PICARD–LEFSCHETZ THEORY AND CHARACTERS OF A SEMISIMPLE LIE GROUP

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Abstract. The paper applies Picard-Lefschetz theory to the distribution characters of infinite dimensional representations of semisimple Lie groups and analyzes their asymptotic behaviour at the identity.

Introduction

In [Rossmann, 1995] we discussed the Picard–Lefschetz theory for the coadjoint quotient of a semisimple Lie algebra from a topological point of view. Here we shall develop the analytic side of the theory and draw from it some consequences concerning characters of a semisimple Lie group.

We recall the general setting. From the analytic point of view, Picard–Lefschetz theory is concerned with the behaviour of integrals of holomorphic forms of top degree over cycles in the fibres of a holomorphic map around a critical value. Write $q : M \to Q, \theta = q(x)$, for the map and

$$I(\Gamma, \theta) = \int_{\Gamma_\theta} \varpi_\theta$$

for the integral: $\{\Gamma_\theta\}$ is a family of cycles on the fibres $\Omega_\theta = q^{-1}(\theta)$, ϖ_θ a family of holomorphic forms of top degree, and $\theta = \theta_o$ the critical value. One is then interested in the behaviour of $I(\Gamma, \theta)$ as a function of θ near $\theta = \theta_o$, especially in asymptotic expansions.

The interest in the fibre integrals in the case of the coadjoint quotient stems from the fact that, *in exponential coordinates, the distribution character of any irreducible admissible representation of a semisimple real Lie group may be given by such an integral,* as will be explained later.

We summarize the main results. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{g}_{\mathbf{R}}$ a real form of $\mathfrak{g}, G = \mathrm{Ad}(\mathfrak{g})$, and $G_{\mathbf{R}} = \mathrm{Ad}(\mathfrak{g}_{\mathbf{R}})$ the adjoint groups. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Let

$$q : \mathfrak{g}^* \to W\backslash\mathfrak{h}^*$$

be the coadjoint quotient, $\Omega_\theta = q^{-1}(\theta)$ its fibre over θ. We also write Ω_λ for Ω_θ if $\lambda \to \theta$ under $\mathfrak{h}^* \to W\backslash\mathfrak{h}^*$; it is a complex variety of even dimension $2n$. For regular

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$\lambda \in \mathfrak{h}^*_{reg}$ the variety Ω_λ is the complex orbit $G \cdot \lambda \approx G/H$ and carries a canonical holomorphic 2–form σ_λ. We consider *orbital contour integrals*

$$\langle I(\Gamma, \lambda), \varphi \rangle = \int_{\Gamma_\lambda} \varphi \sigma_\lambda^n \quad (\lambda \in \mathfrak{h}^*_{reg}) \tag{1}$$

where φ is an entire holomorphic function on $\mathfrak{g}^*$ which is rapidly decreasing in every strip

$$\|\mathrm{Re}\, \xi\| \le \text{const.}, \quad \|q(\xi)\| \le \text{const.} \tag{2}$$

and Γ_λ is a $2n$–cycle on Ω_λ which is contained in some such strip. We refer to such a $2n$–cycle Γ_λ as a contour on Ω_λ.

As explained in [Rossmann, 1995], these contours form a homology group, denoted $'H_{2n}(\Omega_\lambda)$. With coefficients in $\mathbf{C}$, the homology groups $'H_{2n}(\Omega_\lambda)$ form a holomorphic vector bundle over $W \backslash \mathfrak{h}^*_{reg}$, which carries a canonical flat connection, the Gauss–Manin connection. The family of cycles Γ_λ is a section of this vector bundle and we shall require that this section be locally constant. If one identifies the fibres Ω_λ with a standard fibre Ω by a family of homeomorphisms $p_\lambda : \Omega \to \Omega_\lambda$ which trivialize the fibration q locally, then $\Gamma_\lambda = p_\lambda \Gamma$ for a fixed contour Γ on Ω. The standard fibre Ω can be taken to be the cotangent bundle $\mathcal{B}^*$ of the flag manifold $\mathcal{B}$ of $\mathfrak{g}$ and for $\lambda = 0$ the map p_λ becomes the *Springer resolution* $\rho : \mathcal{B}^* \to \mathcal{N}$ of the nilpotent variety $\mathcal{N}$ in $\mathfrak{g}$. The cycles Γ can be realized on the conormal variety $\mathcal{S} \subset \mathcal{B}^* of$ the $G_{\mathbf{R}}$–action on $\mathcal{B}$, and this gives an isomorphism $H_{2n}(\mathcal{S}) \to {'H_{2n}}(\Omega_\lambda), \Gamma \to p_\lambda \Gamma$, where $H_{2n}(\mathcal{S})$ is the homology with arbitrary supports. We prove the following result.

Theorem A. *Let $\Gamma \in H_{2n}(\mathcal{S})$ and φ holomorphic and rapidly decreasing in strips (2).*
a) The integral $\langle I(\Gamma, \lambda), \varphi \rangle$ is entire holomorphic in $\lambda \in \mathfrak{h}^*$ and $G_{\mathbf{R}}$–invariant in φ.
b) The Taylor series of $\langle I(\Gamma, \lambda), \varphi \rangle$ at $\lambda = 0$ takes the form

$$\langle I(\Gamma, \lambda), \varphi \rangle = \sum_{O,k} \int_O P_{O,k}(\Gamma, \lambda) \varphi \mu_O \tag{3}$$

where O runs over $G_{\mathbf{R}} \backslash \mathcal{N}_{\mathbf{R}}$ and $P_{O,k}(\Gamma, \lambda)$ is a differential operator along O, a homogeneous polynomial of degree k in λ. The terms of the series (3) corresponding to a given a O are obtained by integration over $\nu \in O$ of a series expansion

$$\int_{\Gamma_\nu} \varphi \, \frac{p_\lambda^* \sigma_\lambda^n}{p_0^* \sigma_\nu^d} \;=\; \sum_k P_{O,k}(\Gamma, \lambda) \varphi(\nu) \tag{4}$$

where Γ_ν is a compactly supported chain on the Springer variety $p_0^{-1}(\nu) \approx \mathcal{B}^\nu$.
c) Let O be a leading nilpotent of the expansion (3), i.e. open in the set of O's which occur. For each k, the map $\Gamma \to P_{O,k}(\Gamma, \cdot)$ intertwines the monodromy representation of W on $H_{2n}(\mathcal{S})$ with the natural representation of W on polynomials on $\mathfrak{h}^*$. Each $P_{O,k}(\Gamma, \lambda)$ transforms according to a sum of Springer characters $\chi_{\nu,\phi}$ where ϕ contains the trivial character of $A_{\nu,\mathbf{R}}$.

The notation will be explained later in more detail. For now we only mention that if ν is a leading nilpotent, then Γ_ν is a cycle of top degree on $\mathcal{B}^\nu$. The integral in (4) can be viewed as kind of mean value integral of φ over the (compact) cycle $p_\lambda(\Gamma_\nu)$ around ν. The differential operators $P_{O,k}(\Gamma,\lambda)$ give the terms of its Taylor series expansion as a function of the "distance" λ from the center ν, in analogy with the usual mean–value integrals.

Considered as generalized function on $i\mathfrak{g}^*_{\mathbf{R}}$, the functional $I(\Gamma,\lambda)$ is the Fourier transform of the distribution $\theta(\Gamma,\lambda)$ on $\mathfrak{g}_{\mathbf{R}}$ given by

$$\theta(\Gamma,\lambda)=\int_{\xi\in p_\lambda\Gamma} e^\xi\sigma^n_\lambda$$

the integral being convergent as a distribution. The expansion (3), obtained as a Taylor expansion in λ, gives also the asymptotic expansion of the generalized function $I(\Gamma,\lambda)$ at infinity and hence of $\theta(\Gamma,\lambda)$ at zero. Using this observation we shall prove:

Theorem B. *Let $\Gamma\in H_{2n}(\mathcal{S})$. The following sets are equal.*
a) The image of $\operatorname{supp}(\Gamma)$ *under $\rho:\mathcal{S}\to\mathcal{N}_{\mathbf{R}}$.*
b) The asymptotic cone $\lim_{t\to 0+}(tp_\lambda\Gamma)$ *of the contour $p_\lambda\Gamma$ on Ω_λ (for any fixed $\lambda\in\mathfrak{h}^*$).*
c) The closure of the union of the supports of orbits O which occur in the expansion (3) of the generalized functions $I(\Gamma,\lambda)$ for generic λ.
d) The wave front set at zero of the distribution $\theta(\Gamma,\lambda)$ for generic λ.

We call this set the asymptotic support of Γ and denote it $\mathrm{AS}(\Gamma)$.

We now explain the relation to characters. For this we now denote by $G_{\mathbf{R}}$ any connected Lie group with Lie algebra $\mathfrak{g}_{\mathbf{R}}$. In exponential coordinates, the distribution character $\operatorname{ch}(\pi)$ of any irreducible admissible representation π of $G_{\mathbf{R}}$ is of the form $\operatorname{ch}(\pi)=\theta(\Gamma,\lambda_o)$ for a unique $W\cdot\lambda_o\in W\backslash\mathfrak{h}^*$ and a unique $\Gamma\in H_{2n}(\mathcal{S})$ which is invariant under the stabilizer $W(\lambda_o)$ of λ_o. (This is stated more precisely in 3.1.) To such a representation π there is associated a wave front set $\mathrm{WF}(\pi)$ by a procedure due to Howe [1981]. As a consequence of Theorem B and a result of Howe we prove:

Theorem C. *Let π be an irreducible admissible representation of $G_{\mathbf{R}}$. Write its character in the form* $\operatorname{ch}(\pi)=\theta(\Gamma,\lambda_o)$. *Then* $\mathrm{WF}(\pi)=\mathrm{AS}(\Gamma)$.

As mentioned, the series (3) can be interpreted as an asymptotic expansion of $\theta(\Gamma,\lambda)$ at zero. The existence of such expansions for characters was deduced by Barbasch and Vogan [1980] from results of Harish–Chandra. Theorem A gives a rather clear picture of the nature of the terms of this expansion. Theorems B and C prove a conjecture of theirs [1980, p.28].

In the situation of Theorem C, a result of [Barbasch–Vogan, 1980, 4.1] implies that $\mathrm{AS}(\Gamma)$ is contained in the variety $V(\mathrm{gr}I_\pi)$ associated to the graded ideal

$\mathrm{gr}I_\pi \subset \mathbf{C}[\mathfrak{g}^*]$, where I_π is the annihilator of π in the universal enveloping algebra $U(\mathfrak{g})$. It follows from results of [Joseph, 1980] and [Borho–Brylinsky, 1982] that $V(\mathrm{gr}I_\pi)$ is the closure of a single complex orbit $\mathcal{O}$. This implies that the leading term of the expansion (3), i.e. the sum of non–zero terms with minimal k, come from $G_{\mathbf{R}}$–orbits O in $\mathcal{O} \cap i\mathfrak{g}^*_{\mathbf{R}}$, but leaves open the possibility that there might be other leading orbits O in $\mathrm{AS}(\Gamma)$, i.e. orbits not contained in the closure of any other $G_{\mathbf{R}}$– orbit in $\mathrm{AS}(\Gamma)$, which contribute to the expansion (3). We prove that this is not the case and give a more precise description of the leading term.

Theorem D. *Let $ch(\pi) = \theta(\Gamma, \lambda_o)$ be the character of an irreducible admissible representation π of $G_{\mathbf{R}}$. Then all leading nilpotents of $\mathrm{AS}(\Gamma)$ are contained in a single complex orbit and the leading term of the asymptotic expansion of $\theta(\Gamma, \lambda_o)$ at zero takes the form*

$$\theta(\Gamma, \lambda_o) \sim \sum_O c_O(\Gamma, \lambda_o)\theta_O \tag{5}$$

where O runs over the leading nilpotents in $\mathrm{AS}(\Gamma)$. Furthermore, θ_O is the Fourier transform of the canonical invariant measure μ_O on O, and $c_O(\Gamma, \lambda)$ is the polynomial on $\mathfrak{h}^*$ defined by

$$c_O(\Gamma, \lambda) = \frac{1}{e!d!}\int_{\Gamma_\nu} \tau^e_\lambda.$$

This polynomial is the harmonic polynomial associated to the image of $\Gamma_\nu = \Gamma \cap \mathcal{B}^\nu$ in $H_{2e}(\mathcal{B})$ under Borel's isomorphism. The value $c_O(\Gamma, \lambda_o)$ of this polynomial at $\lambda = \lambda_o$ is nonzero.

The proof of Theorem D depends on a result concerning the restrictions of the distributions $\theta(\Gamma, \lambda)$ on $\mathfrak{g}_{\mathbf{R}}$ to a maximal compact subalgebra $\mathfrak{k}$, which seems of independent interest. It can be stated as follows (unexplained terms are defined in 4.1).

Theorem E. *For any $\Gamma \in H_{2n}(\mathcal{S})$ and any $\lambda \in \mathfrak{h}^*$ the distribution $\theta(\Gamma, \lambda)$ on $\mathfrak{g}_{\mathbf{R}}$ admits a restriction to $\mathfrak{k}$, denoted $\theta_{\mathfrak{k}}(\Gamma, \lambda)$. If $\theta_{\mathfrak{k}}(\Gamma, \lambda) = 0$ for a $\mathfrak{g}_{\mathbf{R}}$–regular λ, then $\theta(\Gamma, \lambda) = 0$.*

In view of the existence of a Picard–Lefschetz theory for characters there naturally arises a question: *does there exist a Picard–Lefschetz theory for the representations themselves?* To some extent, the theory of coherent continuation of representations provides an affirmative answer; but it would be interesting to have a theory which operates more directly with the geometry of the coadjoint quotient. Such a theory should involve the fundamental group of $\mathfrak{g}^*_r$ itself, the Artin Braid group of the root system, not just its quotient, the Weyl group.

The results of this paper depend on those of [Rossmann, 1995], but I have tried to make the exposition (if not the proofs) as self–contained as possible, since the analytic setting here has a rather different flavour from the topological setting there. I do not repeat, however, various definitions and constructions to be found there, but give detailed references instead; a reference such as [1994, 1.9] refers to the item

1.9 of that paper.

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1. The integrals $I(\Gamma, \lambda)$

1.1 **Fiber contour integrals**. We momentarily place ourselves in the general setting of Picard–Lefschetz theory [1995, §1]. Thus we assume given a holomorphic map $q : M \to Q$ with fibres $\Omega_\theta = q^{-1}(\theta)$ which is a locally trivial fibration $q : M_r \to Q_r$ over an open subset Q_r of Q. For $\theta \in Q_r$ we consider integrals of the form

$$\int_{\Gamma_\theta} \varpi_\theta \tag{1}$$

where ϖ_θ is a holomorphic form of top degree $m = \dim_{\mathbf{C}} \Omega_\theta$ on Ω_θ and $\{\Gamma_\theta\}$ is a family of m–cycles on the regular fibres Ω_θ, locally constant for the Gauss–Manin connection. The cycles are assumed subanalytic, and the integral over subanalytic chains is the natural extension of the integral over oriented manifolds, as in [Kashiwara–Shapira, 1990, Exercise IX.3, p.407] for example.

We assume further that the map $q : M \to Q$ admits a simultaneous resolution as in [1995, 1.10],

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\rho} & M \\ \tilde{q}\downarrow & & \downarrow q \\ \tilde{Q} & \xrightarrow{\pi} & Q \end{array} \tag{2}$$

This implies in particular that the map $\tilde{q}$ is topologically trivial, so that its fibres $\tilde{\Omega}_\lambda$ ($\lambda \in \tilde{Q}$) may be identified with a standard fibre Ω by a homeomorphism $p_\lambda : \Omega \to \tilde{\Omega}_\lambda$. When $\theta = \pi(\lambda) \in Q_r$ we may identify $\tilde{\Omega}_\lambda$ with Ω_θ and then write Ω_λ for it. The locally constant family $\{\Gamma_\theta\}$ on the Ω_θ is then of the form $p_\lambda \Gamma$ for a fixed cycle Γ on the standard fibre Ω and the integrals (1) take the form

$$I(\Gamma, \lambda) = \int_{p_\lambda \Gamma} \varpi_\lambda. \tag{3}$$

The integral depends only on the homology class of Γ since a holomorphic form of top degree is closed. We require that the form ϖ_λ on Ω_λ depend holomorphically on $\lambda \in \tilde{Q}$; this has a meaning since $\tilde{M} \to \tilde{Q}$ is a holomorphic submersion: locally on $\tilde{M}$ there is a holomorphic form ϖ so that $\varpi | \Omega_\lambda = \varpi_\lambda$. (This convention is not standard in Picard–Lefschetz theory, since it involves the resolution $\tilde{M} \to \tilde{Q}$.) Assuming suitable convergence (a point to be discussed), the integral is a holomorphic function, single–valued as a function of $\lambda \in \tilde{Q}$ and multiple–valued as a function of $\theta = \pi(\lambda) \in Q$. We shall be concerned with the Taylor expansion of particular integrals of this type as functions of λ .

1.2 **Orbital contour integrals**. We now turn to the case of the coadjoint quotient $q:\mathfrak{g}^*\to W\backslash\mathfrak{h}^*$ of a semisimple complex Lie algebra. Then diagram (2) is the Springer–Grothendieck simultaneous resolution [1995, 2.2]

$$\begin{array}{ccc} \tilde{\mathfrak{g}}^* & \xrightarrow{\rho} & \mathfrak{g}^* \\ \tilde{q}\downarrow & & \downarrow q \\ \mathfrak{h}^* & \to & W\backslash\mathfrak{h}^* \end{array} \tag{4}$$

We shall use the notation introduced in [1995, §2] and refer thereto for the definitions. The standard fibre Ω is the cotangent bundle $\mathcal{B}^*$ of the flag manifold $\mathcal{B}$ and the trivialization of $\tilde{q}$ is given by a map

$$p_\lambda:\mathcal{B}^*\to\Omega_\lambda \tag{5}$$

as in [1995, 1.5, eq.(6)]. For the subalgebra $\mathfrak{g}_o$ introduced in [1995, 2.5] we now take a real form $\mathfrak{g}_{\mathbf{R}}$ of $\mathfrak{g}$. The orthogonal ${\mathfrak{g}_o}^\perp$ to $\mathfrak{g}_o$ in $\mathfrak{g}^*$ with respect to the real pairing $\mathrm{Re}\langle\xi,X\rangle$ on $\mathfrak{g}^*\times\mathfrak{g}$ is then $i\mathfrak{g}^*_{\mathbf{R}}$. As a family of supports on $\mathfrak{g}^*$ we take the closed subsets of $\mathfrak{g}^*$ which are contained in strips of the form

$$\|\mathrm{Re}(\xi)\|\le\text{const.},\quad \|q(\xi)\|\le\text{const.} \tag{6}$$

This induces a family of supports on each Ω_λ, and the corresponding homology group is denoted $'H_{.}(\Omega_\lambda)$. The second condition in (6) is vacuous on Ω_λ, but will play a rôle later.

The homology of the standard fibre is $'H_{2n}(\mathcal{B}^*)\approx H_{2n}(\mathcal{S})$ where $\mathcal{S}$ is the conormal variety of the $G_{\mathbf{R}}$ –action on $\mathcal{B}$ [1995, 2.6.1]. The fibre integrals (3) which we shall be interested in are those mentioned in the introduction:

$$\langle I(\Gamma,\lambda),\varphi\rangle=\int_{p_\lambda\Gamma}\varphi\sigma_\lambda^n\quad(\lambda\in\mathfrak{h}^*_{reg}) \tag{7}$$

We explain the hypotheses and the notation.
a) $\lambda\in\mathfrak{h}^*_{reg}$ and σ_λ is the canonical holomorphic 2–form on Ω_λ.
b) φ is an entire holomorphic function on $\mathfrak{g}^*$, rapidly decreasing in strips (6)
c) $\Gamma\in H_{2n}(\mathcal{S})$, a $2n$–cycle with arbitrary support on the conormal variety $\mathcal{S}$.
The definition of the form σ_λ in (a) will be recalled below. Condition (b) means that φ belongs to the space $'\mathcal{H}(\mathfrak{g}^*)$ defined as follows.

1.2.1 **Definition**. $'\mathcal{H}(\mathfrak{g}^*)$ is the space of entire holomorphic functions φ on $\mathfrak{g}^*$ which in any given strip (6) and for any $N=0,1,2,\cdots$ satisfy an estimate of the form

$$|\varphi(\xi)|\le\frac{C}{1+\|\xi\|^N}$$

where C is a constant, which may depend on N and on the constant in (6).

We require that the maps (5) preserve the condition (6), uniformly for λ in a neighbourhood of 0 : for $\xi\in\mathfrak{g}^*$ and $\lambda\in\mathfrak{h}^*$,

$$\|\mathrm{Re}\ \xi\|\le a \text{ and } \|\lambda\|\le b,\ \text{implies } \|\mathrm{Re}\ p_\lambda(\xi)\|\le c. \tag{8}$$

This insures that the p_λ induce maps in the homology for the family of supports consisting of closed subsets of $\mathfrak{g}^*$ contained in strips (6). One particular choice for p_λ is the map

$$p_\lambda : \mathcal{B}^* \to \Omega_\lambda, (x,\nu) \to u(x)\cdot\lambda+\nu \tag{9}$$

defined in [1995, 2.3, eq. (5)]. Note that $\|\mathrm{Re}(u(x)\cdot\lambda+\nu)\| \leq \text{const.}\|\lambda\| + \|\mathrm{Re}(\nu)\|$, so that (8) is satisfied.

If Γ is a cycle as stated, the integral (7) is independent of the trivialization which defines p_λ; but in the form (7), the integral also makes sense if Γ is any $2n$–chain on $\mathcal{S}$ (not necessarily a cycle), although it will then depend on the map (4). For technical reasons, we consider that case as well.

1.3 **Differential forms**. We recall that any G–orbit Ω in $\mathfrak{g}^*$ carries a canonical 2–form σ_Ω given by

$$\sigma_\Omega(X\cdot\xi, Y\cdot\xi) = \langle \xi, [X,Y]\rangle$$

for $\xi \in \Omega$ and $X, Y \in \mathfrak{g}$. When $\lambda \in \mathfrak{h}^*_{reg}$ is regular, then $\Omega_\lambda = G\cdot\lambda$ and σ_λ is this form on Ω_λ. For a complex nilpotent orbit $G\cdot\nu$ we denote this form by σ_ν and for a real nilpotent orbit $O = G_{\mathbf{R}}\cdot\nu$ we also write σ_O for its restriction to O. For any $\lambda \in \mathfrak{h}^*$, regular or not, we denote by τ_λ the restriction to $U\cdot\lambda$ of the canonical 2–form on $G\cdot\lambda$. We shall also consider τ_λ as a form on $\mathcal{B} \approx U/T$ through the pull–back by the map $u\cdot x_1 \to u\cdot\lambda$ (x_1 a base–point for $\mathcal{B}$) and as a form on $\mathcal{B}^*$ through the pull–back by the projection $\mathcal{B}^* \to \mathcal{B}$. As cotangent bundle of a complex manifold, $\mathcal{B}^*$ carries a canonical holomorphic 2–form, which we denote β. The following lemma, proved in [Rossmann, 1991(II), Lemma 7.2], summarizes the relations among these forms.

1.3.1 **Lemma**. *Let $\lambda \in \mathfrak{h}^*_{reg}$ and $p_\lambda : \mathcal{B}^* \to \Omega_\lambda$ be the map defined by (9).*
a) The form τ_λ is a linear function of $\lambda \in \mathfrak{h}^*$ and for $\lambda \in \mathfrak{h}^*_{reg}$ we have

$$p^*_\lambda \sigma_\lambda = \tau_\lambda + \beta.$$

b) For any $O \in G_{\mathbf{R}}\backslash\mathcal{N}_{\mathbf{R}}$ we have $p^*_0\sigma_O = \beta$ on $\mathcal{S}_O$.

This lemma has the important consequence that the pull–backs of the forms σ_λ by $\tilde{\mathfrak{g}}^* \xrightarrow{\rho} \mathfrak{g}^*$ have the holomorphicity properties required in Picard–Lefschetz theory.

1.3.2 **Corollary**. *The family of 2–forms $\lambda \to \rho^*\sigma_\lambda$ on the fibres $\tilde{\Omega}_\lambda$ of $\tilde{\mathfrak{g}}^* \xrightarrow{\rho} \mathfrak{h}^*$, defined for $\lambda \in \mathfrak{h}^*_{reg}$, extends to holomorphic family of holomorphic 2–forms $\lambda \to \tilde{\sigma}_\lambda$ defined on all of $\mathfrak{h}^*$.*

Proof. We use the real–analytic isomorphism $\mathcal{B}^*\times\mathfrak{h}^* \to \tilde{\mathfrak{g}}^*$, defined by the maps (9). Then the form $\rho^*\sigma_\lambda$ on $\tilde{\Omega}_\lambda$ corresponds to the form $p^*_\lambda\sigma_\lambda = \tau_\lambda+\beta$ on $\mathcal{B}^*\times\{\lambda\}$. It follows that $\lambda \to \rho^*\sigma_\lambda$ extends to a real–analytic function of $\lambda \in \mathfrak{h}^*$ with values in the bundle of real–analytic 2–forms on the fibres $\tilde{\Omega}_\lambda$ of the holomorphic submersion $\tilde{\mathfrak{g}}^* \xrightarrow{\tilde{q}} \mathfrak{h}^*$. Since this function has the stated holomorphicity properties on $\mathfrak{h}^*_{reg}$ it has them everywhere. $\square$

1.4 **Quotient forms and differential operators**. We need two auxiliary notions.

1.4.1 **Lemma and definition**. *Let $p : X \to Y$ be a submersion of real manifolds, $m = \dim X$, $n = \dim Y$. Let α, β be forms of top degree m, n on X, Y respectively and assume that β is non–zero everywhere. Then there exists an $m-n$ form γ on X so that*

$$\alpha = \gamma \wedge p^*\beta.$$

The restrictions γ_y of γ to fibres $p^{-1}(y)$ are uniquely determined by this equation and satisfy

$$\int_X \alpha = \int_{y\in Y} \left\{ \int_{p^{-1}(y)} \gamma_y \right\} \beta$$

with convergence as in Fubini's theorem. The family of forms $\{\gamma_y\}$ on the fibres $p^{-1}(y)$ will be denoted $\gamma = \alpha/p^*\beta$.

This is a consequence of Fubini's theorem applied to the measures $|\alpha|$, $|\gamma_y|$, and $|\beta|$ defined by these forms. In particular, absolute convergence of the integral on the left implies absolute convergence of the inner integral on the right for $|\beta|$–almost all $y \in Y$.

1.4.2 **Example.** We give an example of the division of forms which will be relevant later. Fix $O \in G_{\mathbf{R}}\backslash\mathcal{N}_{\mathbf{R}}$ and $\nu \in O$. Let $2d = \dim_{\mathbf{R}} O$ and $2e = \dim_{\mathbf{R}} \mathcal{B}^\nu$, so that $e + d = n$. Let p_λ be the map (9). For any $\nu \in O$, we define the quotient $p^*_\lambda\sigma^n_\lambda/p^*_0\sigma^d_\nu$ as a $2e$-form on the regular set of $\mathcal{B}^\nu$ using the map $p_0 = \rho : \mathcal{S}_O \to O$ and the restriction of $p^*_\lambda\sigma_\lambda$ to $\mathcal{S}_O \subset \mathcal{B}^*$. This quotient form is a polynomial function of $\lambda \in \mathfrak{h}^*$ of degree $\leq n$ of the form

$$\frac{p^*_\lambda\sigma^n_\lambda}{\rho^*\sigma^d_\nu} = \frac{n!}{e!d!}\tau^e_\lambda + \text{ terms of higher degree in } \lambda. \tag{10}$$

Indeed from 1.3.1 one gets

$$p^*_\lambda\sigma^n_\lambda = \sum_{k+l=n} \frac{n!}{k!l!}\tau^k_\lambda\beta^l.$$

The division of forms is defined in terms of a C^∞ splitting of $\rho : \mathcal{S}_O \to O$, locally around a point where it is a submersion, say $\mathcal{S}_O \approx O \times \mathcal{B}^\nu$ with local coordinates $(x_i,\, y_j)$. The form $\beta = \rho^*\sigma_\nu$ on $\mathcal{S}_O$ involves dx_i's only. The quotient form $\tau^k_\lambda\beta^l/\beta^d$ comes from a form γ satisfying

$$\tau^k_\lambda\beta^{d+e-k} = \gamma\beta^d$$

Hence γ is a combination of monomials in the dx_i and dy_j of degree at least $2(e-k)$ in the dx_i and therefore vanishes on $\mathcal{B}^\nu = \{x_i = 0\}$ if $2(e-k) > 0$.

1.4.3 **Definition**. Let $f : X \to Y$ be a C^∞ map between C^∞ manifolds. By a *differential operator on* Y *along* f we mean an operator P from C^∞ functions on Y to C^∞ functions on X which is locally of the form

$$P\varphi(x) = \sum_k c_k(x)(\partial_y^k \varphi)(f(x)) \quad \text{(finite sum)}$$

when x and y are expressed in local coordinates. The $c_k(x)$ are understood to be C^∞ functions and the $\partial_y^k\varphi$ denote the higher–order partial derivatives with respect to the coordinates y (the usual multi–index notation). It is understood that P operates in this way also on functions φ defined only on some open subset of Y. A *differential operator along a submanifold* X of Y is a differential operator P along the inclusion $X \to Y$. When $X = \{x\}$ is a point we call P a *differential operator at* $x \in Y$. (It is a distribution supported at x.) We shall use the notation $\mathcal{D}(Y, X)$ and $\mathcal{D}(Y, x)$ for the differential operators along X and at x, respectively.

1.5 **Definition**. a)Let $\nu \in \mathcal{N}_{\mathbf{R}}$, $2e$–chain on $\mathcal{B}^\nu$, and $\lambda \in \mathfrak{h}^*_{reg}$. Define a differential operator $P_{\nu,k}(\Gamma_\nu, \lambda) \in \mathcal{D}(\mathfrak{g}^*, \nu)$ at ν on $\mathfrak{g}^*$ by the formula

$$P_{\nu,k}(\Gamma_\nu, \lambda)\varphi(\nu) := \int_{\Gamma_\nu} \frac{1}{k!} \frac{[d^k p^*_{t\lambda}(\varphi\sigma^n_{t\lambda})/dt^k]_{t=0}}{\rho^*(\sigma^d_\nu)}. \tag{11}$$

provided the integral converges. (The quotient form is defined as in 1.4.1 for the map $\mathcal{S}_O \xrightarrow{\rho} O, O = G_{\mathbf{R}} \cdot \nu$ and the derivatives at $t = 0$ are defined in view of 1.3.2.)
b)Let $O \in G_{\mathbf{R}} \backslash \mathcal{N}_{\mathbf{R}}$ and Γ_O a $G_{\mathbf{R}}$–stable $2n$–chain on $\mathcal{S}_O$. Define a differential operator $P_{O,k}(\Gamma_O, \lambda) \in \mathcal{D}(\mathfrak{g}^*, O)$ along O on $\mathfrak{g}^*$ by the formula

$$P_{O,k}(\Gamma_O, \lambda)\varphi(\nu) = P_{\nu,k}(\Gamma_\nu, \lambda)\varphi(\nu)$$

where (cf. [1995, 3.3]).

1.5.1 **Remarks**. a) For $\varphi \in C_c^\infty(\mathfrak{g})$, the numerator of the quotient form in (11) is a compactly supported C^∞ form on $\mathcal{B}^*$. The quotient form in (11) is defined by restriction of this form to the set of points of the set $\mathcal{S}_O$ which are smooth on $\mathcal{S}_O$ and at which $\mathcal{S}_O \to O$ is a submersion. These points form an open, dense, $G_{\mathbf{R}}$–stable subset of $\mathcal{S}_O$, hence the quotient form is defined on an open, dense subset of $\mathcal{B}^\nu$ for all $\nu \in O$. It need not be regular on all of $\mathcal{B}^\nu$, but for the case when $\Gamma_\nu = \Gamma_O \cap \mathcal{B}^\nu$ as above the convergence of the integral in (11) for almost all $\nu \in O$ follows from Fubini's theorem, as remarked in connection with 1.4.1. Since such a Γ_O is a linear combination of components of $\mathcal{S}$ the convergence for all $\nu \in O$ then follows from the G–equivariance of the map ρ. A more explicit formula for these operators is given in (14) below.
b) The $2e$–chain Γ_ν depends on the choice of an orientation on O. We shall take the orientation on O which makes the form σ^d_O positive.
c) It is clear that $P_{\nu,k}(\Gamma_\nu, \lambda)$ is a differential operator of degree $\leq k$ at ν on $\mathfrak{g}^*$, hence (11) depends only on the Taylor polynomial of degree k of φ at ν. It can then also be considered as a differential operator at ν on the real form $i\mathfrak{g}^*_{\mathbf{R}}$ of $\mathfrak{g}^*$: if φ is a C^∞ function on $i\mathfrak{g}^*_{\mathbf{R}}$ defined near $\nu \in \mathcal{N}_{\mathbf{R}}$, then (11) is interpreted by replacing φ on the right of (11) by its Taylor polynomial of degree k at ν.

1.5.2 **Lemma**. *As function of λ, the operator $P_{\nu,k}(\Gamma_\nu,\lambda)$ is a homogeneous polynomial of degree k, i.e. $P_{\nu,k}(\Gamma_\nu,\cdot)\in \mathbf{C}[\mathfrak{h}^*]^{(k)}\otimes\mathcal{D}(\mathfrak{g}^*,\nu)$. If φ is a holomorphic function on $\mathfrak{g}^*$ defined near ν, then*

$$\int_{\Gamma_\nu}\frac{p_\lambda^*(\varphi\sigma_\lambda^n)}{\rho^*(\sigma_\nu^d)}=\sum_{k=0}^{\infty}P_{\nu,k}(\Gamma_\nu,\lambda)\varphi(\nu) \tag{12}$$

is the Taylor series expansion at $\lambda=0$ of the holomorphic function of λ on the left.

Proof. The fact that the left–side of (12) is holomorphic in λ follows from 1.3.2. That the right side is its power series expansion follows from the definition (11) by differentiation under the integral sign. □

1.5.3 **Example**. For the map p_λ defined by (9) one has more explicitly

$$\int_{\Gamma_\nu}\frac{p_\lambda^*(\varphi\sigma_\lambda^n)}{\rho^*(\sigma_\nu^d)}=\int_{x\in\Gamma_\nu}\varphi(\nu+u(x)\cdot\lambda)\frac{(\tau_\lambda+\beta)^n}{\rho^*(\sigma_\nu^d)}. \tag{13}$$

This gives

$$P_{O,k}(\Gamma_O,\lambda)\varphi(\nu)=\sum_{j=0}^{k}\frac{1}{j!(k-j)!}\int_{x\in\Gamma_\nu}D^{k-j}_{u(x)\cdot\lambda}\varphi(\nu)\frac{\tau_\lambda^j\beta^{n-j}}{\rho^*(\sigma_\nu^d)}. \tag{14}$$

Here D_η is the directional derivative along the constant vector field $\eta\in\mathfrak{g}^*$, $e=\dim_{\mathbf{C}}\mathcal{B}^\nu$, and $2d=\dim_{\mathbf{C}}G{\cdot}\nu$. The formula (14) means in particular that $P_{O,k}(\Gamma_O,\lambda)=0$ for $k<e$, as is clear from (13) and the discussion in 1.5.3 concerning the vanishing of the quotient forms.

1.6 **Taylor series expansions of orbital contour integrals**. We recall some notation from [1995, §3]. For each $A\subset G_{\mathbf{R}}\backslash\mathcal{N}_{\mathbf{R}}$ (identified with a $G_{\mathbf{R}}$–stable subset of $\mathcal{N}_{\mathbf{R}}$ we denote by $\mathcal{S}_A$ its preimage under $\rho:\mathcal{S}\to\mathcal{N}_{\mathbf{R}}$. Any $\Gamma\in H_{2n}(\mathcal{S})$ can be uniquely written in the form

$$\Gamma=\sum_{O\in G_{\mathbf{R}}\backslash\mathcal{N}_{\mathbf{R}}}\Gamma_O \tag{15}$$

with $\mathrm{supp}\Gamma_O$ contained in the closure of $\mathcal{S}_O$. (Thus Γ_O is the chain on $\mathcal{S}$ obtained by restricting Γ to $\mathcal{S}_O$, as explained in [1995, §3].) A leading nilpotent in $A\subset G_{\mathbf{R}}\backslash\mathcal{N}_{\mathbf{R}}$ is an orbit $O\in G_{\mathbf{R}}\backslash\mathcal{N}_{\mathbf{R}}$ which is not contained in the closure of any other orbit in A.

1.6.1 **Theorem**. *Let $\Gamma\in H_{2n}(\mathcal{S})$, $\lambda\in\mathfrak{h}^*$, and $\varphi\in{}'\mathcal{H}(\mathfrak{g}^*)$.*
a) The integral $\langle I(\Gamma,\lambda),\varphi\rangle$ is entire holomorphic in $\lambda\in\mathfrak{h}^*$ and $G_{\mathbf{R}}$–invariant in $\varphi\in{}'\mathcal{H}(\mathfrak{g}^*)$.
b) Write $\Gamma=\sum_O\Gamma_O$ as in (15). Then

$$\langle I(\Gamma,\lambda),\varphi\rangle=\sum_{O,k}\int_O P_{O,k}(\Gamma_O,\lambda)\varphi\mu_O \tag{16}$$

where O runs over $G_{\mathbf{R}}\backslash\mathcal{N}_{\mathbf{R}}$ and $k=0,1,\cdots$.
c) Let A be a closed subset of $G_{\mathbf{R}\backslash\mathcal{N}_{\mathbf{R}}}$and O be a leading nilpotent for $A(\Gamma)$. For each k, the maps $\Gamma\mapsto\Gamma_O\mapsto P_{O,k}(\Gamma_O,\cdot)$,

$$H_{2n}(\mathcal{S}_A)\to H_{2n}(\mathcal{S}_O)\approx H_{2e}(\mathcal{B}^\nu)^{A_{\nu,\mathbf{R}}}\to\mathbf{C}[\mathfrak{h}^*]\otimes\mathcal{D}(\mathfrak{g}^*,\nu)$$

are W–maps. In particular, each polynomial $P_{O,k}(\Gamma_O,\cdot)$ transforms according to a sum of Springer characters $\chi_{\nu,\phi}$ where ϕ contains the trivial character of $A_{\nu,\mathbf{R}}$.

Here $A_{\nu,\mathbf{R}}$ is the component group of the stabilizer of $\nu\in O$. The representations of W referred to are the natural ones [1995, §4]: the restricted monodromy representations on $H_{2n}(\mathcal{S}_A)$ and $H_{2n}(\mathcal{S}_O)$, the Springer representation on $H_{2e}(\mathcal{B}^\nu)^{A_{\nu,\mathbf{R}}}$, and the natural representation of W on $\mathbf{C}[\mathfrak{h}^*]$ tensored with the trivial representation on $\mathcal{D}(\mathfrak{g}^*,\nu)$.

Proof. a) We choose a trivialization of $\tilde q:\tilde{\mathfrak{g}}^*\to\mathfrak{h}^*$, say

$$\tilde p:\mathcal{B}^*\times\mathfrak{h}^*\xrightarrow{\approx}\tilde{\mathfrak{g}}^*.$$

In terms of the associated maps $\tilde p_\lambda:\mathcal{B}^*\to\tilde\Omega_\lambda=\tilde q^{-1}(\lambda)$ and $p_\lambda=\rho\circ\tilde p_\lambda:\mathcal{B}^*\to\Omega_\lambda$ as in [1995, 1.5] the integral becomes

$$\langle I(\Gamma,\lambda),\varphi\rangle=\int_{\tilde p_\lambda\Gamma}\varpi(\lambda) \tag{17}$$

where we have put $\varpi(\lambda)=\rho^*(\varphi\sigma_\lambda^n)$, a holomorphic $2n$–form on $\tilde\Omega_\lambda$, which depends holomorphically on $\lambda\in\mathfrak{h}^*$, by 1.3.2.

We first deal with the convergence of the integral. Any subanalytic $2n$–chain on $\mathcal{S}$ is a finite linear combination of oriented $2n$–dimensional submanifolds of $\mathcal{S}$. Hence it suffices to show convergence when Γ is replaced by $\mathcal{S}$ itself and the form $p_\lambda^*(\varphi\sigma_\lambda^n)$ by the corresponding measure $|p_\lambda^*(\varphi\sigma_\lambda^n)|$. We recall that $\mathcal{S}$ is the union of the conormal bundles of the $G_{\mathbf{R}}$–orbits on $\mathcal{B}$. The form $p_\lambda^*(\varphi\sigma_\lambda^n)$ is rapidly decreasing along the fibres. Thus using a partition of unity on $\mathcal{B}$ we are reduced to integrals of the form

$$\int_{x\in\mathbf{R}^a,\|x\|\le r}\int_{y\in\mathbf{R}^b}\psi(\lambda,x,y)dydx$$

where $\psi(\lambda,x,y)$ is a C^∞ function of (x,y) on the domain of integration in $\mathbf{R}^a\times\mathbf{R}^b$ which is rapidly decreasing in y, uniformly for λ in compact subsets of $\mathfrak{h}^*$. This shows that the integral in (17) converges, uniformly for λ in compact subsets of $\mathfrak{h}^*$.

The holomorphicity of (17) as a function of λ would also follow were it not for the dependence on λ of the cycle $\tilde p_\lambda\Gamma$. To deal with that, we use a classical argument from Picard–Lefschetz theory. It suffices to show that for any $\lambda,\mu\in\mathfrak{h}^*$, the integral $\langle I(\Gamma,\,\lambda+z\mu),\,\varphi\rangle$ is a holomorphic function of $z\in\mathbf{C}$. Fix $\lambda,\mu\in\mathfrak{h}^*$, $z\in\mathbf{C}$, and let $D=\{\lambda+w\mu:|w-z|\le\epsilon\}$. Using the trivialization $\mathcal{B}^*\times\mathfrak{h}^*\xrightarrow{\approx}\tilde{\mathfrak{g}}^*$ to momentarily identify $\tilde{\mathfrak{g}}^*$ with $\mathcal{B}^* \times \mathfrak{h}^*$, let $\Delta = \Gamma \times \partial D$, considered as $(2n+1)$–chain on $\tilde{\mathfrak{g}}^*$ and let $\varpi \wedge dw = \varpi(\lambda + w\mu) \wedge dw$ considered as form on $\mathcal{B}^* \times D$. Then

$$\begin{aligned}\frac{1}{2\pi i}\int_\Delta \frac{\varpi \wedge dw}{w-z} &= \frac{1}{2\pi i}\int_{\partial D}\left(\int_{\Gamma\times\{w\}}\varpi\right)\frac{dw}{w-z}\\ &= \frac{1}{2\pi i}\int_{\partial D}\left(\int_{\Gamma\times\{z\}}\varpi\right)\frac{dw}{w-z}\\ &\qquad + \frac{1}{2\pi i}\int_{\partial D}\left(\int_{\Gamma\times\{w\}} - \int_{\Gamma\times\{z\}}\right)\varpi\frac{dw}{w-z}\\ &= \int_{\tilde{p}_{\lambda+z\mu}(\Gamma)}\varpi + (\cdots).\end{aligned}$$

The integral on the left side is a holomorphic function of z and independent of ϵ. The first term on the right side of the last equation is $I(\lambda + z\mu)$; the second one tends to 0 as ϵ tends to zero. The required holomorphic dependence on z follows.

To prove the $G_{\mathbf{R}}$–invariance of $\langle I(\Gamma,\lambda),\varphi\rangle$ as functional in $\varphi \in {}'\mathcal{H}(\mathfrak{g}^*)$ we take $g \in G_{\mathbf{R}}$ and set $(g^*\cdot\varphi)(\xi) = \varphi(g\cdot\xi)$. Since $G_{\mathbf{R}}$ preserves the conditions (6), $g^*\varphi \in {}'\mathcal{H}(\mathfrak{g}^*)$ and

$$\langle I(\Gamma,\lambda),\varphi\rangle = \int_{p_\lambda\Gamma} g^*(\varphi\sigma_\lambda^n) = \int_{g\cdot p_\lambda\Gamma}\varphi\sigma_\lambda^n.$$

Since $G_{\mathbf{R}}$ is connected, the cycle $g\cdot p_\lambda\Gamma$ on Ω_λ is homotopic to $p_\lambda\Gamma$, the homotopy preserving the support condition (6). Hence

$$\int_{g\cdot p_\lambda\Gamma}\varphi\sigma_\lambda^n = \int_{p_\lambda\Gamma}\varphi\sigma_\lambda^n$$

as required.

b) With Γ as in the statement of the theorem,

$$\langle I(\Gamma,\lambda),\varphi\rangle = \sum_O\int_{\Gamma_O} p_\lambda^*(\varphi\sigma_\lambda^n) = \sum_O\int_{\nu\in O}\left\{\int_{\Gamma_\nu}\frac{p_\lambda^*(\varphi\sigma_\lambda^n)}{\rho^*(\sigma_\nu^d)}\right\}\sigma_O^d.$$

The integral over O may be taken as an integral with respect to the measure μ_O, in view of the convention concerning the orientation of O (cf. 1.5.1 (a)). Lemma 1.5.2, applied to the inner integral, gives the desired expansion (16).

c) The fact that the first map in (17) is W–equivariant is a consequence of [1995, 4.4.1]. To see that the second map is also W–equivariant we first verify that

$$\langle I(w^{-1}\cdot\Gamma,\lambda),\varphi\rangle = \langle I(\Gamma, w\cdot\lambda),\varphi\rangle \tag{18}$$

with $w \in W$ acting on $H_{2n}(\mathcal{S})$ by the restricted monodromy representation. By definition [1995, 1.9.3], this representation is implemented by the transformations $a_\lambda(w) = p_{w\lambda}^{-1}\circ p_\lambda$ of $\mathcal{B}^*$. Hence

$$\langle I(w^{-1}\cdot\Gamma,\lambda),\varphi\rangle = \int_{p_\lambda\circ a_\lambda(w)^{-1}\Gamma}\varphi\sigma_\lambda^n = \int_{p_{w\lambda}\Gamma}\varphi\sigma_\lambda^n = \langle I(\Gamma,w\cdot\lambda),\varphi\rangle$$

since $\sigma_{w\lambda} = \sigma_\lambda$. If one takes the Taylor expansion at $\lambda = 0$ of both sides on (18) one finds that the second map in (17) is W–equivariant as well. $\square$

1.7 **Homogeneity properties, asymptotic expansions**. The Taylor series expansion of $I(\Gamma,\lambda)$ at $\lambda=0$ in the theorem can also be viewed as an asymptotic expansion at $\xi=\infty$ of $I(\Gamma,\lambda)$ as generalized function on $i\mathfrak{g}^*_{\mathbf{R}}$, as we shall now explain. Denote by m_t the multiplication by $t\in\mathbf{C}^\times$ on $\mathfrak{g}^*$ or on $\tilde{\mathfrak{g}}^*$:
for $\xi\in\mathfrak{g}^*$, $m_t\cdot\xi:=t\xi$,
for $(x,\xi)\in\tilde{\mathfrak{g}}^*, m_t\cdot(x,\xi):=(x,t\xi)$,
For forms or functions ψ we write $m_t^*\ \psi$ for the pull–back by m_t.

1.7.1 **Lemma**. *Let $\lambda\in\mathfrak{h}^*_{reg}, p_\lambda$ the map (6), and $t\in\mathbf{R}^\times_+$.*
a) For any G–orbit Ω in $\mathfrak{g}^*$, $m_t^*\sigma_{t\Omega}=t\sigma_\Omega$; in particular,

$$m_t^*\sigma^n_{t\lambda}=t^n\sigma^n_\lambda,\quad m_t^*\sigma^d_O=t^d\sigma^d_O.$$

b) $m_t\circ p_\lambda=p_{t\lambda}\circ m_t$.
c) For any $\Gamma\in H_{2n}(\mathcal{S})$ and $\varphi\in{}'\mathcal{H}(\mathfrak{g}^*), \langle I(\Gamma,\lambda),m_t^*\varphi\rangle=t^{-n}\langle I(\Gamma,t\lambda),\ \varphi\rangle$.
d) For each $O,k$the differential operator $P_{O,k}(\Gamma_O,\lambda)$ along O satisfies

$$P_{O,k}(\Gamma_O,\lambda)\circ m_t^*=t^{-n+d+k}m_t^*\circ P_{O,k}(\Gamma_O,\lambda).$$

Proof. We omit the simple verification. $\square$

1.7.2 **Corollary**. *The terms in the expansion (6) are homogeneous temperate distributions on $i\mathfrak{g}^*_{\mathbf{R}}$of degree $-\dim_{\mathbf{R}}\mathfrak{g}_{\mathbf{R}}+n-k$ in the sense of [Hörmander, 1983, 3.2.2], i.e.*

$$\int_O P_{O,k}(\Gamma_O,\lambda)(m_t^*\varphi)\mu_O=t^{-n+k}\int_O P_{O,k}(\Gamma_O,\lambda)\varphi\mu_O.$$

This means that the expansion (16) can be viewed as an asymptotic expansion at infinity of the generalized function $\varphi\to\langle I(\Gamma,\lambda),\varphi\rangle$.

1.8 **Notes**. The study of *fibre contour integrals* of the type (1) goes back to the beginnings of the theory, to Picard and Lefschetz. There is an extensive literature dealing with the asymptotic behaviour of such integrals, for example [Malgrange, 1974]. An overview and further references can be found in [Arnold et al., 1988].

2. Fourier transforms, wave front sets

2.1 **Fourier transforms**. Let Γ be any $2n$–chain on $\mathcal{S}$ and $\lambda\in\mathfrak{h}^*_{reg}$. For any $f\in C_c^\infty(\mathfrak{g}_{\mathbf{R}})$, define

$$\langle\theta(\Gamma,\lambda),f\rangle=\langle I(\Gamma,\lambda),\hat{f}\rangle$$

where $\hat{f}$ is the Fourier transform, defined by

$$\hat{f}(\xi) = \int_{X \in \mathfrak{g}_{\mathbf{R}}} e^{\langle \xi, X \rangle} f(X) \qquad (\xi \in \mathfrak{g}^*).$$

(We omit the customary factor $-i$ from the exponent.) According to the Paley–Wiener theorem [Hörmander, 1983, 7.3.1], these functions $\hat{f}$ are characterized by an estimate of the form

$$|\hat{f}(\xi)| \leq \frac{Ae^{B\|\mathrm{Re}\ (\xi)\|}}{1 + \|\xi\|^N} \tag{1}$$

for any $N = 0, 1, 2, \cdots$ with A depending on N. Thus $\hat{f} \in' \mathcal{H}(\mathfrak{g}^*)$ and $\theta(\Gamma, \lambda)$ is a distribution on $\mathfrak{g}_{\mathbf{R}}$. It can be written in the form

$$\theta(\Gamma, \lambda) = \int_{\xi \in \Gamma_\lambda} e^{\xi} \sigma_\lambda^n \tag{2}$$

the integral being convergent as distribution on $\mathfrak{g}_{\mathbf{R}}$.

2.2 **Families of invariant eigendistributions**. If p is a polynomial function on $\mathfrak{g}^*$, denote by $p(\partial)$ the constant coefficient operator on $\mathfrak{g}$ satisfying

$$p(\partial) e^{\xi} = p(\xi) e^{\xi} \tag{3}$$

for all $\xi \in \mathfrak{g}^*$. Since $p(\xi) = p(\lambda)$ for $\xi \in \Omega_\lambda$, it is clear from (2) that

$$p(\partial)\theta(\Gamma, \lambda) = p(\lambda)\theta(\Gamma, \lambda). \tag{4}$$

For $\lambda_o \in \mathfrak{h}^*$ we denote by $W(\lambda_o)$ its stabilizer in W and by $H_{2n}(\mathcal{S})^{W(\lambda_o)}$ the subspace of $H_{2n}(\mathcal{S})$ fixed by $W(\lambda_o)$. In this context we shall always assume that the order of $W(\lambda_o)$ is invertible in the coefficient ring for the homology, so that the $W(\lambda_o)$–projection onto $H_{2n}(\mathcal{S})^{W(\lambda_o)}$ is defined. We denote by Ch the space of families of distributions $\theta(\cdot)$ of the form $\theta(\lambda) = \theta(\Gamma, \lambda)$ with $\Gamma \in H_{2n}(\mathcal{S})$. It follows from (18) of §1 that $\theta(w \cdot \Gamma, \lambda) = \theta(\Gamma, w^{-1} \cdot \lambda)$ so that Ch is W–stable and $\Gamma \to \theta(\Gamma, \cdot)$ is equivariant for the natural action of W, defined by $(w \cdot \theta)(\lambda) = \theta(w^{-1} \cdot \lambda)$. For any $\lambda_o \in \mathfrak{h}^*$ we denote by $\mathrm{Ch}(\lambda_o)$ the space of distributions $\theta(\lambda_o)$ with $\theta(\cdot) \in \mathrm{Ch}$ and by $\mathrm{Ch}^{W(\lambda_o)}$ the subspace of Ch fixed by $W(\lambda_o)$.

2.2.1 **Theorem**.*a) For regular $\lambda_o \in \mathfrak{h}^*$ every $G_{\mathbf{R}}$–invariant distribution on $\mathfrak{g}_{\mathbf{R}}$ satisfying*

$$p(\partial)\theta(\lambda_o) = p(\lambda_o)\theta(\lambda_o) \tag{5}$$

for all G–invariant polynomials p on $\mathfrak{g}^*$ is of the form $\theta(\lambda_o) = \theta(\Gamma, \lambda_o)$ for a unique $\Gamma \in H_{2n}(\mathcal{S})$.
b) For any $\lambda_o \in \mathfrak{h}^*$ the map $\Gamma \to \theta(\Gamma, \lambda_o)$ is an isomorphism of $H_{2n}(\mathcal{S})^{W(\lambda_o)}$ onto $\mathrm{Ch}^{W(\lambda_o)}$.

2.2.2 **Remark**. The families of distributions $\theta(\Gamma, \cdot)$ in Ch correspond to coherent families in the sense of representation theory (cf. [Hecht–Schmid, 1983], for example): Ch has a basis consisting of the standard families defined in §4 and these correspond to the coherent families of standard characters in exponential coordinates, as explained in §3 and §4.

Proof of the theorem. Part (a) follows from Theorem 1.4 of [Rossmann, 1990]. To prove (b) we have to recall part of the proof: there is a grading on Ch,

$$\mathrm{grCh} \approx \sum_c \mathrm{gr}_c \mathrm{Ch}$$

where the index "c" runs over a system of representatives of the $G_{\mathbf{R}}$–conjugacy classes of Cartan subalgebras of $\mathfrak{g}_{\mathbf{R}}$. The graded component $\mathrm{gr}_c\mathrm{Ch}$ is isomorphic to the space of exponential sums

$$\sum_{w \in W} a_{c,w} e^{w^{-1}} \lambda \tag{6}$$

whose coefficients $a_{c,w}$ satisfy

$$a_{c,wz} = a_{c,w} \mathrm{sgn}_{c,\mathbf{I}}(z) \quad \text{ for } z \in W_{c,\mathbf{R}}.$$

Here $W_{c,\mathbf{R}}$ is the subgroup W realized in $G_{\mathbf{R}}$ when W is identified with the Weyl group of the Cartan subalgebra corresponding to c and $\mathrm{sgn}_{c,\mathbf{I}}$ its sign character on the imaginary roots. The action of $y \in W$ on Ch, given by $\lambda \to y^{-1}\lambda$, corresponds to the analogous action on the exponential sums.

Similarly one has a grading on $\mathrm{Ch}(\lambda_o)$ whose graded component $\mathrm{gr}_c\mathrm{Ch}(\lambda_o)$ is isomorphic to sums (6) with $\lambda = \lambda_o$. The evaluation map $\mathrm{Ch} \to \mathrm{Ch}(\lambda_o)$ is compatible with the grading and the induced map on the spaces of formal sums (6) is given by the evaluation $\lambda \to \lambda_o$. The assertion is now evident. □

2.3 **Asymptotic expansions**. Under the Fourier transform, the expansion 1.6.1, eq.(16), becomes

$$\theta(\Gamma, \lambda) = \sum_{k=0}^{\infty} \sum_{O \in G_{\mathbf{R}} \backslash \mathcal{N}_{\mathbf{R}}} \theta_{O,k}(\Gamma_O, \lambda) \tag{7}$$

where $\theta_{O,k}(\Gamma, \lambda)$ has the temperate distribution ${}^t P_{O,k}(\Gamma_O, \lambda)\mu_O$ as Fourier transform. It follows from 1.7.2 that $\theta_{O,k}(\Gamma, \lambda)$ is homogeneous of degree $-n + k$ [Hörmander, 1983, 7.1.16]. Hence (7) gives the asymptotic expansion of the distribution $\theta(\Gamma, \lambda)$ on $\mathfrak{g}_{\mathbf{R}}$ at 0 in the sense of [Barbasch–Vogan, 1980].

2.4 **Asymptotic supports**. For a $2n$–cycle Γ on the conormal variety $\mathcal{S}$ one can define in several apparently rather different ways a subset A on the nilpotent cone $\mathcal{N}_{\mathbf{R}}$, each of which could be construed as some kind of "asymptotic support" associated to Γ.

a) $A(\Gamma)$ = the image of the support of Γ under the map $\rho : \mathcal{S} \to \mathcal{N}_{\mathbf{R}}$:

$$A(\Gamma) = \rho(\mathrm{supp}\Gamma).$$

It is a closed, $G_{\mathbf{R}}$–stable subset of $\mathcal{N}_{\mathbf{R}}$ (closed, because ρ is proper). Since $\rho = p_0$ one can write for any fixed λ,

$$A(\Gamma) = \lim_{t\to 0+} p_{t\lambda}(\mathrm{supp}\Gamma),$$

which exhibits $A(\Gamma)$ as a kind of asymptotic support of the family of contours $\Gamma_{t\lambda} = p_{t\lambda}(\Gamma)$.
b) $\mathrm{AC}(p_{\cdot}(\Gamma))$ = *the asymptotic cone* of $p_{\cdot}(\Gamma)$. To define it we use the particular maps p_λ given by (9), §1, fix any λ, and set

$$\mathrm{AC}(p_\lambda(\Gamma)) = \lim_{t\to 0+} tp_\lambda(\mathrm{supp}\Gamma),$$

where the right side is defined as the set of all limits of sequences $\{t_k p_\lambda(z_k)\}$ with $z_k \in \mathrm{supp}\Gamma, t_k \to 0+$, for any fixed λ. We shall see presently that this set is the same as the one in (a) and therefore independent of λ.
c) $\mathrm{AS}(I(\Gamma,\cdot))$ = *the generic asymptotic support at* ∞ of the generalized functions $I(\Gamma,\lambda)$ on $i\mathfrak{g}^*_{\mathbf{R}}$. This is defined as the union of the supports of the homogeneous distributions which occur in the expansions (7): it is the closure of the union of the orbits O for which the distribution ${}^tP_{O,k}(\Gamma_O,\lambda)\mu_O$ in 1.6.1 is non–zero for some k and λ. So $\mathrm{AS}(I(\Gamma,\cdot))$ is the union of sets $\mathrm{AS}(I(\Gamma,\lambda))$ defined in the same way for fixed λ. In fact, one evidently has $\mathrm{AS}(I(\Gamma,\cdot)) = \mathrm{AS}(I(\Gamma,\lambda))$ for λ outside of the algebraic subvariety of $\mathfrak{h}^*$ where all the terms in the asymptotic expansion corresponding to some O vanish.
d) $\mathrm{WF}(\theta(\Gamma,\cdot))$ = *the generic wave front set at 0* of the distributions $\theta(\Gamma,\lambda)$ on $\mathfrak{g}_{\mathbf{R}}$. This is defined as the union of the wave front sets $\mathrm{W}F_0(\theta(\Gamma,\lambda))$ for all λ.

Since it is needed for the proofs, we recall the definition of $\mathrm{W}F_0(\theta)$ for a distribution θ on a real vector space E : $\mathrm{W}F_0(\theta)$ is the subset of $iE^* - \{0\}$ characterized by the following property. Let $0 \neq \zeta_o \in iE^*$. Then

$$\zeta_o \notin \mathrm{WF}_0(\theta) \text{ iff for all } N, \lim_{t\to 0+} t^{-N}\langle\theta, e^{-\zeta/t}g\rangle = 0$$

$$\textit{uniformly} \text{ in } \zeta \text{ on some neighbourhood of } \zeta_o \tag{8}$$

for some $g \in C_c^\infty(E)$ with $g(0) = 1$. Because of the missing "$-i$" in our definition of the Fourier transform, $\mathrm{W}F_0(\theta)$ is here a subset of iE^*. We refer to [Hörmander, 1983] for further details.

2.4.1 **Remark**. In analogy with the definition of $\mathrm{W}F_0(\theta)$, one can give a definition of asymptotic support at ∞, denoted $\mathrm{A}S_\infty(\hat\theta)$ which makes sense for any distribution θ on a real vector space E. ($\hat{\theta}$ is the Fourier transform of θ.) Let $0 \neq \zeta_o \in iE^*$. Then

$$\zeta_o \notin \mathrm{AS}_\infty(\hat{\theta}) \text{ iff for all } N, \lim_{t\to 0+} t^{-N}\langle\theta, e^{-\zeta/t}g\rangle = 0$$

$$\textit{as distribution in } \zeta \text{ on some neighbourhood of } \zeta_o \tag{9}$$

for some $g \in C_c^\infty(E)$ with $g(0) = 1$. When $\hat{\theta}$ admits an asymptotic expansion at ∞ as in 1.6.1(b) then $\mathrm{AS}_\infty(\hat{\theta})$ is the union of the supports of the homogeneous distributions which occur. In this context it should be remembered that a homogeneous distribution is necessarily temperate [Hörmander, 1983, 7.1.18]; the proof of this fact given there also proves the assertion about the supports. It is clear from (8) and (9) that

$$\mathrm{AS}_\infty(\hat{\theta}) \subset \mathrm{WF}_0(\theta) \tag{10}$$

as mentioned in [Barbasch–Vogan, 1980, p.25].

Intrinsically, $\mathrm{AS}_\infty(\hat{\theta})$ should be viewed as subset of the sphere at infinity in the spherical completion $\mathbf{R}_+^\times\backslash[iE^* \times \mathbf{R} - (0,0)]$ of iE^*; the same is true of the other sets defined above. The condition $\zeta \neq 0$ is therefore natural, but will be ignored, here and elsewhere, to avoid trivial notational complications.

2.4.2 **Theorem**. *Let* $\Gamma \in H_{2n}(\mathcal{S})$. *The four sets* $A(\Gamma), \mathrm{AC}(p_\cdot(\Gamma)), \mathrm{AS}(I(\Gamma,\cdot)),$ *and* $\mathrm{WF}(\theta(\Gamma,\cdot))$ *are equal.*

We break down the proof into several lemmas. Throughout we fix $\Gamma \in H_{2n}(\mathcal{S})$.

2.4.3 **Lemma**. *For any* $\lambda \in \mathfrak{h}^*, \mathrm{AC}(p_\lambda(\Gamma)) = A(\Gamma)$.

Proof. We recall [1995, 3.2] that supp Γ is the closure of a union of connected components of the smooth part $\mathcal{S}_{sm}$, hence is stable under $\mathbf{R}_+^\times$. We show that for any closed $\mathbf{R}_+^\times$–stable subset S of $\mathcal{S}$ and any $\lambda \in \mathfrak{h}^*$,

$$\lim_{t\to 0+}(tp_\lambda(S)) = \lim_{t\to 0+}(p_{t\lambda}(S)) = p_0(S). \tag{11}$$

One has $tp_\lambda(u\cdot(x_o,\nu)) = u\cdot(t\lambda + t\nu)$. Thus for $t > 0$

$$tp_\lambda(S) = p_{t\lambda}(tS) = p_{t\lambda}(S)$$

from which the first equality is evident.

The inclusion $\lim_{t\to 0+}(p_{t\lambda}(S)) \supset p_0(S)$ is clear, since for $u\cdot(x_o,\nu) \in S$ one has

$$p_0(u\cdot(x_o,\nu)) = u\cdot\nu = \lim_{t\to 0+} u\cdot(t\lambda+\nu) = \lim_{t\to 0+} p_{t\lambda}(u\cdot(x_o,\nu)$$

The inclusion $\lim_{t\to 0+}(p_{t\lambda}(S)) \subset p_0(S)$ is seen as follows. An element of the left side looks like

$$\xi = \lim_{k\to\infty}(u_k\cdot(t_k\cdot\lambda + \nu_k)) \tag{12}$$

where $\{u_k \cdot (x_o, \nu_k)\}$ is a sequence in S and $t_k \to 0+$. After passing to a subsequence one may assume that $\{u_k\}$ converges, say to u_o. Then $\nu_k = u_k^{-1}[(u_k \cdot (t_k\lambda + \nu_k)) - (t_k u_k \cdot \lambda_k)]$ converges as well, say to ν_o. Hence $u_k(x_o, \nu_k) \in S$ converges to $u_o \cdot (x_o, \nu_o)$, which is in S since S is closed, so $\xi = u_o \cdot (\lambda + \nu_o)$ belongs to $p_o(S)$.

This proves the relation (11). Applied to $S = \text{supp}\Gamma$, this shows that $\text{AC}(p_\lambda(\Gamma)) = \text{AC}(p_0(\Gamma)) = A(\Gamma)$. $\square$

The following lemma is an adaptation of Lemma 8.1.7 in [Hörmander, 1983]. A similar result is Proposition 2.1 of [Howe, 1981].

2.4.4 **Lemma.** *For any* $\lambda \in \mathfrak{h}^*, \text{WF}_0(\theta(\Gamma, \lambda)) \subset \text{AC}(p_\lambda(\Gamma))$.

Proof. In the context of (8) one has

$$\langle \theta(\Gamma, \lambda), e^{-\zeta/t} g \rangle = \langle I(\Gamma, \lambda), \hat{g}(\cdot - \zeta/t) \rangle.$$

So to show that $\zeta_o \notin \text{WF}_0(\theta(\Gamma, \lambda))$ it suffices to show that there is $g \in C_c^\infty(\mathfrak{g}_{\mathbf{R}})$ with $g(0) = 1$ and a neighbourhood V of ζ_o in $i\mathfrak{g}_{\mathbf{R}}^*$ so that for $\psi = \hat{g}$ and all N we have an estimate,

$$|\langle I(\Gamma, \lambda), \psi(\cdot - \zeta) \rangle| \leq \frac{C}{1 + \|\zeta\|^N} \quad \text{if } \zeta \in V. \tag{13}$$

We show more: for any $\psi \in {}'\mathcal{H}(\mathfrak{g}^*)$ and for any closed cone $V \subset i\mathfrak{g}_{\mathbf{R}}^*$ with $\text{AC}(\Gamma, \lambda)) \cap V = \{0\}$ the inequality (13) holds for any given N with a suitable constant $C = C(\psi, V, N)$. We note that there are $\epsilon > 0$ and R so that

$$\|\xi - \zeta\| \geq \epsilon \|\xi\| \quad \text{if } \xi \in p_\lambda(\text{supp}\Gamma), \|\xi\| \geq R, \zeta \in V. \tag{14}$$

For otherwise one could choose $\zeta_k \in V$ and $\xi_k \in p_\lambda(\text{supp}\Gamma)$ so that $\xi_k/\|\xi_k\| - \zeta/\|\xi_k\| \to 0$ and $\|\xi_k\| \to \infty$. Passing to a subsequence one could arrange that $\xi_k/\|\xi_k\|$, and hence $\zeta/\|\xi_k\|$, converge, which is impossible, since the common limit would be a non–zero element of $\text{AC}(p_\lambda(\Gamma)) \cap V$.

We use the map p_λ defined by eq.(9), §1 in order to apply the formula $p_\lambda^* \sigma_\lambda = \tau_\lambda + \beta$ of 1.3.1. Let μ_λ be the measure on $\mathcal{S}$ defined by the restriction on the form $(\tau_\lambda + \beta)^n$. Since the map $p_\lambda : \mathcal{B}^* \to \Omega_\lambda$ is proper, there is N_o so that

$$\int_{\mathcal{S}} (1 + \|p_\lambda\|^{-N_o}) \mu_\lambda < \infty. \tag{15}$$

This can again be seen by using the interpretation of $\mathcal{S}$ as a conormal variety, as in the proof of Theorem 1.6.1. For any $\zeta \in V$ and $N \geq 0$, we have

$$\begin{aligned}
|\langle I(\Gamma,\lambda), \psi(\cdot - \zeta)\rangle| &= \left|\int_\Gamma \psi(p_\lambda(\cdot) - \zeta)(\tau_\lambda + \beta)^n\right| \\
&\leq \text{const.} \int_{\mathrm{supp}\Gamma} (1 + \|p_\lambda(\cdot) - \zeta\|)^{-N-N_o}\mu_\lambda \\
&= \text{const.} \int_{\mathrm{supp}\Gamma, \|p_\lambda\| \leq R} \cdots \\
&\qquad + \int_{\mathrm{supp}\Gamma, \|p_\lambda\| \geq R} +(1 + \|p_\lambda(\cdot) - \zeta\|)^{-N-N_o}\mu_\lambda \\
&\leq \text{const.} \|\zeta\|^{-N-N_o} \int_{\mathrm{supp}\Gamma, \|p_\lambda\| \leq R} \mu_\lambda \\
&\qquad + \text{const.} \|\zeta\|^{-N} \int_{\mathrm{supp}\Gamma, \|p_\lambda\| \geq R} (1 + \|p_\lambda(\cdot) - \zeta\|)^{-N_o}\mu_\lambda.
\end{aligned}$$

The first integral is finite because p_λ is proper. In the second integral we use (14) and (15). We find that the above expression is

$$\leq \text{const.} \|\zeta\|^{-N-N_o} + \text{const.} \|\zeta\|^{-N}$$

Hence (13) holds. $\square$

2.4.5 **Lemma**. $\mathrm{AS}(I(\Gamma,\cdot)) = A(\Gamma)$

Proof. Recall the expansion 1.6.1(b):

$$I(\Gamma,\lambda) = \sum_{O,k} \int_O {}^tP_{O,k}(\Gamma_O,\lambda)\mu_O \tag{16}$$

Since $\mathrm{AS}(I(\Gamma,\cdot))$ is the closure of union of O for which some${}^tP_{O,k}(\Gamma_O,\lambda)\mu_O \neq 0$ and $A(\Gamma)$ the union of the O with $\Gamma_O \neq 0$ the inclusion $\mathrm{AS}(I(\Gamma,\cdot)) \subset A(\Gamma)$ is obvious.

To prove the reverse inclusion, it suffices show that if O is a leading orbit in $A(\Gamma)$ (i.e. O is not in the closure of any other orbit in $A(\Gamma)$) then ${}^tP_{O,k}(\Gamma_O,\,\lambda)\mu_O \neq 0$ for some λ and k. Fix such an O. By definition, $\Gamma_O \in H_{2n}(\mathcal{S}_O)$ is the image of $\Gamma \in H_{2n}(\mathcal{S})$ under the natural map $H_{2n}(\mathcal{S}) \to H_{2n}(\mathcal{S}_O)$ and by [1995, 4.3.1] this map is surjective, and

$$H_{2n}(\mathcal{S}_O) = H_{2n}(\mathcal{S}_{\bar{O}})/H_{2n}(\mathcal{S}_{\partial O}). \tag{17}$$

Thus there is $\Gamma' \in H_{2n}(\mathcal{S}_{\bar{O}})$ with $\Gamma'_O = \Gamma_O$. To show that ${}^tP_{O,k}(\Gamma_O,\lambda)\mu_O \neq 0$ for some λ and k we may therefore replace Γ by Γ' in the first place, so that now $\Gamma \in H_{2n}(\mathcal{S}_{\bar{O}})$ and O is the unique leading nilpotent in $A(\Gamma)$. We now use the $W-$ action. The representation of W on the subquotient (17) of $H_{2n}(\mathcal{S}_{\bar{O}})$ decomposes as a direct sum of irreducible characters $\chi_{\nu,\phi}$ with ν representing the complex orbit containing O; the other irreducible characters in $H_{2n}(\mathcal{S}_{\bar{O}})$ are of the form $\chi_{\nu',\phi'}$ with ν' belonging to ∂O.

Arguing by contradiction, we suppose that ${}^tP_{O,k}(\Gamma_O,\,\lambda)\mu_O = 0$ for all k and λ. All non–zero terms in the expansion 1.6.1(b) belong to orbits $O' \neq O$ in $A(\Gamma) = \bar{O}$ i.e. $O' \subset \partial O$. They transform according to characters $\chi_{\nu',\phi'} \neq \chi_{\nu,\phi}$. Hence $\chi_{\nu,\phi}$ (any ϕ) does not occur in the W–module spanned by the terms of the expansions of $I(\Gamma,\cdot)$ as functions of λ. Since $I(\Gamma, w\lambda) = I(w^{-1}\Gamma, \lambda)$, by eq. (18), §1, this means that $\chi_{\nu,\phi}$ (any ϕ) does not occur in the W–module generated by Γ in $H_{2n}(\mathcal{S})$. Hence the component Γ_O of Γ in the subquotient (17) which transforms according to $\chi_{\nu,\phi}$ (various ϕ) must be zero, i.e. $\Gamma_O = 0$. This contradicts the assumption that O belongs to A(Γ). $\square$

Putting together the relations

$$\mathrm{AC}(p.(\Gamma) = A(\Gamma) = \mathrm{AS}(I(\Gamma,\cdot) \subset \mathrm{WF}(\theta(\Gamma,\cdot)) \subset \mathrm{AC}(p_{.}(\Gamma))$$

in 2.4.3, 2.4.5, (10), and 2.4.4 we get the statement of the theorem.

2.4.6 **Convention**. To have a one term for one object we shall call the set figuring in the theorem the asymptotic support of Γ and denote it by AS(Γ).

2.4.7 **Remark**. For any $\lambda \subset \mathfrak{h}^*$ we have the inclusions

$$\mathrm{AS}(I(\Gamma,\lambda)) \subset \mathrm{WF}_0(\theta(\Gamma,\lambda)) \subset \mathrm{AS}(\Gamma)$$

and we know that $\mathrm{AS}(I(\Gamma,\lambda)) = \mathrm{AS}(\Gamma)$ for generic λ (outside of a proper algebraic subvariety of $\mathfrak{h}^*$). Hence also $\mathrm{W}F_0(\theta(\Gamma,\lambda)) = \mathrm{AS}(\Gamma)$ for generic λ.

3. Wave front sets of characters and of representations

3.1 **Characters**. We shall explain the relation of the distributions $\theta(\Gamma,\lambda)$ to characters of infinite–dimensional representations. For this we now denote by $G_{\mathbf{R}}$ any connected Lie group with Lie algebra $\mathfrak{g}_{\mathbf{R}}$ and finite centre. We also fix a maximal compact subgroup K of $G_{\mathbf{R}}$. A (continuous, but not necessarily unitary) representation π of $G_{\mathbf{R}}$ on a Hilbert space H_π is *admissible* if its restriction to K decomposes with finite multiplicities, cf.[Wallach, 1988; the group $G_{\mathbf{R}}$ could be more generally a reductive group in the sense defined there]. Define a function $j^{1/2}$ on $\mathfrak{g}$ by

$$j^{1/2}(X) = \frac{\det^{1/2}\sinh \mathrm{ad}(X/2)}{\mathrm{ad}(X/2)}.$$

It is well–known that this is an entire function on $\mathfrak{g}$, in spite of the root cf. [Rossmann, 1984, p.377], for example. Its square j relates the Haar measures on $G_{\mathbf{R}}$ and $\mathfrak{g}_{\mathbf{R}}$: $d(\exp X) = j(X)dX$.
Like 2.2.1, the following theorem follows from Theorem 1.4 of [Rossmann, 1990] together with well–known facts about characters.

3.2 **Theorem** *Let π be an irreducible admissible representation of $G_{\mathbf{R}}$. For $f \in C_c^\infty(\mathfrak{g}_{\mathbf{R}})$ define*

$$\pi(f) = \int_{X \in \mathfrak{g}_{\mathbf{R}}} f(X)\pi(\exp X) j^{1/2}(X).$$

Then $\pi(f)$ is of trace class and $\mathrm{ch}(\pi) : f \to \mathrm{tr}\ \pi(f)$ is a distribution of the form

$$\mathrm{ch}(\pi) = \theta(\Gamma, \lambda_o) \tag{1}$$

for a unique $W \cdot \lambda_o \in W \backslash \mathfrak{h}^*$ and a unique $\Gamma \in H_{2n}(\mathcal{S})^{W(\lambda_o)}$.

For a given Γ and λ_o we shall call $\theta(\Gamma, \lambda_o)$ a character of $G_{\mathbf{R}}$ if it is of the formch(π). It should be noted, however, that it determines the distribution character of π only on the image of the exponential map, not necessarily on all of $G_{\mathbf{R}}$.

3.3 **The wave front set of a representation**. According to Howe [1981] the wave front set of an admissible representation π of $G_{\mathbf{R}}$, denoted WF(π), is defined as the closure of the union of the wave front sets of continuous matrix coefficients of π, i.e. functions of the form $g \to \mathrm{tr}(\pi(g)T)$ where T is a trace class operator. As explained there, WF(π) may be identified with a subset of $i\mathfrak{g}^*_{\mathbf{R}}$ (the factor i coming from our conventions). We shall need the following result of [Howe, 1981, Theorem 1.8].

3.4 **Theorem** (Howe). WF (π) *coincides with the wave–front set at the identity of the distribution character of π.*

Actually, in loc. cit. the representation is assumed to be unitary. We briefly outline the modification required in the present setting, without repeating the rest of the argument. For a unitary representation π one has the relation $\pi(\varphi)^*\pi(\varphi) = \pi(\varphi^* * \varphi)$, which is used in equation (1.32), p.128 , loc. cit. To find a replacement for it, we write out explicitly:

$$\begin{aligned} \pi(\varphi)^*\pi(\varphi) &= \int_g \int_h \bar{\varphi}(g)\varphi(h)\pi(g)^*\pi(h) dg dh \\ &= \int_g \bar{\varphi}(g)[\pi(g)^*\pi(g)] \left(\int_h \varphi(gh)\pi(h) dh \right) dg \\ &= \int_g \bar{\varphi}(g)[\pi(g)^*\pi(g)]\pi(L(g^{-1})\varphi) dg \end{aligned} \tag{2}$$

where $L(g^{-1})\varphi(h) = \varphi(g^{-1}h)$ and $\pi(\varphi) = \int \varphi(g)\pi(g)dg$.

As in loc. cit., fix a small neighbourhood V of 1 in $G_{\mathbf{R}}$ and a bounded open set $U \subset i\mathfrak{g}^*_{\mathbf{R}}$. For $\varphi \in C_c^\infty(V)$ and $\xi \in \mathfrak{g}^*$ set

$$\varphi_\xi(\exp X) = e^{\langle \xi, X \rangle}\varphi(\exp X).$$

Assume that for $\varphi \in C_c^\infty(V), \xi \in U$, and all N,

$$\lim_{t\to 0} t^{-N} \mathrm{tr}\ \pi(\varphi_{\xi/t}) = 0 \tag{3}$$

uniformly in $\xi \in U$. We have to show that in the same sense (with V replaced by a possibly smaller V_1) also

$$\lim_{t\to 0} t^{-N} \mathrm{tr}\ \pi(\varphi_{\xi/t})^* \pi(\varphi_{\xi/t}) = 0. \tag{4}$$

For $g \in V$ and $t \in \mathbf{R}$, set $g^t = \exp tX$ if $g = \exp X$. Change variables $g \to g^t$ in the integral (2) to find that the left side of (4) equals

$$\lim_{t\to 0} t^{-N+\dim\mathfrak{g}}\ \mathrm{tr} \int_g \bar{\varphi}_\xi(g^t)[\pi(g^t)^*\pi(g^t)]\pi(L(g^{-t})\varphi_{\xi/t})dg \tag{5}$$

The assumption (3) implies that without the term in brackets the limit is zero (for suitable V_1). Since the term in brackets $\to 1$ as $t \to 0$, it does not affect the limit, which is therefore still zero.
As an application of Theorem 2.4.4 we prove

3.5 **Theorem**. *Let π be an irreducible admissible representation of $G_{\mathbf{R}}$. Write its character in the form* $\mathrm{ch}(\pi) = \theta(\Gamma, \lambda_o)$. *Then* $\mathrm{WF}(\pi) = \mathrm{AS}(\Gamma)$.

Proof. The distributions $\theta(\Gamma, \lambda)$ with $\theta(\Gamma, \lambda_o) = \mathrm{ch}(\Gamma, \lambda_o)$ as in the statement of the theorem form a coherent family in the sense of representation theory (cf. [Hecht–Schmidt, 1983], for example; the distributions on the group $G_{\mathbf{R}}$ used there are pulled–back to the Lie algebra $\mathfrak{g}_{\mathbf{R}}$, and only those value of λ for which $\theta(\Gamma, \lambda)$ comes in this way from a distribution on the group are considered.) We assume that λ_o is chosen dominant with respect to the Borel subalgebra $\mathfrak{b}$ entering into the definition of p_λ. For dominant, integral $\mu \in \mathfrak{h}^*$ the distribution $\theta(\Gamma, \lambda_o + \mu)$ is then the character, as in (1), of a subquotient π' of the tensor product $\pi \otimes \Phi(\mu)$ of π with the finite–dimensional representation $\Phi(\mu)$ of highest weight μ. It follows from the definition of $\mathrm{WF}(\pi)$, recalled in 2.4 (d) above, that

$$\mathrm{WF}(\pi') \subset \mathrm{WF}(\pi \otimes \Phi(\mu)) = \mathrm{WF}(\pi).$$

By theorem 3.4, this gives $\mathrm{WF}(ch\pi') \subset \mathrm{WF}(\mathrm{ch}\pi)$, i.e.

$$\mathrm{WF}_0(\theta(\Gamma, \lambda_o + \mu)) \subset \mathrm{WF}_0(\theta(\Gamma, \lambda_o)). \tag{6}$$

From theorem 2.4.4 we know that $\mathrm{WF}_0(\theta(\Gamma, \lambda)) \subset \mathrm{AS}(\Gamma)$ for all λ, with equality for λ outside of an algebraic subvariety of $\mathfrak{h}^*$. It follows that $\mathrm{WF}_0(\theta(\Gamma, \lambda_o)) = \mathrm{AS}(\Gamma)$. $\square$
This proves a conjecture of Barbasch and Vogan [1980, p. 28].

3.5.1 It follows from (6) and 2.4.7 that $\mathrm{AS}(I(\Gamma, \lambda_o)) = \mathrm{AS}(\Gamma)$ as well.

4. K–character and leading nilpotents

4.1 **Restrictions to $\mathfrak{k}$**. We momentarily return to the generalized functions $I(\Gamma,\lambda)$ on $i\mathfrak{g}_{\mathbf{R}}^*$, without reference to characters. Let $\mathfrak{k}=\mathfrak{g}_{\mathbf{R}}\cap\mathfrak{u}$ be the maximal compact subalgebra of $\mathfrak{g}_{\mathbf{R}}$ fixed by the Cartan involution τ defining U. We recall that τ is assumed to commute with the conjugation σ with respect to $\mathfrak{g}_{\mathbf{R}}$ in $\mathfrak{g}$. We write

$$\xi^{\pm\sigma}=\frac{1}{2}(\xi\pm\sigma\xi)$$

for the projection of $\xi\in\mathfrak{g}^*$ onto the (± 1)–eigenspace of σ, and define $\xi^{\pm\tau},\xi^{\pm\sigma,\pm\tau}$ analogously. We write $Q(\xi)$ for the real quadratic form on $\mathfrak{g}^*$ induced by the Killing form of $\mathfrak{g}$ as real Lie algebra.

4.1.1 **Lemma**. *As functions of $\xi\in\mathfrak{g}^*$,*

$$|Q(\xi)|,\|\xi^\sigma\|\le c \textit{ implies } \|\xi\|^2\le a\|\xi^{-\sigma,\epsilon\tau}\|^2+b$$

with constants a,b depending only on c and given $\epsilon=\pm 1$.

Proof. This follows immediately from the orthogonality of the components $\xi^{\pm\tau,\pm\sigma}$. We omit the details. $\square$

As a consequence of this lemma we get:

4.1.2 **Lemma**. *Let $f\in C_c^\infty(\mathfrak{k})$. Let $\hat f$ be its Fourier transform, considered as function on $\mathfrak{g}^*$:*

$$\hat f(\xi)=\int_{X\in\mathfrak{k}}e^{\langle\xi,X\rangle}f(X).$$

Then $\hat f\in{}'\mathcal{H}(\mathfrak{g}^*)$.

Proof. By the Paley–Wiener theorem for $\mathfrak{k}$ we have estimates of the form 2.1 eq.(1):

$$|\hat f(\xi)|\le\frac{Ae^{B\|\xi^{\sigma,\tau}\|}}{1+\|\xi^{\sigma\tau}\|^N}.$$

In strips of the form 1.2 eq.(6) we have $\|\xi^\sigma\|\le$ const. and $|Q(\xi)|\le$ const. From the first inequality it follows that $\|\xi^{\sigma,\tau}\|\le$ const. and together with the second and 4.1.1 that further $\|\xi^{\sigma\tau}\|\ge$ const.$\|\xi\|+$const. This implies that one has in fact estimates of the form

$$|\hat f(\xi)|\le\frac{C}{1+\|\xi\|^N}$$

in these strips. $\square$

We shall need a regularity condition on the Ω_λ which depends on the real form $\mathfrak{g}_{\mathbf{R}}$. This can be formulated intrinsically in terms of Ω_λ as well as in terms of the intersections $\Omega_\lambda \cap \mathfrak{h}^*$ with σ–stable Cartan subalgebras $\mathfrak{h}$. The definition requires some preliminaries.

4.1.3 **Notation**. If $\mathfrak{h}$ is any σ–stable Cartan subalgebra of $\mathfrak{g}$, then

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{a} \tag{1}$$

denotes the unique σ–stable decomposition so that $\mathfrak{t}_{\mathbf{R}}$ is compact and$\mathfrak{a}_{\mathbf{R}}$ is split in $\mathfrak{g}_{\mathbf{R}}$. We write $\mathfrak{m}$ for the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. The imaginary roots of $\mathfrak{h}$ are those of $(\mathfrak{m}, \mathfrak{h})$; $W(\mathfrak{m}, \mathfrak{h})$ is naturally a subgroup of $W(\mathfrak{g}, \mathfrak{h})$. We denote by $W(\mathfrak{g}_{\mathbf{R}}, \mathfrak{h})$ the subgroup of $W(\mathfrak{g}, \mathfrak{h})$ realized in $G_{\mathbf{R}}$; it preserves the decomposition (1). We write $W(\mathfrak{g}_{\mathbf{R}}, \mathfrak{t})$ for the group of transformations of $\mathfrak{t}$ induced by the action of $W(\mathfrak{g}_{\mathbf{R}}, \mathfrak{h})$. We recall that any σ –stable Cartan $\mathfrak{h}$ is $G_{\mathbf{R}}$–conjugate to a σ, τ–stable one; then $W(\mathfrak{g}_{\mathbf{R}}, \mathfrak{h}) = W(\mathfrak{k}, \mathfrak{h})$ and hence $W(\mathfrak{g}_{\mathbf{R}}, \mathfrak{t}) = W(\mathfrak{k}, \mathfrak{t})$ for the corresponding groups of transformations of $\mathfrak{t}$. If several σ–stable Cartan subalgebras $\mathfrak{h}', \mathfrak{h}'', \cdots$ are under consideration, as in the following lemma, we write $\mathfrak{t}', \mathfrak{t}'', \cdots$ etc. To avoid confusion arising from the presence of several Cartan subalgebras we do not identify $G\backslash\backslash\mathfrak{g}^*$ with $W\backslash\mathfrak{h}^*$ and use the notation Ω_θ instead of Ω_λ.

4.1.4 **Lemma**. *Let $\theta \in G\backslash\backslash\mathfrak{g}^*$ be a regular element (i.e. the stabilizers of elements of Ω_θ are Cartan subalgebras). The following conditions on θ are equivalent.*
a) If $\lambda', \lambda'' \in \Omega_\theta$ have σ–stable stabilizers $\mathfrak{h}', \mathfrak{h}''$ and if $(\lambda'|\mathfrak{t}')$ and $(\lambda''|\mathfrak{t}'')$ are $G_{\mathbf{R}}$ –conjugate, then λ', λ'' are $G_{\mathbf{R}}$ –conjugate.
b) Let $\mathfrak{h}$ be any σ–stable Cartan subalgebra. If $\lambda', \lambda'' \in \Omega_\theta \cap \mathfrak{h}^*$ and if $(\lambda'|\mathfrak{t})$ and $(\lambda''|\mathfrak{t})$ are $W(\mathfrak{g}_{\mathbf{R}}, \mathfrak{t})$–conjugate, then λ' and λ'' are $W(\mathfrak{g}_{\mathbf{R}}, \mathfrak{h})$ –conjugate.
c) Let $\mathfrak{h}$ be any σ, τ–stable Cartan subalgebra. If $\lambda', \lambda'' \in \Omega_\theta \cap \mathfrak{h}^*$ and if $(\lambda'|\mathfrak{t})$ and $(\lambda''|\mathfrak{t})$ are $W(\mathfrak{k}, \mathfrak{t})$–conjugate, then λ' and λ'' are $W(\mathfrak{k}, \mathfrak{h})$–conjugate.
These elements θ form a Zariski open and dense subset of $G\backslash\backslash\mathfrak{g}^*$.

Proof. (a) $\Rightarrow$ (b) is clear and (b)$\Leftrightarrow$ (c) is standard.
(c) $\Rightarrow$ (a). Assume (c) holds. Suppose $\lambda', \lambda'' \in \Omega_\theta$ with σ–stable centralizers $\mathfrak{h}', \mathfrak{h}''$ and assume $(\lambda'|\mathfrak{t}')$ and $(\lambda''|\mathfrak{t}')$ are $G_{\mathbf{R}}$–conjugate. Since λ' and λ'' are $\mathfrak{g}$–regular, $(\lambda'|\mathfrak{t}')$ and $(\lambda''|\mathfrak{t}')$ are $\mathfrak{m}'$–regular and $\mathfrak{m}''$–regular, respectively. After conjugation by $G_{\mathbf{R}}$, we may assume that $\mathfrak{t}' = \mathfrak{t}'' =: \mathfrak{t}$. Then $\mathfrak{h}'$ and $\mathfrak{h}''$ are split Cartan subalgebras of the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$, hence conjugate by the centralizer of $\mathfrak{t}$ in $G_{\mathbf{R}}$. We may then assume that $\mathfrak{h}' = \mathfrak{h}'' = \mathfrak{h}$. We may further assume that $\mathfrak{h}$ is also τ –stable. Since $(\lambda'|\mathfrak{t}')$ and $(\lambda''|\mathfrak{t}')$ are $G_{\mathbf{R}}$–conjugate, they are in fact $W(\mathfrak{k}, \mathfrak{t})$–conjugate and (c) implies that λ' and λ'' are $W(\mathfrak{k}, \mathfrak{h})$ –conjugate as well. This shows that (a) holds. It remains to prove the last assertion. We shall use the characterization (c). Since there are only finitely many K–conjugacy classes of σ, τ–stable Cartan subalgebras, it suffices deal with only one, say $\mathfrak{h}$. It suffices to show that the $\lambda \in \mathfrak{h}^*$ satisfying

$$\text{if } w \in W(\mathfrak{g}, \mathfrak{h}) \text{ and } (w \cdot \lambda)|\mathfrak{t} = \lambda|\mathfrak{t}, \text{ then } w \in W(\mathfrak{k}, \mathfrak{h}) \tag{2}$$

are Zariski–open and dense in $\mathfrak{h}^*$: if $\Omega_\theta \cap \mathfrak{h}^*$ is of the form $W \cdot \lambda$ for such a λ, then (c) holds. (The $\lambda \in \mathfrak{h}^*$ are here not required to be regular.) That the set of $\lambda \in \mathfrak{h}^*$ satisfying (2) is open is clear; it remains to show that it is non–empty. Take in particular $\lambda \in \mathfrak{t}^*$; then equation in (2) then implies that $w \cdot \lambda = \lambda$, as one sees by considering components according to $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$. Choose $\lambda \in \mathfrak{t}^*$ so that an element of $W(\mathfrak{g}, \mathfrak{h})$ which fixes λ fixes all of $\mathfrak{t}^*$. Then w belongs to the centralizer of $\mathfrak{t}$, which contains $\mathfrak{h}$ as a split Cartan subalgebra. It follows that $w \in W(\mathfrak{g}_{\mathbf{R}}, \mathfrak{h}) = W(\mathfrak{k}, \mathfrak{h})$, as required. $\square$

4.1.5 **Definition**. An element $\theta \in G\backslash\backslash\mathfrak{g}^*$ is $\mathfrak{g}_{\mathbf{R}}$–regular if it is regular and satisfies the equivalent conditions of the lemma. We apply the same term to any representative for θ in $\mathfrak{g}^*$.

We shall need the following result concerning restrictions to $\mathfrak{k}$.

4.1.6 **Theorem**. *For any $\Gamma \in H_{2n}(\mathcal{S})$ and any $\lambda \in \mathfrak{h}^*$ the distribution $\theta(\Gamma, \lambda)$ on $\mathfrak{g}_{\mathbf{R}}$ admits a restriction to $\mathfrak{k}$, denoted $\theta_{\mathfrak{k}}(\Gamma, \lambda)$ and given by*

$$\langle \theta_{\mathfrak{k}}(\Gamma, \lambda), f \rangle = \langle I(\Gamma, \lambda), \hat{f} \rangle$$

for $f \in C_c^\infty(\mathfrak{k})$. If $\theta_{\mathfrak{k}}(\Gamma, \lambda) = 0$ for a $\mathfrak{g}_{\mathbf{R}}$–regular λ, then $\theta(\Gamma, \lambda) = 0$.

Proof. The existence of the restriction and the formula for it follow from 4.1.2 (cf. [Hörmander, 1983] for the definition of the restriction). To prove the second assertion we need to recall the basis for the $\theta(\Gamma, \lambda)$'s consisting of the standard families $\theta(C, \lambda)$, as in [Rossmann, 1990]. They are indexed by a set of representatives $\{C\}$ for the K–conjugacy classes of chambers $C \subset i\mathfrak{h}^*_{\mathbf{R}}$ cut out by the imaginary roots for the various σ, τ–stable Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$. Fix such a Cartan subalgebra $\mathfrak{h}$. Write $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ as in (1) and let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ a σ–stable parabolic subalgebra with Levi factor $\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{a})$. Let $\theta_{\mathfrak{m}}(C, \lambda)$ be the family for $\mathfrak{m}_{\mathbf{R}}$ which corresponds to the family if contours $\Gamma_{\mathfrak{m}, C, \lambda} = p_\lambda \Gamma_{\mathfrak{m}, C}$ characterized by

$$\Gamma_{\mathfrak{m}, C, \lambda} = M_{\mathbf{R}} \cdot \lambda \text{ when } \lambda \in C. \tag{3}$$

On the Cartan subalgebra $\mathfrak{t}_{\mathbf{R}}$ of $\mathfrak{m}_{\mathbf{R}}$ the distribution $\theta_{\mathfrak{m}}(C, \lambda)$ is given by Harish–Chandra's formula for the discrete series, i.e. up to a constant,

$$\theta_{\mathfrak{m}}(C, \lambda) = \frac{1}{\pi_{\mathfrak{m}}} \sum_{w \in W(\mathfrak{m} \cap \mathfrak{k}, \mathfrak{t})} \operatorname{sgn}_{\mathfrak{m}}(w) e^{w w_C \lambda}. \tag{4}$$

Here $w_C \in W(\mathfrak{g}, \mathfrak{h})$ maps the Borel subalgebra of $\mathfrak{m}$ defined by C into the Borel subalgebra of $\mathfrak{g}$ entering into the definition of p_λ. The denominator $\pi_{\mathfrak{m}}$ is the product of a system of positive roots for $(\mathfrak{m}, \mathfrak{t})$.
The standard family $\theta = \theta(C, \lambda)$ is induced from $\theta_{\mathfrak{m}} = \theta_{\mathfrak{m}}(C, \lambda)$ in the following

sense. Pull back $\theta_{\mathfrak{m}}$ to a $P_{\mathbf{R}}$ –invariant distribution $\theta_{\mathfrak{p}}$ on $\mathfrak{p}_{\mathbf{R}}$ by $\mathfrak{p}_{\mathbf{R}} \to \mathfrak{m}_{\mathbf{R}} \approx \mathfrak{p}_{\mathbf{R}}/\mathfrak{n}_{\mathbf{R}}$. Let $j_{\mathfrak{g}}/\mathfrak{p} = j_{\mathfrak{g}}/j_{\mathfrak{p}}$ with the j–functions defined as in 3.1. Then

$$\theta = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}\theta_{\mathfrak{p}} := \int_{g\in G_{\mathbf{R}}/P_{\mathbf{R}}} g\cdot (j_{\mathfrak{g}/\mathfrak{p}}^{-1/2}\theta_{\mathfrak{p}}). \tag{5}$$

In this integral $j_{\mathfrak{g}/\mathfrak{p}}^{-1/2}\theta_{\mathfrak{p}}$ is considered a distribution on $\mathfrak{g}_{\mathbf{R}}$ concentrated on $\mathfrak{p}_{\mathbf{R}}$ and $g \in G_{\mathbf{R}}$ acts by the adjoint representation. As a function of $g \in G_{\mathbf{R}}$, the integrand is then a density on $G_{\mathbf{R}}/P_{\mathbf{R}}$ so that the integral is defined. The equation (5) defines the induction of invariant distributions in general; the fact the standard families (as defined here) are obtained by this procedure follows from [Rossmann, 1984, p.377]. We shall need a lemma.

4.1.7 **Lemma**. *The restriction of the distribution $\theta = \theta(C,\lambda)$ to $\mathfrak{k}$ is*

$$\theta_{\mathfrak{k}} = \mathrm{Ind}_{\mathfrak{m}\cap\mathfrak{k}}^{\mathfrak{k}}\theta_{\mathfrak{m}\cap\mathfrak{k}} = \int_{k\in K/M\cap K} k\cdot (j_{\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k}}^{-1/2}\theta_{\mathfrak{m}\cap\mathfrak{k}}) \tag{6}$$

where $\theta_{\mathfrak{m}\cap\mathfrak{k}}$ is the restriction of $\theta_{\mathfrak{m}}$ to $\mathfrak{m}\cap\mathfrak{k}$.

Proof of 4.1.7. For $\lambda \in C \subset i\mathfrak{h}_{\mathbf{R}}^*$, the contour $\Gamma_\lambda \subset \Omega_\lambda$ which defines $\theta = \theta(C,\lambda)$ is

$$G_{\mathbf{R}}\cdot\lambda = K\cdot (M_{\mathbf{R}}\cdot\lambda + i\mathfrak{p}_{\mathbf{R}}^{\perp}),$$

hence

$$\Gamma_{C,\lambda} = K\cdot(\Gamma_{\mathfrak{m},C,\lambda} + i\mathfrak{p}_{\mathbf{R}}^{\perp}),$$

and this then holds for all regular λ. Let $f \in C_c^\infty(\mathfrak{k})$. One computes (we omit the computations of some Jacobians, cf. [Rossmann, 1984, p.378] for similar computations):

$$\begin{aligned}
\langle\theta_{\mathfrak{k}}, f\rangle &= \int_{\Gamma_{C,\lambda}} \hat{f}\sigma_\lambda^n \\
&= \int_{K\cdot(\Gamma_{\mathfrak{m},C,\lambda}+i\mathfrak{p}_{\mathbf{R}}^{\perp})} \hat{f}\sigma_\lambda^n \\
&= \int_{k\in K/M\cap K} k\cdot\left(j_{\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k}}^{-1/2} \int_{\Gamma_{\mathfrak{m},C,\lambda}} (f|_{\mathfrak{m}\cap\mathfrak{k}})\hat{}\,\sigma_{\mathfrak{m},\lambda}^{n_{\mathfrak{m}}}\right) \\
&= \int_{k\in K/M\cap K} \langle k\cdot (j_{\mathfrak{k}/\mathfrak{k}\cap\mathfrak{m}}^{-1/2}\theta_{\mathfrak{m}\cap\mathfrak{k}}), f\rangle.
\end{aligned}$$

This is just the equation (6). $\square$(4.1.7)

The distribution $\theta_{\mathfrak{m}\cap\mathfrak{k}}$ is a function which on $\mathfrak{t}_{\mathbf{R}}$ is still given by Harish–Chandra's formula (4), interpreted as in “Blattner's conjecture” (cf. [Duflo et al., 1984]). We write this out more explicitly. If $f \in C_c^\infty(\mathfrak{k})$, then

$$\langle\theta_{\mathfrak{k}}, f\rangle = \langle j_{\mathfrak{k}/\mathfrak{m}\cap\mathfrak{k}}^{-1/2}\,\theta_{\mathfrak{m}\cap\mathfrak{k}}, f^K\rangle$$

where $f \to f^K := \int k \cdot f$ is the projection onto the K–invariant functions. The restriction of f^K to $\mathfrak{t}_{\mathbf{R}}$ is then invariant under $W(\mathfrak{k}, \mathfrak{t})$ and in view of Harish–Chandra's formula (4) we get

$$\langle \theta_{\mathfrak{k}}(C, \lambda), f \rangle = \sum_{w \in W(\mathfrak{k}, \mathfrak{t})} \operatorname{sgn}_{\mathfrak{m}}(w) \langle j_{\mathfrak{k}/\mathfrak{m} \cap \mathfrak{k}}^{-1/2} \, \pi_{\mathfrak{m}}^{-1} e^{(w w_C \lambda)|\mathfrak{t}, f^K|\mathfrak{t}_{\mathbf{R}}} \rangle. \tag{7}$$

The pairing in this equation is integration over $\mathfrak{t}_{\mathbf{R}}$.
Suppose now that the $\theta_{\mathfrak{k}}(C, \lambda)$ are linearly dependent, say

$$\sum_C a_C \langle \theta_{\mathfrak{k}}(C, \lambda), f \rangle = 0 \tag{8}$$

for all $f \in C_c^\infty(\mathfrak{k})$. We first assume that this happens already with C running over the chambers in the given Cartan subalgebra $\mathfrak{h}$. It follows from (7) that then

$$\sum_C a_C \sum_{w \in W(\mathfrak{k}, \mathfrak{t})} \operatorname{sgn}_{\mathfrak{m}}(w) e^{w(w_C \lambda | \mathfrak{t})} = 0. \tag{9}$$

Assume now that λ is $\mathfrak{g}_{\mathbf{R}}$–regular for $\mathfrak{g}$. This implies that the exponents belong to distinct $W(\mathfrak{k}, \mathfrak{t})$–orbits: otherwise one gets an equation

$$w' w_{C'} \cdot \lambda = w'' w_{C''} \cdot \lambda, \text{ with } w', w'' \in W(\mathfrak{k}, \mathfrak{h})$$

which implies that $w_{C'}$ and $w_{C''}$ lie in the same coset in $W(\mathfrak{k}, \mathfrak{h}) \backslash W(\mathfrak{g}, \mathfrak{h})$, contrary to their definition. Hence $a_C = 0$ for all C.
It remains to show that the distributions $\theta_{\mathfrak{k}}(C, \lambda)$ remain independent even if one lets C run over the chambers in a whole set of representatives of the K–conjugacy classes of Cartan subalgebras $\mathfrak{h}$. For this we need another lemma.

4.1.8 **Lemma**. *Let $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ and $\mathfrak{h}' = \mathfrak{t}' + \mathfrak{a}'$ be two σ, τ–stable Cartans in $\mathfrak{g}$. If $\mathfrak{m} \cap \mathfrak{k}$ and $\mathfrak{m}' \cap \mathfrak{k}$ are K–conjugate, then $\mathfrak{h}$ and $\mathfrak{h}'$ are K–conjugate.*

Proof of 4.1.8. Recall that $\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{a})$. Thus $\mathfrak{h}_{\mathbf{R}}$ is a compact Cartan subalgebra of $\mathfrak{m}_{\mathbf{R}}$ modulo its centre and $\mathfrak{t}_{\mathbf{R}}$ is a maximal torus in $\mathfrak{m} \cap \mathfrak{k}$. Suppose $\mathfrak{m} \cap \mathfrak{k}$ and $\mathfrak{m}' \cap \mathfrak{k}$ are K–conjugate. We may as well assume that $\mathfrak{m} \cap \mathfrak{k} = \mathfrak{m}' \cap \mathfrak{k}$. Then $\mathfrak{t}_{\mathbf{R}}$ and $\mathfrak{t}'_{\mathbf{R}}$ are maximal tori in $\mathfrak{m} \cap \mathfrak{k}$, hence we may assume that $\mathfrak{t} = \mathfrak{t}'$. Both $\mathfrak{h}$ and $\mathfrak{h}'$ are then σ, τ–stable split Cartan subalgebras in the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$, hence $\mathfrak{h}$ and $\mathfrak{h}'$ are conjugate by the centralizer of $\mathfrak{t}$ in K. $\square$(4.1.8)

The distribution $\theta_{\mathfrak{k}}(C, \lambda)$ in (8) is a delta–function concentrated on $K \cdot (\mathfrak{m} \cap \mathfrak{k})$ where $\mathfrak{m}$ belongs to the Cartan $\mathfrak{h}$ containing C. It follows from 4.1.8 that already the part of the sum in (8) corresponding to a given $\mathfrak{h}$ must be zero. This finishes the proof of the theorem. $\square$

4.2 **Leading nilpotents of irreducible characters**. We now turn to characters and continue with the setup of §3. Thus π is an irreducible admissible representation of $G_{\mathbf{R}}$. We write π_K for its restriction to the maximal compact subgroup K. In analogy, we also write $\theta_K(\Gamma, \lambda)$ for $\theta_{\mathfrak{k}}(\Gamma, \lambda)$ and j_K for the j–function for $\mathfrak{k}$ (cf. 3.1).

4.2.1 **Lemma**. *Let π be an irreducible admissible representation of $G_{\mathbf{R}}$. For any $f \in C_c^\infty(\mathfrak{k})$ the operator*

$$\pi_K(f) = \int_{X \in \mathfrak{k}} f(X)\pi(\exp X) j_K^{1/2}(X)$$

is of trace class and $\mathrm{ch}(\pi_K) : f \to \mathrm{tr}\pi_K(f)$ *is the distribution on $\mathfrak{k}$ given by*

$$\mathrm{ch}(\pi_K) = j_{G/K}^{-1/2}\, \theta_K(\Gamma, \lambda_o)$$

where $\theta(\Gamma, \lambda_o) = \mathrm{ch}(\pi)$ *as in 3.2.*

Proof. This result is due to Harish–Chandra; a proof can be found in [Duflo et al., 1984, A.5] or [Barbasch–Vogan, 1980, 3.4], for example. □

4.2.2 **Theorem**. *Let* $\mathrm{ch}(\pi) = \theta(\Gamma, \lambda_o)$ *be the character of an irreducible admissible representation π of $G_{\mathbf{R}}$ as in 3.2. Then all leading nilpotents of* $\mathrm{AS}(\Gamma)$ *are contained in a single complex orbit and the leading term of the asymptotic expansion of $\theta(\Gamma, \lambda_o)$ at zero (cf. 2.3, eq (7)) takes the form*

$$\theta(\Gamma, \lambda_o) \sim \sum_O c_O(\Gamma, \lambda_o)\theta_O \tag{10}$$

where O runs over the leading nilpotents in $\mathrm{AS}(\Gamma)$. *Furthermore, θ_O is the Fourier transform of the canonical invariant measure μ_O on O, and $c_O(\Gamma, \lambda)$ is the polynomial on $\mathfrak{h}^*$ defined by*

$$c_O(\Gamma, \lambda) = \frac{1}{e!d!}\int_{\Gamma_\nu} \tau_\lambda^e. \tag{11}$$

This polynomial is the harmonic polynomial associated to the image of $\Gamma_\nu = \Gamma \cap \mathcal{B}^\nu$ in $H_{2e}(\mathcal{B})$ under Borel's isomorphism. The value $c_O(\Gamma, \lambda_o)$ of this polynomial at $\lambda = \lambda_o$ is non–zero.

4.2.3 **Remarks and explanations**. a) Recall that the leading nilpotents in $\mathrm{AS}(\Gamma)$ are the $G_{\mathbf{R}}$–orbits in $\mathrm{AS}(\Gamma)$ which are not contained in the closure of any other orbit in $\mathrm{AS}(\Gamma)$. As mentioned in the introduction, the assertion of the theorem is stronger than the same assertion with "leading nilpotents of $\mathrm{AS}(\Gamma)$" replaced by "nilpotent orbits of maximal dimension in $\mathrm{AS}(\Gamma)$": the theorem identifies the leading terms of the series (16) in 1.6.1 relative to the closure order on the O's, not just relative to the dimension order (or equivalently homogeneity degree).

b) In the formula for $c_O(\Gamma,\lambda)$ we take $\nu \in O$ and set $\Gamma_\nu = \Gamma_O \cap \mathcal{B}^\nu \in H_{2e}(\mathcal{B}^\nu)^{A_{\nu,\mathbf{R}}}$ (cf. [1995, 3.3]). We have also set $e = \dim_{\mathbf{C}} \mathcal{B}^\nu$, as usual.
c) The form τ_λ on $\mathcal{B}$ is the one defined in 1.3. Borel's isomorphism associates to each $\gamma \in H_{.}(\mathcal{B})$ the polynomial c_γ on $\mathfrak{h}^*$ given by

$$c_\gamma(\lambda) = \int_\gamma e^{(-1/2\pi i)\tau_\lambda}.$$

See [Rossmann, 1991(II), p.170] for more details and references. In the theorem we have omitted the factor $(-1/2\pi i)$, which could have been incorporated in the definition of the integrals $I(\Gamma,\lambda)$.

Proof. Let I_π be the ideal of the universal enveloping algebra $U(\mathfrak{g})$ which annihilates π. Then the variety $V(\mathrm{gr} I_\pi) \subset \mathfrak{g}^*$ defined by the graded ideal $\mathrm{gr} I_\pi \subset \mathrm{gr} U(\mathfrak{g}) = \mathbf{C}[\mathfrak{g}^*]$ is contained in the closure of a single complex orbit $\mathcal{O}$: this follows from the irreducibility of Joseph's Weyl group representations [Joseph, 1980], as shown in [Borho–Brylinsky, 1982]. It follows from [Barbasch–Vogan, 1979, Theorem 4.1] or [Howe, 1982, proof of Proposition 1.2] that $\mathrm{WF}_0(ch(\pi)) \subset V(\mathrm{gr} I_\pi)$. To show that all of the leading nilpotents in $\mathrm{AS}(\Gamma)$ are contained in the complex orbit $\mathcal{O}$ it suffices to show that any leading nilpotent O in $\mathrm{WF}_0(\mathrm{ch}(\pi))$ satisfies

$$\dim_{\mathbf{R}} O = \dim_{\mathbf{C}} \mathcal{O}. \tag{12}$$

We shall prove this using a result of Vogan [1978, Theorem 1.2], the results of §2 and §3, and an adaptation of an argument in [Barbasch–Vogan, 1980].
The irreducible representations ρ_μ of K can be indexed by elements μ of a lattice in $\mathfrak{h}^* \cap i\mathfrak{k}^*$ (assuming that $\mathfrak{h} \cap \mathfrak{k}$ is a maximal torus in $\mathfrak{k}$) so that the character $\chi_\mu = \mathrm{ch}(\rho_\mu)$ of the corresponding representation ρ_μ of K is given by a formula as in Theorem 3.2. To be precise, for $f \in C_c^\infty(\mathfrak{k})$ set

$$\rho_\mu(f) = \int_{X\in\mathfrak{k}} f(X)\rho_\mu(\exp X) j_K^{1/2}(X).$$

Then

$$\langle \chi_\mu, f\rangle := \mathrm{tr}\rho_\mu(f) = \int_{K\cdot\mu} \hat{f}\sigma_\mu^m \tag{13}$$

where σ_μ is the canonical two–form on $K\cdot\mu, m = \dim_{\mathbf{R}} K\cdot\mu$. Write $[\pi_K : \chi_\mu]$ for the multiplicity of χ_μ in π_K. The result of Vogan mentioned implies that there is $C \neq 0$ so that for t sufficiently large

$$\sum_{\mu:\|\mu\|\le t^{-1}} [\pi_K : \chi_\mu]\chi_\mu(1) \ge Ct^{-d} \tag{14}$$

with $d = (1/2)\dim_{\mathbf{C}} V(I_\pi){=}(1/2)\dim_{\mathbf{C}} \mathcal{O}$. (The fact that the d in (14) is (1/2) $\dim_{\mathbf{C}}$ $(V(I_\pi))$ follows from [Joseph, 1978].)

Let $f \in C_c^\infty(\mathfrak{k})^K$ be a K–invariant function. It follows from (10) that apart from a positive constant,

$$\langle \chi_\mu, f\rangle = \hat{f}(\mu)\chi_\mu(1).$$

Hence

$$\begin{aligned}\langle \mathrm{ch}(\pi_K), f\rangle &= \sum_\mu [\pi_K : \chi_\mu]\langle \chi_\mu, f\rangle \\ &= \sum_\mu [\pi_K : \chi_\mu]\chi_\mu(1)\hat{f}(\mu)\end{aligned}$$

Assume now that

$$\hat{f}(0) \neq 0, \tag{15}$$

say $\mathrm{Re}f(0) > 0$. Then there is $\epsilon > 0$ and $r > 0$ so that $\mathrm{Re}\hat{f}(\xi) \geq \epsilon$ for $\|\xi\| \leq r$. Write

$$f_t(\cdot) = t^{-\dim \mathfrak{k}} f(t^{-1}\cdot),$$

so that $(f_t)\hat{} = \hat{f}(t\cdot)$. We get

$$\begin{aligned}\mathrm{Re}\langle \mathrm{ch}(\pi_K), f_t\rangle &= \sum_\mu [\pi_K : \chi_\mu]\chi_\mu(1)\mathrm{Re}\hat{f}(t\cdot\mu) \\ &\geq \epsilon \sum_{\|t\mu\|\leq r} [\pi_K : \chi_\mu]\chi_\mu(1) \\ &\geq Ct^{-d}\end{aligned}$$

for some non–zero constant C. Thus

$$\langle \mathrm{ch}(\pi_K), f_t\rangle = Ct^{-d} + O(t^{-d+1}) \tag{16}$$

as $t \to 0$. Lemma 4.1.7 implies that this holds also with $\mathrm{ch}(\pi_K)$ replaced by $\theta(\Gamma, \lambda_o)$ because the factor j/j_K takes the value 1 at zero, hence does not affect the first term of the asymptotic expansion. We shall need the following assertion.

Let $f \in C_c^\infty(\mathfrak{k})^K$ with $f(0) \neq 0$. There is a homogeneous polynomial $C(\lambda)$ in λ with $C(\lambda_o) \neq 0$ so that for any $\lambda \in \mathfrak{h}^*$,

$$\langle I(\Gamma, \lambda), \hat{f}(t\cdot)\rangle = C(\lambda)t^{-d} + O(t^{-d+1}) \tag{17}$$

That this holds for $\lambda = \lambda_o$ follows from (16). We compare (17) with the expansion of Theorem 1.6.1. Since we already know that $\mathrm{AS}(\Gamma)$ is contained in the closure of $\mathcal{O}$, the exponent $d = (1/2)\dim_{\mathbf{C}}(\mathcal{O})$ occurring in (17) for $\lambda = \lambda_o$ is the maximal such exponent which can occur for any λ. The assertion (17) follows.
We claim that (12) will follow from the following statement.

If O is a leading nilpotent of $A := AS(\Gamma)$, then there is $f \in C_c^\infty(\mathfrak{k})$ so that

$$\langle \hat{f}(0) \neq 0, \text{ but } \langle I(\Gamma', \cdot), \hat{f}\rangle = 0 \text{ for } \Gamma' \in {}'H_{2n}(\mathcal{S}_{A-O}). \tag{18}$$

Indeed, if such an f exists, then it may be chosen to be K–invariant, by the K–invariance the condition (18) on f, so that (17) applies. On the other hand, the second condition in (18) implies that the only leading nilpotent which contributes to the expansion of the left side in (17) for such an f is O. By 1.6.1 this expansion then looks like

$$\langle I(\Gamma, \lambda), f(t\cdot)\rangle = C_O(\lambda)t^{-d_O} + O(t^{-d_O+1}) \tag{19}$$

where

$$C_O(\lambda) = c_O(\Gamma, \lambda)\langle \mu_O, \hat{f}\rangle$$

with $c_O(\Gamma, \lambda)$ and μ_O as stated in the theorem. Here $d_O = (1/2)\dim_{\mathbf{R}} O$ and $e_O + d_O = n$ as usual. Since $C(\cdot) \neq 0$, (17) implies that $-d = -d_O + k$ for some $k \geq 0$, since the degrees of all terms belonging to O in the expansion have this property, cf. 1.6.1. Hence $d \leq d_O$. Since we already know that O is in the closure of $\mathcal{O}$ this gives $d = d_O$, which gives (12).
We now prove (18). For this we momentarily fix a $\mathfrak{g}_{\mathbf{R}}$–regular $\lambda \in \mathfrak{h}^*$ and consider the map

$$\theta_{\mathfrak{k}}(\cdot, \lambda) : H_{2n}(\mathcal{S}) \to C_c^\infty(\mathfrak{k})^* \tag{20}$$

given by Theorem 4.1.7. It follows from that theorem that (20) is injective. The same is then true for the induced map

$$\theta_{\mathfrak{k}}(\cdot, \lambda) : H_{2n}(\mathcal{S}_B) \overset{\subseteq}{\to} C_c^\infty(\mathfrak{k})^* \tag{21}$$

for any closed subset $B \in G_{\mathbf{R}}\backslash\mathcal{N}_{\mathbf{R}}$, because the natural map $H_{2n}(\mathcal{S}_B) \to H_{2n}(\mathcal{S})$ is injective for such B, by [1995, 4.3.1]. This applies in particular to $B = A - O$.
By way of contradiction, suppose that (18) is false. This means that in $C_c^\infty(\mathfrak{k})$ we have

$$\theta_{\mathfrak{k}}(H_{2n}(\mathcal{S}_B), \lambda)^\perp \subset \delta^\perp$$

where $\delta \in C_c^\infty(\mathfrak{k})^*$ is the functional $f \to f(0)$ and the left side is the orthogonal to the image of (21) with $B = A - O$. It follows that $\delta \in \theta_{\mathfrak{k}}(H_{2n}(\mathcal{S}_B))$, so that there is $\Delta \in H_{2n}(\mathcal{S}_{A-O})$ satisfying

$$\theta_{\mathfrak{k}}(\Delta, \lambda) = \delta \tag{22}$$

as distribution on $\mathfrak{k}$. Compare this equation with the formula (7) for the basis $\theta_{\mathfrak{k}}(C, \lambda)$ of these distributions: the support of $\theta_{\mathfrak{k}}(C, \lambda)$ is contained in the K–orbit of $\mathfrak{m} \cap \mathfrak{k}$ and these supports are filtered according to the inclusion relation among the $\mathfrak{m} \cap \mathfrak{k} = C_{\mathfrak{k}}(\mathfrak{a})$. It follows that there can be an equation such as (22) only if there is an $\mathfrak{m}$ with $\mathfrak{m} \cap \mathfrak{k} = \{0\}$, i.e. the corresponding Cartan subalgebra $\mathfrak{h}$ has $\mathfrak{t} = \{0\}$, i.e. is split for $\mathfrak{g}_{\mathbf{R}}$. The parabolic subalgebra $\mathfrak{p}$ is (8) is then a split Borel subalgebra and the distribution $\theta(C, \lambda)$ in (8) is of the form $\theta(\Gamma_1, \lambda)$ where Γ_1 is the oriented conormal variety of the closed $G_{\mathbf{R}}$–orbit on $\mathcal{B}$. (This follows from [Rossmann, 1984].) But then $\mathrm{A}(\Gamma_1) = \mathcal{N}_{\mathbf{R}}$ so Γ_1 is not contained in $H_{2n}(\mathcal{S}_B)$ for any proper closed subset B of $G_{\mathbf{R}} \backslash \mathcal{N}_{\mathbf{R}}$. This contradiction proves that (18) holds. This completes the proof of (12) and thereby the first part of the theorem, including the expansion (10). The fact that the term $c_O(\Gamma, \lambda_o)\theta_O$ in this formula has the stated properties was shown during the proof as well. $\square$

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