-1}$, then we may define $y(t) = c$, where c is any constant, for $\tau_k - \alpha \leq t < \tau_k$.

of techniques of impulsive differential equations that are applicable is slightly limited. Specifically, since the jump operator is linear and non-invertible, impulsive Floquet theory [1] is not applicable. Consequently, inferring stability or instability of a periodic orbit in a nonlinear system becomes more complicated.

3.3 Infinite-dimensional formulation

Equation (2) can naturally be interpreted as a delay differential equation with variable delay. Specifically, we write the differential equation in the equivalent form

$$\frac{dx}{dt} = f(t, x(t)) + \sum_{k=-\infty}^{\infty} \mathbb{1}_{\mathcal{S}_k}(t)\varphi_k(t, x(t - r(t))), \tag{8}$$

where $r : \mathbb{R} \to [0, \sup a_k]$ is defined by

$$r(t) = \begin{cases} t - \tau_k & t \in \mathcal{S}_k \\ 0 & t \notin \mathcal{S}. \end{cases}$$

Definition 3.3. *A function $\phi : I \to \mathbb{R}^n$ defined on an interval of the form $I = [t_0 - \sup a_k, t_0 + A]$ for some $t_0 \in \mathbb{R}$ and $A > 0$ is a* quasi-solution *of the impulse extension equation* (2) *if it satisfies the following conditions.*

1′. ϕ is absolutely continuous,

2′. $(t, \phi(t)) \in \mathbb{R} \times \Omega$,

3′. $\phi(t)$ is differentiable almost everywhere in $[t_0, t_0 + A]$ and satisfies equation (8) *almost everywhere on this interval.*

Existence of a solution at $t_0 \in \mathbb{R}$ with initial data $x_0 \in C([-\sup a_k, 0], \mathbb{R}^n)$ can then be considered in the typical way; see [10], for example. It is also easy to see that every classical solution generates a family of quasi-solutions.

This construction is somewhat artificial and is presented mainly because it is one of the most general ways in which an initial-value problem for the impulse extension equation (2) can be posed.

4 Linear impulse extension equations

For the rest of this paper, we will be working with the one-point formulation of the initial-value problem. We now consider linear impulse extension equations, beginning with a definition.

Definition 4.1. *An impulse extension equation is* linear *is it is of the form*

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t) & t &\notin \mathcal{S} \\ \frac{dx}{dt} &= A(t)x + g(t) + \varphi_k^B(t)x(\tau_k) + \varphi_k^h(t) & t &\in \mathcal{S}_k, \end{aligned} \tag{9}$$

$\varphi_k(t,x) = \varphi_k^B(t)x + \varphi_k^h(t)$ *is an impulse extension, and the following conditions are met:*

- *The functions* $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ *and* $g : \mathbb{R} \to \mathbb{R}^n$ *are bounded and measurable on all compact subsets of* $\mathbb{R}$*.*
- *The functions* $\varphi_k^B : \mathcal{S}_k \to \mathbb{R}^{n\times n}$ *and* $\varphi_k^h : \mathcal{S}_k \to \mathbb{R}^n$ *are bounded and measurable for all* $k \in \mathbb{Z}$*.*

Let us present some examples of linear impulse extensions. It will be assumed that the phase space is $\mathbb{R}^n$. For these examples, we will let $\Delta x = I_k(x) = B_k x + h_k$ be a linear jump condition to be described by an impulse extension, with impulse times τ_k.

Example 4.1. *Perhaps the simplest impulse extension is the* constant extension*. Given a compatible step sequence* a_k*, the constant extension is the pair* $(\varphi_k^B, \varphi_k^h)$ *defined by*

$$\varphi_k^B(t) = \frac{1}{a_k}B_k, \qquad \varphi_k^h(t) = \frac{1}{a_k}h_k.$$

It is easy to verify that the function $\varphi_k(t,x) = \varphi_k^B(t)x + \varphi_k^h(t)$ *constitutes an impulse extension for any impulsive differential equation with the affine jump* $\Delta x = B_k x + h_k$*. The extension is linear because of the above decomposition.*

Example 4.2. *Impulse extensions can be constructed from matrix exponentials. For each* $k \in \mathbb{Z}$*, let* M_k *and* N_k *be* $n \times n$ *matrices all of whose eigenvalues have positive real part. Given a square matrix* H*, define the function*

$$C_k(t;H) = \frac{a_k}{(\tau_k + a_k - t)^2} H \exp\left(-H\left(\frac{t-\tau_k}{\tau_k + a_k - t}\right)\right).$$

Then we can construct a linear extension $(\varphi_k^B, \varphi_k^h)$ *as follows:*

$$\varphi_k^B(t) = C_k(t;M_k)B_k \qquad \varphi_k^h(t) = C_k(t;N_k)h_k.$$

We now show that this defines a valid impulse extension. These functions are each bounded on $\mathcal{S}_k$*, so it suffices to show that* $\int_{\mathcal{S}_k} C_k(t;H)dt = E$ *whenever all eigenvalues of* H *have positive real part, for then, we will have*

$$\int_{\mathcal{S}_k} \varphi_k^B(t)x + \varphi_k^h(t)dt = EB_k x + Eh_k = B_k x + h_k$$

as required. We calculate

$$\begin{aligned}\int_{\mathcal{S}_k} C_k(t;H)dt &= H\int_{\tau_k}^{\tau_k+a_k} \frac{a_k}{(\tau_k+a_k-t)^2}\exp\left(-H\left(\frac{t-\tau_k}{\tau_k+a_k-t}\right)\right)dt\\ &= H\int_0^{a_k}\frac{a_k}{(a_k-u)^2}\exp\left(-H\left(\frac{u}{a_k-u}\right)\right)du\\ &= H\int_0^{\infty}\exp(-Hr)dr\\ &= E,\end{aligned}$$

where the fourth equality follows by the eigenvalues of H having positive real part. We conclude that the pair $(\varphi_k^B,\varphi_k^h)$ defines a linear impulse extension. In fact, for each k, we have a choice; the function $\varphi_k^B(t) = B_kC_k(t;M_k)$ is another suitable candidate for the matrix-valued part φ_k^B.

4.1 Existence and uniqueness of solutions

We establish here the existence and uniqueness of solutions of the linear impulse extension equation (9). Lemma 3.1 is not applicable since the functions appearing in (9) are in general not continuous, so we must use alternative techniques. This being said, the proofs are simple but tedious. They may be found in the appendix.

Lemma 4.1. *Let (t_0,x_0) be admissible for (9). Then the initial-value problem for (9) with $\tau_{k-1}+a_{k-1}<t_0\le\tau_k$ with initial condition $x(t_0)=x_0$ has a unique solution defined for all $t>\tau_{k-1}+a_{k-1}$.*

With impulsive differential equations, the condition required for uniqueness of solutions is $\det(E+B_k)\neq 0$, where the impulse condition is $\Delta x = B_kx+h_k$ at time $t=\tau_k$. The condition for (9) is similar.

Lemma 4.2. *Let (t_0,x_0) be k-indeterminate for (9). The initial-value problem for (9) with initial condition $x(t_0)=x_0$ has a solution if and only if*

$$X^{-1}(t_0;\tau_k)x_0 - \int_{\tau_k}^{t_0} X^{-1}(s;\tau_k)\big[g(s)+\varphi_k^h(s)\big]ds$$

is in the range of $L(t_0;\tau_k)$, where for $t_0\in\overline{\mathcal{S}_k}$, $L(t_0,\tau_k)$ is defined by

$$L(t_0;\tau_k)\equiv E+\int_{\tau_k}^{t_0} X^{-1}(s;\tau_k)\varphi_k^B(s)ds, \tag{10}$$

and $X(t;s)$ is the Cauchy matrix of the homogeneous ordinary differential equation $x'(t)=A(t)x$. If a solution exists, then it exists for all $t>\tau_{k-1}+a_{k-1}$. The solution is unique for $t\ge\tau_{k-1}+a_{k-1}$ if and only if $\det L(t_0;\tau_k)\neq 0$.

By combining the above two lemmas, we can state a global existence, uniqueness and continuability result.

Theorem 4.1. *For* $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$*, the initial-value problem for* (9)*with initial condition* $x(t_0) = x_0$ *has a unique solution defined for all* $t \in \mathbb{R}$ *if and only if one of the following is satisfied:*

- $t_0 \in [\tau_k + a_k, \tau_{k+1}]$ *and* $\det L(\tau_j + a_j; \tau_j) \neq 0$ *for all* $j \leq k$*,*
- $t_0 \in \mathcal{S}_k^\circ$*,* $\det L(t_0; \tau_k) \neq 0$ *and* $\det L(\tau_j + a_j; \tau_j) \neq 0$ *for all* $j < k$*.*

Moreover, all solutions defined locally are uniquely continuable to $\mathbb{R}$ *if and only if* $\det L(\tau_k + a_k; \tau_k) \neq 0$ *for all* $k \in \mathbb{Z}$*.*

With linear impulsive differential equations, to have global existence and uniqueness of solutions for all initial-value problems, we require

$$\det(E + B_k) \neq 0$$

for all k [1], where the "homogeneous part" of the jump condition is given by $\Delta x = B_k x$. With impulse extension equations, we require

$$\det\left(E + \int_{\tau_k}^{t} X^{-1}(s; \tau_k)\varphi_k^B(s)ds\right)$$

for all k and all $t \in (\tau_k, \tau_k + a_k]$, where the "homogeneous part" of the impulse extension is given by $\varphi_k^B(t)$. Formally, in the determinant conditions above, the integrals in the extension case play the role of the matrix B_k in the impulsive case. In the limiting case where

$$\varphi_k^B(t) = \delta(t - \tau_k)B_k,$$

and $\delta(t)$ is the Dirac delta function, we have

$$\begin{aligned}\int_{\tau_k}^{t} X^{-1}(s; \tau_k)\varphi_k^B(s)ds &= \int_{\tau_k}^{t} \delta(s - \tau_k)X^{-1}(s; \tau_k)B_k ds \\ &= X^{-1}(\tau_k; \tau_k)B_k \\ &= EB_k \\ &= B_k\end{aligned}$$

for $t > \tau_k$. Therefore, in the limiting case of an impulsive differential equation, the existence and uniqueness criteria reduce to the impulsive existence and uniqueness conditions. In this sense, the conditions are consistent.

4.2 The homogeneous equation, matrix solutions and the predictable set

In this section, we will be interested in matrix solutions of the homogeneous impulse extension equation (11),

$$\begin{aligned} \frac{dx}{dt} &= A(t)x & t \notin \mathcal{S} \\ \frac{dx}{dt} &= A(t)x + \varphi_k^B(t)x(\tau_k) & t \in \mathcal{S}_k. \end{aligned} \tag{11}$$

associated to the inhomogeneous equation (9). By Theorem 3.5, existence of a unique solution to a k-admissible initial-value problem $x(t_0) = x_0$ is equivalent to the invertibility of $L(t_0; \tau_k)$. In general, this is not satisfied for all times $t_0 \in \mathcal{S}_k$, as can be verified by considering Example 3.1, where one finds that $L(1; 0) = 0$. Since we will soon be interested in the stability of periodic equations, this observation suggests our matrix solutions will not, in general, be invertible everywhere. We begin with a definition.

Definition 4.2. *A function $U : I \subset \mathbb{R} \to \mathbb{R}^{n\times n}$ is a* matrix solution *of the homogeneous impulse extension equation* (11) *if the following conditions are satisfied.*

1. *For all $x_0 \in \mathbb{R}^n$, $x(t) = U(t)x_0$ is a solution of* (11).
2. *$U(t^*)$ is nonsingular for some $t^* \in I$.*

A matrix solution $U(t)$ is maximal *if there is no matrix solution $V : I^+ \to \mathbb{R}^{n\times n}$ with $I^+ \supsetneq I$ such that $U(t) = V(t)$ on I.*

We remark that $U(t)$ is a matrix solution if and only if it satisfies the matrix impulse extension equation

$$\begin{aligned} U'(t) &= A(t)U(t) & t \notin \mathcal{S} \\ U'(t) &= A(t)U(t) + \varphi_k^B(t)U(\tau_k) & t \in \mathcal{S}_k, \end{aligned} \tag{12}$$

and there is at least one time t^* at which $U(t^*)$ is invertible. Notice that matrix solutions always exist, because one can construct one by choosing any $t_0 \in \mathbb{R} \setminus \mathcal{S}^+$ and solving the set of n initial-value problems

$$x_i(t_0) = e^i,$$

and then, $U(t) = [x_1(t) \cdots x_n(t)]$ is a matrix solution. The standard basis $\{e^i\}$ can of course be replaced by any basis, and the result holds.

One property of fundamental matrix solutions used in the classical proof of Floquet's theorem (of which we prove an analogue later) is that, for any two fundamental matrix solutions $X(t)$ and $Y(t)$, there is a unique invertible matrix C such that $Y(t) = X(t)C$ for all $t \in \mathbb{R}$. In the classical case, fundamental matrices are invertible everywhere, so one can simply choose $C = X^{-1}(0)Y(0)$. However, it is not the case that a matrix solution need be invertible everywhere, so this may not be possible. Because this property is so crucial, we state a definition.

Definition 4.3. *Let $\mathcal{U}$ denote the set of all maximal matrix solutions for the homogeneous equation* (11)*. $\mathcal{U}$ satisfies the* uniqueness property *if, for each pair $X(t), Y(t) \in \mathcal{U}$, there exists a unique invertible matrix C such that $Y(t) = X(t)C$ whenever t is in the domain of both X and Y.*

The uniqueness property is satisfied under fairly mild conditions. The following notion will be central to both this, and later, stability results.

Definition 4.4. *The* predictable set*, $\mathcal{P} \subset \mathbb{R}$, of a linear impulse extension equation* (9)*, is the set of all $t_0 \in \mathbb{R}$ such that, for any $x_0 \in \mathbb{R}^n$,* (9) *has a unique local solution $x : I \to \mathbb{R}^n$ with $\inf I < t_0$, satisfying $x(t_0) = x_0$.*

We have the following proposition, which follows directly from Lemma 4.2.

Proposition 4.1. *Define the map $p : \mathcal{S}^+ \to \mathbb{R}$ by*

$$p(t) = \det L(t; \max\{\tau_k : \tau_k \leq t\}),$$

where one will recall that $\mathcal{S}_k^+ = (\tau_k, \tau_k + a_k]$ and $\mathcal{S}^+ = \bigcup_k \mathcal{S}_k^+$. Then

$$\mathcal{P} = \mathbb{R} \setminus p^{-1}(0).$$

The utility of the predictable set is that it it precisely the set on which any matrix solution can hope to be invertible.

Lemma 4.3. *Let $U : I \to \mathbb{R}^{n\times n}$ be a matrix solution of the homogeneous IEE* (11)*. If $U(t_0)$ is invertible and $\inf I < t_0$, then t_0 is* predictable*; that is, $t_0 \in \mathcal{P}$. Also, if this is the case, then $U(t)U^{-1}(t_0)x_0$ is, locally, the unique solution of the initial-value problem $x(t_0) = x_0$.*

Proof. Suppose $U(t_0)$ is invertible. Then, for all $x_0 \in \mathbb{R}^n$, the function

$$x(t; x_0) = U(t)U^{-1}(t_0)x_0$$

is a solution of the homogeneous equation satisfying $x(t_0; x_0) = x_0$. If (t_0, x_0) is k-indeterminate and U is defined for some $t < t_0$ (which, by hypothesis, it is) it follows by Lemma 4.2 that $L(t_0; \tau_k)$ must have full rank because the indeterminate initial-value problem $x(t_0) = x_0$ has a solution for any $x_0 \in \mathbb{R}^n$. Therefore $\det L(t_0; \tau_k) \neq 0$, so that $t_0 \in \mathcal{P}$. Conversely, if $t_0 \notin \mathcal{S}^+$, then $t_0 \in \mathcal{P}$; see Proposition 4.1. It follows that $U(t)U^{-1}(t_0)x_0$ is, locally, the unique solution of the IVP $x(t_0) = x_0$ because of Lemma 4.2. □

The converse to this statement is not, in general, true. It is possible for t_0 to be predictable and for a matrix solution $U(\cdot)$ to be non-invertible at t_0, even if $U(t)$ exists for some $t < t_0$. There is a situation in which the converse holds, however. If each "endpoint" of impulse effect, $\tau_k + a_k$, is predictable, then the predictable set is precisely where any maximal matrix solution will be invertible. As an added benefit, the uniqueness property holds and any maximal matrix solution is defined on the entire real line.

Theorem 4.2 (Predictable Endpoints)**.** *The following are equivalent.*

1. $\tau_k + a_k \in \mathcal{P}$ *for all* $k \in \mathbb{Z}$*; that is, every endpoint of the impulse effect is predictable.*

2. *Every maximal matrix solution of the homogeneous equation* (11) *has domain equal to* $\mathbb{R}$*.*

3. *The set of maximal matrix solutions of the homogeneous equation* (11) *satisfies the uniqueness property.*

4. *Any matrix solution is invertible at* $t \in \mathbb{R}$ *if and only if* $t \in \mathcal{P}$*.*

Proof. We demonstrate several equivalences and implications. First, we show that Statements 1 and 2 are equivalent. We then show that these two together imply Statement 4. We then proceed to show that Statement 4 impllies Statement 3, which implies Statement 2, which implies Statement 1. These results prove the theorem.

($1 \Leftrightarrow 2$) First, by Theorem 4.1 and Proposition 4.1, Statements 1 and 2 are equivalent. Indeed, the first of these results states that any local solution of (11) is uniquely continuable to $\mathbb{R}$ if and only if $\det L(\tau_k + a_k; \tau_k) \neq 0$ for all $k \in \mathbb{Z}$. This is precisely the statement that $\tau_k + a_k \in \mathcal{P}$ for all k, by the latter proposition. Since the columns of a matrix solution are all solutions of (11), these statements remain true if "solution" is replaced with "matrix solution".

(1 and $2 \Rightarrow 4$) Suppose $t_0 \in \mathcal{P} \cap \mathcal{S}_k^+$. By Statements 1 and 2, we have $\det L(t_0; \tau_k) \neq 0$ and $\det L(\tau_j + a_j; \tau_j) \neq 0$ for all $j \in \mathbb{Z}$. By Theorem 4.1, there is a unique, global solution of the initial-value problem $x(t_0) = 0$. Suppose the conclusion is false, so that $U(t_0)$ is not invertible. Then there are two distinct $x, y \in \mathbb{R}^n$ for which $U(t_0)x = 0 = U(t_0)y$. By definition of the matrix solution, both $U(t)x$ and $U(t)y$ are solutions of the initial-value problem $x(t_0) = 0$, and, since $U(t^*)$ is invertible for some t^*, these solutions are distinct because $U(t^*)x \neq U(t^*)y$, which is a contradiction. We conclude that $U(t_0)$ is invertible. An identical argument serves to prove the result for $t_0 \in \mathcal{P} \setminus \mathcal{S}^+$. Therefore $U(t)$ is invertible on $\mathcal{P}$. The converse has previously been established in Lemma 4.3.

($4 \Rightarrow 3$) Let $X(t)$ and $Y(t)$ be two maximal matrix solutions. Define $C = X^{-1}(\tau_0)Y(\tau_0)$, which exists and is invertible by Statement 4. We will show that $Y(t) = X(t)C$. Define the function $Z(t) = Y(t) - X(t)C$. It is easily verified that $Z(t)$ is a solution of the matrix impulse extension equation (12) and, by construction, that $Z(\tau_0) = 0$. The unique solution of the initial-value problem $x(\tau_0) = 0$ of the homogeneous equation is precisely the trivial solution, $x(t) = 0$; see Theorem 4.1. Since each column $z(t)$ of $Z(t)$ is a solution of the homogeneous equation and $z(\tau_0) = 0$, we conclude that $Z(t) = 0$ and, consequently, that $Y(t) = X(t)C$. Uniqueness is clear.

($3 \Rightarrow 1$) Finally, we demonstrate that Statement 3 implies Statement 1. Suppose not; that is, the set of maximal matrix solutions satisfies the uniqueness property, but there exists an integer k such that $\tau_k + a_k \notin \mathcal{P}$. Let $Y(t)$ be a matrix solution that satisfies $Y(\tau_k + a_k) = E$, and let $X(t)$ be the matrix solution that satisfies $X(\tau_k) = E$, where E is the appropriate identity matrix. By the discussion following Definition 4.2, these matrix solutions exists. By the uniqueness property, there exists an invertible matrix C such that $X(t) = Y(t)C$. However, one finds by Theorem 4.1 that $X(\tau_k + a_k)$ does not have full rank, which is a contradiction, because $Y(\tau_k + a_k)C = EC = C$ has full rank. □

Another reason the predictable endpoints condition is useful is the following obvious corollary.

Corollary 4.3. *Let $\tau_k + a_k \in \mathcal{P}$ for all $k \in \mathbb{Z}$. The set of all maximal solutions of the homogeneous impulse extension equation* (11) *is an n-dimensional, real vector space with the usual operations.*

Finally, the structure of solutions of the inhomogeneous equation (9) can be described in terms of those of the homogeneous equation (11). The following proposition should be familiar, and we omit the proof.

Proposition 4.2. *The following statements hold.*

1. *Let u, v denote two solutions of the inhomogeneous equation* (9)*. Then $u - v$ is a solution of the corresponding homogeneous equation* (11)*.*

2. *Let x be a solution of the inhomogeneous equation* (11) *and let t^* be in its domain. Then*
$$x(t) = x_h(t) + x_0(t),$$
where x_h is a solution of the homogeneous equation satisfying $x_h(t^*) = x(t^*)$ and x_0 is a solution of the inhomogeneous equation satisfying $x_0(t^*) = 0$. If $U(\cdot)$ is a matrix solution invertible at t^*, then x_h and x_0 are unique, and
$$x_h(t) = U(t)U^{-1}(t^*)x(t^*).$$

5 Linear periodic equations

Definition 5.1. *A linear impulse extension equation is* periodic with period T and cycle number c *if the following identities are satisfied:*

P5.1 *$A(t)$ and $g(t)$ are T periodic,*

P5.2 *$c \in \mathbb{N}$ is the smallest integer for which $\tau_{k+c} = \tau_k + T$, $a_{k+c} = a_k$ and the shift property*

$$\varphi^{\alpha}_{k+c}(t+T) = \varphi^{\alpha}_{k}(t)$$

holds for all $t \in \mathcal{S}_k$ and integers k, where $\alpha \in \{B, h\}$ and $(\varphi^B_k, \varphi^h_k)$ is the linear extension pair.

We will call these statements together *condition* [P]. This definition can be obviously extended to the general impulse extension equation (2). Before delving into the main results, we show that condition [P] simplifies the description of the predictable set.

Lemma 5.1. *If condition* [P] *holds, then $t \in \mathbb{R}$ is predictable if and only if $t + T$ is.*

Proof. We have, using the properties [P] described in Definition 5.1,

$$\begin{aligned}
L(t+T;\tau_{k+c}) &= E + \int_{\tau_{k+c}}^{t+T} X^{-1}(s;\tau_{k+c})\varphi^B_{k+c}(s)ds \\
&= E + \int_{\tau_k+T}^{t+T} X^{-1}(s;\tau_k+T)\varphi^B_{k+c}(s)ds \\
&= E + \int_{\tau_k}^{t} X^{-1}(s+T;\tau_k+T)\varphi^B_{k+c}(s+T)ds \\
&= E + \int_{\tau_k}^{t} X^{-1}(s;\tau_k)\varphi^B_{k}(s)ds \\
&= L(t;\tau_k).
\end{aligned}$$

Therefore, by Proposition 4.1, if $t \in \mathcal{S}^+$, then $t \in \mathcal{P}$ if and only if $t + T \in \mathcal{P}$. Since $\mathbb{R} \setminus \mathcal{S}^+ \subset \mathcal{P}$ and $(\mathbb{R} \setminus \mathcal{S}^+) + T = \mathbb{R} \setminus \mathcal{S}^+$, the lemma is proven. $\square$

5.1 Homogeneous equations

We will prove a generalization of Floquet's theorem for the periodic homogeneous impulse extension equation (11).

Theorem 5.1. *Let the condition [P] hold and suppose $\tau_k + a_k$ is predictable for all $k \in \mathbb{Z}$. Then each matrix solution $U(t)$ of the homogeneous periodic impulse extension equation* (11) *can be respresented in the form*

$$U(t) = \phi(t)e^{\Lambda t}, \tag{13}$$

where $\Lambda \in \mathbb{C}^{n\times n}$ is constant and the matrix $\phi(\cdot)$ is differentiable almost everywhere, complex-valued, T-periodic and non-singular on $\mathcal{P}$.

Proof. Let $U(t)$ be a matrix solution. We claim first that $U(t+T)$ is also a matrix solution. We show this by demonstrating that $Y(t) = U(t+T)$ satisfies (12). For $t \notin \mathcal{S}$, we have

$$\frac{dY}{dt}(t) = A(t+T)U(t+T) = A(t)Y(t),$$

and, for $t \in \mathcal{S}_k$, if we remark that $t + T \in \mathcal{S}_{k+c}$ and $\tau_{k+c} = \tau_k + T$ (see Definition 4.1), then

$$\frac{dY}{dt}(t) = A(t+T)U(t+T) + \varphi^B_{k+c}(t+T)U(\tau_{k+c}) = A(t)Y(t) + \varphi^B_k(t)Y(\tau_k).$$

So $Y(t)$ satisfies (12) and therefore $Y(t) = U(t+T)$ is a matrix solution. By the uniqueness property, let M be the unique invertible matrix such that $U(t+T) = U(t)M$. Define the following:

$$\begin{aligned} \Lambda &= \frac{1}{T}\ln M \\ \phi(t) &= U(t)e^{-\Lambda t}. \end{aligned} \tag{14}$$

Note that Λ exists since M is non-singular [9], although it need not be unique (however, any logarithm will suffice). With these representations, formula (13) holds, and we have

$$\phi(t+T) = U(t+T)e^{-\Lambda(t+T)} = U(t)Me^{-\Lambda T}e^{-\Lambda t} = U(t)e^{-\Lambda t} = \phi(t),$$

so $\phi(t)$ is T-periodic. It has the same regularity as $U(t)$, and is therefore differentiable almost everywhere. Since $U(t)$ is invertible on $\mathcal{P}$, $\phi(t)$ is as well. □

As usual, if one wishes to have this representation in terms of real matrices, then the choice of

$$\Lambda = \frac{1}{2T}\ln M^2$$

and

$$\phi(t) = U(t)e^{-\Lambda t}$$

would satisfy (13) and these matrices would be real; however, $\phi(t)$ would be $2T$-periodic.

Definition 5.2. *The eigenvalues of the matrix M appearing in* (14) *are called the* Floquet multipliers *of the periodic IEE.*

It is not difficult to verify that the Floquet multipliers do not depend on the choice of matrix solution used to calculate M; choosing another matrix solution $X_1(t)$ and using this to derive the matrix M_1, one finds that M and M_1 are similar.

As they do with ordinary and impulsive differential equations, the Floquet multipliers of a linear, homogeneous periodic equation characterize both its stability and the existence of periodic solutions.

Corollary 5.2. *Let the IEE* (11) *satisfy condition [P] with period T. This IEE has a kT-periodic solution if and only if there exists a multiplier $\mu \in \sigma(M)$ such that $\mu^k = 1$.*

Proof. Without loss of generality, let $\tau_0 = 0$. We seek conditions under which $U(kT)x_0 = U(0)x_0$ for some x_0, for this is precisely the condition under which we have a kT-periodic solution. By Theorem 5.1, this is equivalent to having $\phi(0)e^{\Lambda kT}x_0 = \phi(0)x_0$. Since $0 \in \mathcal{P}$ and $\Lambda = \frac{1}{T}\ln M$, this is equivalent to $M^k x_0 = x_0$. The result follows. □

Corollary 5.3. *Let $\tau_k + a_k \in \mathcal{P}$ for all $k \in \mathbb{Z}$. Let $N \subseteq \mathcal{P}$ be a subset of the predictable set. The linear homogeneous IEE is*

- *asymptotically stable*[2] *on N and uniformly attracting on $\mathbb{R}$ if and only if $\rho(M) < 1$,*
- *stable on N if and only if $\rho(M) \leq 1$ and, for any eigenvalue μ of M with unit modulus, the geometric and algebraic multiplicities of μ coincide.*
- *Suppose N is bounded and separated from $\mathbb{R} \setminus \mathcal{P}$. The above results hold with uniform stability.*
- *Suppose, for all $t_0 \in N$, there exists $\hat{t}_0 \in [\tau_0, \tau_0 + T] \cap N$ such that $t_0 = \hat{t}_0 + jT$ for some $j \in \mathbb{Z}$. If N and $\mathbb{R} \setminus \mathcal{P}$ are separated, the above results hold with uniform stability.*

Proof. We will prove the uniform attractivity first. Let $x(t)$ and $y(t)$ be two solutions of (11). Since $U(\tau_0)$ is nonsingular, we must have $x(t) = U(t)U^{-1}(\tau_0)x(\tau_0)$ and $y(t) = U(t)U^{-1}(\tau_0)y(\tau_0)$ by Lemma 4.3. Then

$$x(t) - y(t) = U(t)U^{-1}(\tau_0) \cdot [(x(\tau_0) - y(\tau_0)].$$

It follows that $x(t)$ is uniformly attracting on $\mathbb{R}$ if and only if $U(t) \to 0$ as $t \to \infty$. By the Floquet factorization $U(t) = \phi(t)e^{\Lambda t}$, since ϕ is continuous and T-periodic, this occurs if and only if $Re(\sigma(\Lambda)) < 0$, or, equivalently, $\rho(M) < 1$.

We now prove stability results. Note that, by the previous calculation, it suffices to check the stability of the trivial solution $x = 0$. Let $N \subseteq \mathcal{P}$. By Lemma 4.3 and Theorem 5.1, the solution of the initial-value problem $x(t_0; t_0, x_0) = x_0$ for $t_0 \in N$ can be written

$$x(t; t_0, x_0) = \phi(t)e^{\Lambda t}K_{t_0}x_0,$$

where $K_{t_0} = e^{-\Lambda t_0}\phi^{-1}(t_0)$. Since $\phi(t)$ is periodic, all solutions remain bounded for all $t \geq t_0$ if and only if $||e^{\Lambda t}|| \leq K$ for some constant $K > 0$,

[2]Note that, here, it is important to specify stability with respect to specific initial times. That is, a solution $x(t)$ is *stable at* t_0 if, given $\epsilon > 0$, solutions $y(t)$ that are within $\delta(\epsilon, t_0)$ of $x(t)$ at time $t = t_0$ remain within ϵ of $x(t)$ for all $t \geq t_0$. Asymptotic stability and attractivity are defined analogously and uniform properties are properties for which δ can be chosen independently of t_0.

which is true if and only if all eigenvalues of Λ have negative real parts and any eigenvalue λ with zero real part has geometric multiplicity equal to its algebraic multiplicity [9]. Since $\exp : \mathbb{C} \to \mathbb{C}$ is holomorphic and $M = \exp(\Lambda T)$, the spectral mapping theorem (see [20], Theorem 2.1.10) implies that any eigenvalue m_j of M is of the form $m_j = e^{\lambda_j T}$ for an eigenvalue λ_j of Λ, with multiplicities preserved. These will all have modulus less than or equal to one if and only if the eigenvalues of Λ all have real part less than or equal to zero, thus establishing the required equivalence of $\rho(M) \leq 1$ with $Re(\sigma(\Lambda)) \leq 0$. If this condition is satisfied, then if $\epsilon > 0$ is given and

$$\delta = \frac{\epsilon}{K||K_{t_0}||\max||\phi(\cdot)||},$$

then $||x_0|| < \delta$ implies $||x(t; t_0, x_0)|| < \epsilon$ for all $t \geq t_0$, so we have stability. Note that δ exists since $\phi(t)$ is bounded and nonzero. Conversely, if any multiplier has modulus greater than 1, then Λ has an eigenvalue with positive real part and at least one column of $e^{\Lambda t}$ grows without bound, so we have instability. Since these results hold for any $t_0 \in N$, the stability results follow.

We now deal with uniform stability, under the assumptions that N is bounded separated from $\mathbb{R} \setminus \mathcal{P}$. As in the proof of (non-uniform) stability, let $x(t; t_0, x_0)$ denote the solution of the initial-value problem $x(t_0; t_0, x_0) = x_0$, for $t_0 \in N$. As before, we write $K_{t_0} = e^{-\Lambda t_0}\phi^{-1}(t_0)$. The quantity $\widehat{K} = \sup_{t_0 \in N} ||K_{t_0}||$ exists, since $t_0 \mapsto \phi(t_0)$ is continuous and is only singular at $t_0 \in \mathbb{R} \setminus \mathcal{P}$, which is separated from the bounded set N. The rest of the proof proceeds as above, where we ensure $||e^{\Lambda t}|| \leq K$ for $t \geq t_0 \geq \inf N$ and take instead $\delta = \epsilon/(K\widehat{K} \max ||\phi(\cdot)||)$. Since δ is independent of t_0, the result is proven.

If N is not bounded but every $t_0 \in N$ can be written $\hat{t}_0 + jT = t_0$ for some $\hat{t}_0 \in [\tau_0, \tau_0 + T] \cap N$, it suffices to prove uniform stability for initial conditions $t_0 \in [\tau_0, \tau_0 + T] \cap N$. This last follows by periodicity of equation (11). The proof is nearly identical to the above, except that we take instead $\widehat{K} = \sup_{t_0 \in [\tau_0, \tau_0 + T] \cap N} ||K_{t_0}||$ and ensure that $||e^{\Lambda t}|| \leq K$ for $t \geq t_0 \geq \tau_0$. □

To determine stability on the predictable set of the periodic IEE, all that is needed is a monodromy matrix M and its eigenvalues. As can be verified by the variation of constants formula for ordinary differential equations (see later the proof of Proposition 5.1 for a similar calculation), the solution of (11) satisfying $x(\tau_0) = x_0$ is given by, for $t \in [\tau_k, \tau_{k+1})$,

$$x(t; x_0) = Y(t)x_0 := X(t; \tau_k)L(\bar{t}; \tau_k) \left[\prod_{r=k-1}^{0} X(\tau_{r+1}; \tau_r)L(\tau_r + a_r; \tau_r) \right] x_0, \tag{15}$$

where $\bar{t} = \min\{t, \tau_k + a_k\}$, $X(t; s)$ is the Cauchy matrix of $x'(t) = A(t)x$ and $L(t; s)$ is defined as in (10). Using this, we can construct a monodromy matrix, since the above expression implies that $Y(t)$ is a matrix solution. We have $Y(\tau_0) = I$, so the monodromy matrix is $M = Y^{-1}(\tau_0)Y(T+\tau_0) = Y(\tau_c)$. Therefore

$$M = \prod_{k=c-1}^{0} X(\tau_{k+1}; \tau_k) L(\tau_k + a_k; \tau_k), \tag{16}$$

where c is the cycle number.

Outside of the predictable set, a subspace of solutions can merge into a single point, so that any of these solutions is within any $\delta > 0$ of another, no matter how small δ is. However, as time progresses and the predictable set is reached, these solutions that have merged must once again split off. However, making δ smaller has no effect on how far apart these solutions can drift. Hence stability is not present at "unpredictable" times.

5.2 Inhomogeneous equation

We next deal with stability of the inhomogeneous equation. This is determined completely by the associated homogeneous equation.

Corollary 5.4. *Consider a linear inhomogeneous periodic equation* (9). *Let condition* [P] *hold, let* $\tau_k + a_k \in \mathcal{P}$ *for all* $k \in \mathbb{Z}$, *and let* M *denote the monodromy matrix of the associated homogeneous equation. Then the inhomogeneous equation* (9) *has the same stability properties as the homogeneous equation* (11), *as determined in Corollary 5.3.*

Proof. Let $x(t)$ and $y(t)$ be solutions satisfying $x(t_0) = x_0$ and $y(t_0) = y_0$ respectively. It is simple to verify that $z(t) = x(t) - y(t)$ is a solution of the associated homogeneous equation (11) satisfying $z(t_0) = x_0 - y_0$. If $t_0 \in \mathcal{P}$ and $U(t)$ is the matrix solution of the homogeneus equation satisfying $U(t_0) = E$, then

$$x(t) - y(t) = U(t)(x_0 - y_0).$$

Therefore stability of the linear equation (9) is completely determined by the stability of the homogeneous equation (11), which is furnished by Corollary 5.3. □

Now we turn our attention to the existence of periodic solutions of the inhomogeneous equation (9). We require an analytical respresentation of its solutions. The following proposition will suffice.

Proposition 5.1. *Consider the linear impulse extension equation* (9). *For each integer* $k \geq 1$, *there exists* $v_k \in \mathbb{R}^n$ *such that the solution* $x(t; x_0)$ *of the initial-value problem* $x(\tau_0) = x_0$ *can be written*

$$x(\tau_k; x_0) = \left[\prod_{j=k-1}^{0} X(\tau_{j+1}, \tau_j) L(\tau_j + a_j, \tau_j) \right] x_0 + v_k = U(\tau_k; \tau_0) x_0 + v_k,$$

where $X(t,s)$ is the Cauchy matrix of the homogeneous ordinary differential equation $\dot{x} = A(t)x$, $L(t,s)$ is as defined in (10) *and $U(t;\tau_0)$ is a principal matrix solution at τ_0 of* (11)*. Specifically, the v_k are independent of x_0 and are generated by the recurrence relation*

$$\begin{aligned} v_0 &= 0 \\ v_{k+1} &= X(\tau_{k+1},\tau_k)L(\tau_k + a_k,\tau_k)v_k + X(\tau_{k+1},\tau_k)Q_k \\ Q_k &= \int_{\tau_k}^{\tau_{k+1}} X^{-1}(s,\tau_k)g(s)ds + \int_{\mathcal{S}_k} X^{-1}(s,\tau_k)\varphi_k^h(s)ds. \end{aligned}$$

We defer the proof of this proposition to the appendix, since the existence of such v_k is guaranteed by Proposition 4.2 and the explicit recurrence relation is provided by a tedious inductive argument along with the variation of constants formula for ordinary differential equations.

Theorem 5.5. *Consider a linear, periodic impulse extension equation* (9) *with period T and cycle number c. This equation has a unique T-periodic solution if and only if* $\det(E - M) \neq 0$*, where M is as in equation* (16).

Proof. By the periodic structure of (9) guaranteed by the conditions [P], $x(t;x_0)$ is a T-periodic solution satisfying $x(\tau_0) = x_0$ if and only if

$$x(\tau_0 + T; x_0) = x_0 = x(\tau_0; x_0).$$

By the identity $\tau_c = \tau_{0+c} = \tau_0 + T$, this is equivalent to having $x(\tau_c; x_0) = x_0$. By Proposition 5.1 and equation (16), we have the representation

$$x_0 = x(\tau_c; x_0) = \left[\prod_{j=c-1}^{0} X(\tau_{j+1},\tau_j)L(\tau_j + a_j,\tau_j)\right] x_0 + v_c = Mx_0 + v_c.$$

Equivalently,

$$(E - M)x_0 = v_c. \tag{17}$$

The above equation has a unique solution x_0 and, consequently, a unique periodic solution exists if and only if $\det(E - M) \neq 0$. Equivalently, M has no eigenvalues equal to 1. □

Notice that the predictable endpoints condition is not necessary here, because any solution $x(t)$ defined on the interval $[\tau_0, \tau_0 + T]$ and satisfying $x(\tau_0) = x(\tau_0 + T)$ is uniquely continuable to a periodic soluton on $\mathbb{R}$. This is because of how condition [P] is specified.

Theorem 5.5 provides an impulse extensions analogue of the "non-critical case" from impulsive differential equations. The "critical case", where $\det(E - M) = 0$, is more difficult. This amounts to determining conditions by which (17) has a solution x_0. To do this, we will now have to make slightly stronger assumptions with respect to the predictable set.

Definition 5.3. *Let $\tau_0 = 0$, let $U(t)$ be the principal matrix solution of* (11) *at $t_0 = 0$, and suppose $\mathcal{P} = \mathbb{R}$. The* adjoint equation *to the homogeneous system* (11) *is the ordinary differential equation*

$$\begin{aligned} \frac{dy}{dt} &= -A^*(t)y && t \notin \mathcal{S} \\ \frac{dy}{dt} &= -A^*(t)y - \left(U(\tau_k)U^{-1}(t)\right)^* [\varphi_k^B]^*(t)y && t \in \mathcal{S}. \end{aligned} \tag{18}$$

We now prove that the above differential equation truly is adjoint to (11). We will refer to these equations as *mutually adjoint.*

Proposition 5.2. *Let $\mathcal{P} = \mathbb{R}$. The homogeneous equation* (11) *and the adjoint equation* (18) *satisfy the following properties.*

1. *For any two solutions $x(t)$, $y(t)$ of the mutually adjoint equations* (11) *and* (18)*, the following identity is valid*

   $$\langle x(t), y(t)\rangle = \langle x(0), y(0)\rangle$$

   for all $t \in \mathbb{R}$. That is, $\langle x(t), y(t)\rangle$ is constant.

2. *Any matrix solutions $U(t)$ and $Y(t)$ of the mutually adjoint equations* (11) *and* (18)*, respectively, satisfy the identity*

   $$Y^*(t)U(t) = C$$

   for some $C \in \mathbb{C}^{n\times n}$.

3. *If the identity in Part 2 is valid for a matrix solution $U(t)$ of* (11) *and C is a non-singular matrix, then $Y(t)$ is a fundamental matrix of* (18).

Proof. To prove part 1, if $t \notin \mathcal{S}$, the proof is the same as for ordinary differential equations. We therefore prove only the (more difficult) case where $t \in \mathcal{S}_k$. Supressing the dependence on t (except where there may be ambiguity), we have

$$\begin{aligned} \frac{d}{dt}\langle x, y\rangle &= \langle x', y\rangle + \langle x, y'\rangle \\ &= \langle Ax + \varphi_k^B x(\tau_k), y\rangle + \langle x, -A^*y - \left(U(\tau_k)U^{-1}(t)\right)^* [\varphi_k^B]^* y\rangle \\ &= \langle Ax, y\rangle + \langle \varphi_k^B x(\tau_k), y\rangle - \langle x, A^*y\rangle - \langle x, \left(U(\tau_k)U^{-1}(t)\right)^* [\varphi_k^B]^* y\rangle \\ &= \langle Ax, y\rangle + \langle \varphi_k^B U(\tau_k)x(0), y\rangle - \langle Ax, y\rangle - \langle U(t)x(0), \left[\varphi_k^B U(\tau_k)U^{-1}(t)\right]^* y\rangle \\ &= \langle \varphi_k^B U(\tau_k)x(0), y\rangle - \langle \varphi_k^B U(\tau_k)U^{-1}(t)U(t)x(0), y\rangle \\ &= \langle \varphi_k^B U(\tau_k)x(0), y\rangle - \langle \varphi_k^B U(\tau_k)x(0), y\rangle = 0 \end{aligned}$$

almost everywhere on $\mathcal{S}_k$. This also holds on $\mathbb{R}\backslash\mathcal{S}$. Since $x(t)$ and $y(t)$ are absolutely continuous, their inner product is as well [21]. Therefore $\langle x(t), y(t)\rangle$ is absolutely continuous with zero derivative almost everywhere. It follows that $\langle x(t), y(t)\rangle$ is constant for all $t \in \mathbb{R}$. Consequently, $\langle x(t), y(t)\rangle = \langle x(0), y(0)\rangle$. This proves Part 1.

Part 2 is a direct consequence of this. Indeed,

$$Y^*U[i,j] = [Y^*]^i X_j = \sum_k (Y^*)^i(k)U_j(k) = \sum_k (Y_i)^*(k)U_j(k) = \langle Y_i, U_j\rangle,$$

which by Part 1 is constant.

To prove Part 3, suppose $Y = (CU^{-1})^*$, so that it is defined by the identity $Y^*U = C$ for a nonsingular matrix C. It is easy to verify that, for $t \notin \mathcal{S}$, we have

$$\frac{dY}{dt} = -A^*Y.$$

Conversely, if $t \in \mathcal{S}_k$, then

$$\begin{aligned}
\frac{dY}{dt} &= \frac{d(U^{-1})^*}{dt} C^* \\
&= -\left(U^{-1}\left[AU + \varphi_k^B U(\tau_k)\right]U^{-1}\right)^* C^* \\
&= -\left(U^{-1}AUU^{-1}\right)^* C^* - \left(U^{-1}\varphi_k^B U(\tau_k)U^{-1}\right)^* C^* \\
&= -\left(U^{-1}A\right)^* C^* - \left(U^{-1}\varphi_k^B U(\tau_k)U^{-1}\right)^* C^* \\
&= -A^*(U^{-1})^*C^* - \left(U(\tau_k)U^{-1}\right)^* \left[\varphi_k^B\right]^* (U^{-1})^*C^* \\
&= -A^*Y - \left(U(\tau_k)U^{-1}\right)^* \left[\varphi_k^B\right]^* Y.
\end{aligned}$$

Therefore $Y(t)$ is a matrix solution of (18) and, since $\det Y \neq 0$, it is a fundamental matrix for (18). □

Notice that, in the limiting case of impulsive differential equations, we have, for $t \in (\tau_k, \tau_{k+1}]$,

$$U(t) = X(t;\tau_k)(E + B_k) \prod_{r=k-1}^{0} X(\tau_{r+1};\tau_r)(E + B_r) \equiv X(t;\tau_k)(E + B_k)R$$

and $U(\tau_k) = R$, so that

$$U(\tau_k)U^{-1}(t) = (E + B_k)^{-1}X^{-1}(t;\tau_k)$$

for $t \in (\tau_k, \tau_{k+1}]$. Therefore

$$\begin{aligned}
\int [U(\tau_k)U^{-1}(t)]^*[\varphi_k^B]^*(t)dt &= \left[\int [(E + B_k)^{-1}X^{-1}(t;\tau_k)]^*\delta(t-\tau_k)dt\right] B_k^* \\
&= [(E + B_k)^{-1}]^* B_k^* \\
&= (E + B_k^*)^{-1}B_k^*.
\end{aligned}$$

Consequently, as the step sequence becomes small[3], the adjoint equation (18) reduces to the impulsive differential equation

$$\begin{aligned} \frac{dy}{dt} &= -A^*(t)y & t \neq \tau_k \\ \Delta y &= -(E + B_k^*)^{-1} B_k^* y & t = \tau_k. \end{aligned} \tag{19}$$

This is precisely the homogeneous adjoint from impulsive differential equations; see [1] for details.

We establish now the existence criteria for periodic solutions in the critical case. The proof is somewhat technical, and at times the notation can be a bit cumbersome; it can be found in the appendix.

Theorem 5.6. *Let condition* [P] *hold and let* $\mathcal{P} = \mathbb{R}$. *Let the homogeneous equation* (11) *have* $m \leq n$ *linearly independent* T*-periodic solutions* $p_1(t), \ldots, p_m(t)$. *Then:*

1. *The adjoint equation* (18) *has* m *linearly-indepenent solutions* $r_i(t)$, $i = 1, \ldots, m$.
2. *Equation* (9) *has a nontrivial* T*-periodic solution if and only if, for* $j = 1, \ldots, m$, *the following condition is satisfied:*

$$\sum_{k=0}^{c-1} \int_{\mathcal{S}_k} r_j^*(t) \mathcal{H}_k(t)[g(t) + \varphi_k^h(t)]dt + \int_{\tau_k + a_k}^{\tau_{k+1}} r_j^*(t) g(t) dt = 0 \tag{20}$$

where $\mathcal{H}_k(t)$, *defined by*

$$\mathcal{H}_k(t) = X(t, \tau_k) L(t, \tau_k) L^{-1}(\tau_k + a_k, \tau_k) X^{-1}(t, \tau_k)$$

is the homogeneity matrix.[4]

3. *If condition* (20) *is met, then each* T*-periodic solution of* (9) *has the form*

$$x(t) = c_1 p_1(t) + \cdots + c_m p_m(t) + x_0(t)$$

for a particular T*-periodic solution* $x_0(t)$ *of* (9).

When $\varphi_k^B = 0$ for all k, the homogeneity matrices $\mathcal{H}_k$ become the identity matrix, and condition (20) reduces to

$$\int_{\tau_0}^{\tau_c} r_j^*(s) g(s) ds + \sum_{k=0}^{c-1} \int_{\mathcal{S}_k} r_j^*(s) \varphi_k^h(s) ds = 0$$

for $j = 1, \ldots, m$. This formula was previously established by Church and Smith? in [5]. Condition (20) provides a generalization of this to the case

[3] Comparison to impulsive differential equations by this type of limit will be rigorously discussed in a forthcoming paper.

[4] We name $\mathcal{H}_k$ as such because when $\varphi_k^B = 0$, the identity $\mathcal{H}_k = E$ is valid.

where we do not necessarily have $\varphi_k^B = 0$. We do, however, require the strong assumption that the predictable set is the entire real line.

In the impulsive case, where, formally,

$$\begin{aligned}\varphi_k^B &= \delta(t-\tau_k)B_k\\ \varphi_k^h &= \delta(t-\tau_k)h_k\end{aligned}$$

and $a_k \to 0$ for all k, condition (20) reduces to

$$\int_{\tau_0}^{\tau_c} \psi_j^*(s)g(s)ds + \sum_{k=0}^{c-1} \psi_j^*(\tau_k^+)h_k = 0$$

for $j = 1, \ldots, m$, where ψ_j are the linearly independent periodic solutions of the homogeneous adjoint (see equation (19)) of the impulsive differential equation

$$\begin{aligned}\frac{dx}{dt} &= A(t)x + g(t) && t \neq \tau_k\\ \Delta x &= B_k x + h_k && t = \tau_k.\end{aligned}$$

This is precisely the condition for existence of a periodic solution in the critical case for impulsive differential equations; see [1]. In this way, Theorem 5.6 generalizes the impulsive case as well.

The Massera theorem holds for the linear impulse extension equation (9). The proof is the same as in the continuous or impulsive case, as it relies almost entirely on elementary results from linear algebra. Moreover, the predictable endpoints requirement is not necessary. The proof is available in the appendix for completeness.

Theorem 5.7. *Let the linear IEE satisfy condition* [P]. *This equation has a bounded solution for* $t \geq \tau_0$ *if and only if it has a non-trivial* T*-periodic solution.*

6 Discussion

The general impulse extension equation (2) has been defined, and it has been shown that this equation is in some sense an "impulsive differential equation with continuous impulses". Specifically, it can be seen as a continuous version of the impulsive differential equation (1) with impulses at fixed times. Results on existence and uniqueness of solutions have been obtained for a class of well-behaved initial-value problems, and one general existence result has been stated that is applicable to all initial-value problems.

Conditions for global existence and uniqueness of solutions for the linear impulse extension equation (9) were determined. This is mostly quantified by the structure that we refer to as the predictable set, $\mathcal{P}$. If $\mathcal{P}$ contains all endpoints of impulse effect, $\{\tau_k + a_k\}_{k\in\mathbb{Z}} \subset \mathcal{P}$, then any matrix solution is uniquely continuable to $\mathbb{R}$ and is unique up to multiplication by an invertible matrix. In the periodic case, this was sufficient to prove an analogue of Floquet's theorem for periodic homogeneous systems. This theorem allowed for the characterization of stability of the homogeneous and inhomogeneous equations in terms of Floquet multipliers, thereby reducing the problem of stability to a problem of finding a monodromy matrix.

The existence or nonexistence of periodic solutions for the periodic linear IEE is also directly related to the monodromy matrix; if M is a monodromy matrix and $\det(E - M) \neq 0$, the inhomogeneous equation has a unique periodic solution and the homogeneous equation has none (although this holds under weaker hypotheses; see Theorem 5.5). In the opposite case, the homogeneous equation has a periodic solution and Theorem 5.6 provides a necessary and sufficient condition for existence of a periodic solution of the inhomogeneous equation phrased in terms of the adjoint equation, provided the predictable set is the entire real line.

Along the way, it was shown that several results obtained including the condition for existence and uniqueness of solutions (Corollary 4.1) for the linear impulse extension equation and the conditions for existence of a periodic solution in the critical case (Theorem 5.6) are consistent with the the analogous results for impulsive differential equations [1] as well as more restricted classes of impulse extension equations [5]. This work therefore extends and generalizes those results.

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Appendix: Proofs

Proof of Lemma 4.1. There are two cases to consider: $t_0 \notin \mathcal{S}$ and $t_0 = \tau_k$ for some k. However, the proof of each is essentially the same, so we will only prove the second, slightly more technical, case. Let $x(\tau_k) = x_0$. We first construct a solution that is valid for all $t \geq t_0$. Since we are only interested in solutions defined for $t \geq t_0$, we may fix

$$x'(t) = A(t)x(t) + g(t) + \varphi_k^B(t)x_0 + \varphi_k^h(t)$$

for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ and ϵ small, since the solution for $t < t_0$ will be disregarded. Locally then, since this equation is linear, a unique solution exists by standard results of ordinary differential equations; see Carathéodory conditions [9]. Moreover, due to linearity, this solution is continuable until $t = \tau_k + a_k$. Let this solution, defined for $\tau_k \leq t < \tau_k + a_k$ be denoted $x(t; x_0)$. Since the solution is continuous, we define

$$x(\tau_k + a_k; x_0) \equiv \lim_{t \to \tau_k^-} x(t; x_0).$$

Now define the auxiliary initial-value problem

$$\begin{aligned} y'(t) &= A(t)y(t) \\ y(\tau_k + a_k) &= x(\tau_k + a_k; x_0). \end{aligned}$$

Again, by standard results, this has a unique solution defined for, in particular, $\tau_k + a_k \leq t < \tau_{k+1}$. By continuity, we define $y(\tau_k)$ by a left limit, similarly to above. Now extend $x(t; x_0)$ as follows:

$$x(t; x_0) = \begin{cases} x(t; x_0) & \tau_k \leq t < \tau_k + a_k \\ y(t) & \tau_k + a_k \leq t \leq \tau_{k+1}. \end{cases}$$

It is easy to verify that $x(t; x_0)$ as defined above is still a solution of the initial-value problem. To extend the solution for $t > \tau_{k+1}$, the procedure can be repeated inductively. Since, at each step, the resulting solution is unique, a unique solution exists and is defined for all $t \geq t_0$.

To continue backward in time, we consider the auxiliary initial-value problem

$$\begin{aligned} z'(t) &= A(t)z(t) \\ z(\tau_k) &= x_0. \end{aligned}$$

By standard existence and uniqueness for linear differential equations, a unique solution exists and is continuable backwards in time until $t = \tau_{k-1} + a_{k+1}$. If $x(t; x_0)$ is the unique maximal forward solution, then its unique backward continuation for $t > \tau_{k-1} + a_{k-1}$ is given by

$$x(t; x_0) = \begin{cases} x(t; x_0) & t \geq \tau_k \\ z(t) & \tau_{k-1} + a_{k-1} < t < \tau_k. \end{cases}$$

This proves the desired result. □

Proof of Lemma 4.2. Suppose a solution exists. Since $t_0 \in \mathcal{S}_k$, this solution must be defined at τ_k. In particular, there must exist some $x_{\tau_k} \in \mathbb{R}^n$ such that the solution $\phi(t)$ of the initial-value problem

$$\begin{aligned} \phi'(t) &= A(t)\phi(t) + g(t) + \varphi_k^B(t)x_{\tau_k} + \varphi_k^h(t) \\ \phi(\tau_k) &= x_{\tau_k} \end{aligned} \tag{21}$$

satisfies $\phi(t_0) = x_0$. Since the above ODE is linear, it has a solution defined, in particular, for all $t \in (\tau_k, \tau_k + a_k)$. By similar arguments to the proof of Lemma 4.1, this solution $\phi(t)$ can be extended backward in time until $t = \tau_{k-1} + a_{k-1}$ such that, for $\tau_{k-1} + a_{k-1} < t < \tau_k$, it is continuous and satisfies the differential equation

$$\phi'(t) = A(t)\phi(t) + g(t).$$

Conversely, if said x_{τ_k} should exist, then the solution $\phi(t)$ defined above is a solution of the initial-value problem in question, defined for $\tau_{k-1} + a_{k-1} < t < \tau_k + a_k$. Extending by continuity to $t = \tau_k + a_k$, we can extend further to all $t > \tau_k + a_k$ by applying Lemma 4.2. We conclude that the existence criteria of the lemma is equivalent to showing that such an x_{τ_k} exists.

Such an x_{τ_k} exists if and only if there exists an x_{τ_k} such that the solution of the initial-value problem (21) satisfies $\phi(t_0) = x_0$. The solution of said initial-value problem is

$$\phi(t) = X(t;\tau_k)x_{\tau_k} + X(t;\tau_k)\int_{\tau_k}^{t} X^{-1}(s;\tau_k)\big[g(s) + \varphi_k^B(s)x_{\tau_k} + \varphi_k^h(s)\big]ds,$$

where $X(t;s)$ is the Cauchy matrix of $x'(t) = A(t)x(t)$. Rearranging the above and imposing the condition $\phi(t_0) = x_0$, we arrive at the condition

$$x_0 - X(t_0;\tau_k)\int_{\tau_k}^{t_0} X^{-1}(s;\tau_k)\big[g(s) + \varphi_k^h(s)\big]ds = X(t_0;\tau_k)L(t_0;\tau_k)x_{\tau_k}.$$

Taking into account the invertibility of the Cauchy matrix, we arrive at

$$X^{-1}(t_0;\tau_k)x_0 - \int_{\tau_k}^{t_0} X^{-1}(s;\tau_k)\big[g(s) + \varphi_k^h(s)\big]ds = L(t_0;\tau_k)x_{\tau_k}.$$

An x_{τ_k} exists that satisfies the above relation if and only if the vector on the left-hand side is in the column space of $L(t_0;\tau_k)$. A unique x_{τ_k} exists if and only if $L(t_0;\tau_k)$ is invertible; or, equivalently, if and only if $\det L(t_0;\tau_k) \neq 0$. This proves the lemma. □

Proof of Theorem 4.1. If $t_0 \in [\tau_k, \tau_k + a_k)$, then by Lemma 4.2, a unique solution exists if and only if $\det L(t_0;\tau_k) \neq 0$; if this is satisfied, then a unique solution $x(t)$ exists for all $t > \tau_{k-1} + a_{k-1}$. Defining the solution at $t = \tau_{k-1} + a_{k-1} \equiv t_1$ by continuity, to continue backwards in time requires solving an initial-value problem of the form $x(t_1) = x_{t_1}$. For $\tau_{k-1} < t < t_1$, we have to solve

$$\begin{aligned} \phi'(t) &= A(t)\phi(t) + g(t) + \varphi^B_{k-1}(t)x_{\tau_{k-1}} + \varphi^h_k(t) \\ \phi(\tau_{k-1}) &= x_{\tau_{k-1}}, \end{aligned}$$

where $x_{\tau_{k-1}}$ has yet to be determined, such that $\lim_{t\to t_1^-}\phi(t)$ coincides with $x(t_1)$. By the proof of Lemma 4.2, existence of a unique $x_{\tau_{k-1}}$ with this property is equivalent to having $\det L(\tau_{k-1} + a_{k-1}; \tau_{k-1}) \neq 0$; when this is satisfied, the solution $x(t)$ exists for all $t > \tau_{k-2} + a_{k-2}$. By an inductive argument, we see that the solution exists for all $t \in \mathbb{R}$ if and only if the first condition $\det L(t_0; \tau_k)$ is satisfied, along with $\det L(\tau_j + a_j; \tau_j) \neq 0$ for all $j < k$.

On the other hand, if $t_0 \in [\tau_k + a_k, \tau_{k+1})$, then Lemma 4.1 ensures that a unique solution exists for all $t \geq \tau_k + a_k$. Then, to uniquely continue backward in time beyond $t = \tau_k + a_k$ is, by the above argument, equivalent to having $\det L(\tau_j + a_j; \tau_j) \neq 0$ for all $j \leq k$. This completes the proof. □

Proof of Proposition 5.1. We proceed by induction on. For $k = 1$, by the variation of constant formula for ordinary differential equations, we have

$$\begin{aligned} x(\tau_1; x_0) &= X(\tau_1,\tau_0)x_0 + X(\tau_1,\tau_0)\int_{\tau_0}^{\tau_1} X^{-1}(s,\tau_0)\left[g(s) + \mathbb{1}_{\mathcal{S}_0}(s)\varphi^h_0(s)\right]ds \ldots \\ &\quad + X(\tau_1,\tau_0)\int_{\tau_0}^{\tau_1} X^{-1}(s,\tau_0)\mathbb{1}_{\mathcal{S}_0}(s)\varphi^B_0(s)x_0 ds \\ &= X(\tau_1,\tau_0)\left[E + \int_{\mathcal{S}_0} X^{-1}(s,\tau_0)\varphi^B_0(s)ds\right]x_0 + X(\tau_1,\tau_0)Q_1 \\ &= X(\tau_1,\tau_0)L(\tau_0 + a_0,\tau_0)x_0 + v_1 \\ &= \prod_{j=1-1}^{0} X(\tau_{j+1},\tau_j)L(\tau_j + a_j,\tau_j) + v_1, \end{aligned}$$

where, indeed, $v_1 = X(\tau_1,\tau_0)Q_1$ as required.

Suppose the result holds for some $k > 1$. Denote $x(\tau_k) = x(\tau_k; x_0)$. Then, by the variation of constants formula, we have

$$\begin{aligned} x(\tau_{k+1}; x_0) &= X(\tau_{k+1},\tau_k)\left[x(\tau_k) + \int_{\tau_k}^{\tau_{k+1}} X^{-1}(s,\tau_k)\big[g(s) + \mathbb{1}_{\mathcal{S}_k}(s)\varphi^h_k(s)\big]ds\right] \ldots \\ &\quad + X(\tau_{k+1},\tau_k)\int_{\tau_k}^{\tau_{k+1}} X^{-1}(s,\tau_k)\mathbb{1}_{\mathcal{S}_k}(s)\varphi^B_k(s)x(\tau_k)ds \\ &= X(\tau_{k+1},\tau_k)\left[E + \int_{\mathcal{S}_k} X^{-1}(s,\tau_k)\varphi^B_k(s)ds\right]x(\tau_k; x_0) + X(\tau_{k+1},\tau_k)Q_k \\ &= X(\tau_{k+1},\tau_k)L(\tau_k + a_k,\tau_k)\left(\left[\prod_{j=k-1}^{0} X(\tau_{j+1},\tau_j)L(\tau_j + a_j,\tau_j)\right]x_0 + v_k\right) \ldots \end{aligned}$$

$$
\begin{aligned}
&+ X(\tau_{k+1},\tau_k)Q_k \\
&= \left[\prod_{j=k}^{0} X(\tau_{j+1},\tau_j)L(\tau_j+a_j,\tau_j)\right] x_0 + X(\tau_{k+1},\tau_k)L(\tau_k+a_k,\tau_k)v_k \ldots \\
&+ X(\tau_{k+1},\tau_k)Q_k \\
&= \left[\prod_{j=k+1-1}^{0} X(\tau_{j+1},\tau_j)L(\tau_j+a_j,\tau_j)\right] x_0 + v_{k+1}.
\end{aligned}
$$

And, as required, v_{k+1} satisfies the required recurrence relation. By induction, the lemma is proven, where the representation in terms of matrix solutions is furnished by equation (15). □

Proof of Theorem 5.6.

1. By the conditions of the theorem, $(E - M)x = 0$ has m linearly independent solutions x_i to which there correspond m linearly independent T-periodic solutions $p_i(t)$ of (11). By elementary linear algebra (see the Fredholm alternative [6]), this is true if and only if $(E - M^*)y = 0$ has m linearly independent solutions y_i. Without loss of generality, by taking $\tau_0 = 0$ and and setting $U(t)$ to be the principal matrix solution at $t_0 = 0$ for (11), we can choose $M = U(T)$ as the monodromy matrix. Now let $Y(t)$ be the principal fundamental matrix of the adjoint equation (18) at $t_0 = 0$. By Proposition 5.2, we have $Y^*(t)U(t) = Y^*(0)U(0) = E$, from which it follows that $M^* = U^*(T) = Y^{-1}(T)$. Then, for each solution y_i of $(E - M^*)y = 0$, we have

$$M^* y_i = y_i \Longrightarrow Y^{-1}(T)y_i = y_i \Longrightarrow Y(T)y_i = y_i.$$

Since $Y(T)$ is a monodromy matrix for (18), we conclude that the adjoint equation has m linearly independent periodic solutions $r_i(t)$ satisfying $r_i(0) = y_i$.

2. We must determine the solvability of $(E - M)x_0 = v_c$. By elementary results from linear algebra, a solution exists if and only if $y_j^* v_c = 0$ for each y_j satisfying $r_j(0) = y_j$, as described in Part 1. We must now describe the vector v_c in more detail. We claim that v_c can be written as a sum of c products. For brevity, let $X_k = X(\tau_k, \tau_{k-1})$ and $L_k = L(\tau_{k-1} + a_{k-1}, \tau_{k-1})$. Then, by the recurrence relation of Proposition 5.1, we find

$$
\begin{aligned}
v_c = &\left(X_c L_c X_{c-1} L_{c-1} \cdots X_1 \int_{\tau_0}^{\tau_1} X^{-1}(s,\tau_0)[g(s) + \varphi_0^h(s)]ds\right) \ldots \\
&+ \left(X_c L_c \cdots X_3 L_3 X_2 \int_{\tau_1}^{\tau_2} X^{-1}(s,\tau_1)[g(s) + \varphi_1^h(s)]ds\right) \ldots \\
&+ \cdots + \left(X_c \int_{\tau_{c-1}}^{\tau_c} X^{-1}(s,\tau_{c-1})[g(s) + \varphi_{c-1}^h(s)]ds\right),
\end{aligned}
$$

where, for succinctness, we abuse notation and write $\varphi_k^h = \mathbb{1}_{\mathcal{S}_k}\varphi_k^h$. The above sum contains precisely c terms. Let v_c^k denote the k'th term. Then

$$v_c^k = X_c L_c X_{c-1} L_{c-1} \cdots X_{k+1} L_{k+1} X_k \int_{\tau_{k-1}}^{\tau_k} X^{-1}(s, \tau_{k-1})[g(s) + \varphi_{k-1}^h(s)]ds.$$

For $a > b$, let the symbol $X(a|b)$ be defined by

$$X(a|b) = X_a L_a X_{a-1} L_{a-1} \cdots X_{b+1} L_{b+1} X_b.$$

With this notation, the identities

$$X(a|b) = X_a L_a X(a{-}1|b), \qquad X(a|b) L_b X(b{-}1|d) = X(a|d) \tag{22}$$

hold when defined, where $a > b > d$. By taking the constant matrices under the integral sign, with the above symbolic notation, we can write v_k as

$$\begin{aligned}
v_c^k &= \int_{\tau_{k-1}}^{\tau_k} X(c|k) \cdot X^{-1}(s, \tau_{k-1})[g(s) + \varphi_{k-1}^h(s)]ds \\
&= \int_{\tau_{k-1}}^{\tau_k} X(c|k)\,[L_k X(k{-}1|1) L_1] \cdot [L_k X(k{-}1|1) L_1]^{-1}\, X^{-1}(s, \tau_{k-1})[g(s) + \varphi_{k-1}^h(s)]ds \\
&= \int_{\tau_{k-1}}^{\tau_k} X_c L_c X(c{-}1|k) L_k X(k{-}1|1) L_1\,[L_k X(k{-}1|1) L_1]^{-1}\, X^{-1}(s, \tau_{k-1})[g(s) + \varphi_{k-1}^h(s)]ds \\
&= \int_{\tau_{k-1}}^{\tau_k} X(c|1) L_1 \cdot [X(s, \tau_{k-1}) L_k X(k{-}1|1) L_1]^{-1}\,[g(s) + \varphi_{k-1}^h(s)]ds \\
&= \int_{\tau_{k-1}}^{\tau_k} U(T)[X(s, \tau_{k-1}) L_k X(k{-}1|1) L_1]^{-1}[g(s) + \varphi_{k-1}^h(s)]ds \\
&= \int_{\mathcal{S}_{k-1}} U(T) U^{-1}(s) \mathcal{H}_{k-1}(s)[g(s) + \varphi_{k-1}^h(s)]ds + \int_{\tau_{k-1}+a_{k-1}}^{\tau_k} U(T) U^{-1}(s) g(s) ds.
\end{aligned} \tag{23}$$

Most of the above calculations involve use of the identities (22). We make a few clarifications, however. By (15), if $t \in [\tau_{q-1} + a_{q-1}, \tau_k)$ for some q, then

$$U(t) = X(t, \tau_{q-1}) L(\tau_{q-1}{+}a_{q-1}) \cdots X(\tau_1, \tau_0) L(\tau_0{+}a_0, \tau_0) = X(t, \tau_{q-1}) L_q X(q{-}1|1) L_1.$$

This implies $U(T) = X(\tau_c, \tau_{c-1}) L_c X(c{-}1|1) L_1 = X(c|1) L_1$. Conversely, if $t \in \mathcal{S}_{k-1}$, then, by (15), we know

$$\begin{aligned}
U(t) &= X(t; \tau_{k-1}) L(t; \tau_{k-1}) \prod_{r=k-1}^{1} X(\tau_r; \tau_{r-1}) L(\tau_{r-1} + a_{r-1}; \tau_{r-1}) \\
&= X(t, \tau_{k-1}) L(t, \tau_{k-1}) X_{k-1} L_{k-1} \cdots X_1 L_1 \\
&= X(t, \tau_{k-1}) L(t, \tau_{k-1}) X(k{-}1|1) L_1.
\end{aligned}$$

It follows that

$$X(t,\tau_{k-1})L_kX(k{-}1|1)L_1 = X(t,\tau_{k-1})L(\tau_{k-1}+a_{k-1},\tau_{k-1})\left[X(t,\tau_{k-1})L(t,\tau_{k-1})\right]^{-1}U(t)$$
$$= \mathcal{H}_{k-1}^{-1}(t)U(t).$$

Therefore

$$[X(t,\tau_{k-1})L_kX(k{-}1|1)L_1]^{-1} = U^{-1}(t)\mathcal{H}_{k-1}(t).$$

We have thus established (23). Now let $r_j(t)$ be one of the T-periodic solutions of (18). We next calculate $r_j^*(0)v_c^k$ for each $k = 1,\dots,c-1$.

By periodicity, we have $r_j^*(0) = r_j^*(T)$. By Lemma 5.2 Part 1, we have $r_j^*(T)U(T) = r_j^*(t)U(t)$ for all $t \in \mathbb{R}$, since each column of $U(t)$ is a solution of (11). This establishes the identity

$$r_j^*(T)U(T)U^{-1}(s) = r_j^*(s)U(s)U^{-1}(s) = r_j^*(s).$$

Therefore, multiplying (23) on the left by $r_j^*(0)$, we obtain

$$\begin{aligned} r_j^*(0)v_c^k &= \int_{\mathcal{S}_{k-1}} r_j^*(T)U(T)U^{-1}(s)\mathcal{H}_{k-1}(s)[g(s)+\varphi_{k-1}^h(s)]ds \\ &\quad + \int_{\tau_{k-1}+a_{k-1}}^{\tau_k} r_j^*(T)U(T)U^{-1}(s)g(s)ds \\ &= \int_{\mathcal{S}_{k-1}} r_j^*(s)\mathcal{H}_{k-1}(s)[g(s)+\varphi_{k-1}^h(s)]ds + \int_{\tau_{k-1}+a_{k-1}}^{\tau_k} r_j^*(s)g(s)ds. \end{aligned}$$

Since $v_c = \sum_{k=1}^{c} v_c^k$, we arrive at

$$r_j^*(0)v_c = \sum_{k=1}^{c}\int_{\mathcal{S}_{k-1}} r_j^*(s)\mathcal{H}_{k-1}(s)[g(s)+\varphi_{k-1}^h(s)]ds + \int_{\tau_{k-1}+a_{k-1}}^{\tau_k} r_j^*(s)g(s)ds,$$

from which we obtain the left-hand side of (20) by shifting summation indices.

3. The proof of this assertion is the same as in the case of ordinary differential equations. As such, we omit it. □

Proof of Theorem 5.7. Let $\hat{y}(t)$ be a bounded solution of (9). By Proposition 5.1 and equation (16), we have $\hat{y}(T) = M\hat{y}(0) + v$ for some $v \in \mathbb{R}^n$. Due to the periodicity hypothesis [P], a straightforward inductive argument shows that

$$\hat{y}(rT) = M^r\hat{y}(0) + \sum_{k=0}^{r-1} M^k v$$

for any integer $r \geq 1$. Now suppose (9) has no nontrivial periodic solutions. This is equivalent to the equation $(E-M)y = v$ having no solutions. By the Fredholm alternative theorem, this is true if and only if there is a solution z

of $(E - M^*)z = 0$ for which $\langle z, v\rangle \neq 0$. Consequently, $M^*z = z$, from which it follows that

$$z = (M^*)^k z = (M^k)^* z$$

for all $k \in \mathbb{Z}$. We take the inner product of $\hat{y}(rT)$ with z:

$$\begin{aligned}
\langle z, \hat{y}(rT)\rangle &= \langle z, M^r \hat{y}(0)\rangle + \sum_{k=0}^{r-1} \langle z, M^k v\rangle \\
&= \langle (M^r)^* z, \hat{y}(0)\rangle + \sum_{k=0}^{r-1} \langle (M^k)^* z, v\rangle \\
&= \langle z, \hat{y}(0)\rangle + \sum_{k=0}^{r-1} \langle z, v\rangle \\
&= \langle z, \hat{y}(0)\rangle + r\langle z, v\rangle.
\end{aligned}$$

It then follows that $\langle z, \hat{y}(rT)\rangle \to \infty$ as $r \to \infty$, contradicting the boundedness of $\hat{y}(t)$. We conclude that (9) must have a nontrivial T-periodic solution. $\square$