PICARD-LEFSCHETZ THEORY FOR THE COADJOINT QUOTIENT OF A SEMISIMPLE LIE ALGEBRA

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Abstract. The paper develops a Picard-Lefschetz theory for the coadjoint quotient of a semisimple Lie algebra and analyzes the resulting monodromy representation of the Weyl group.

Introduction

The coadjoint quotient of a complex semisimple Lie algebra $\mathfrak{g}$ is the quotient map in the sense of algebraic geometry,

$$q : \mathfrak{g}^* \to G\backslash\backslash\mathfrak{g}^*,$$

for the action of $G = \mathrm{Ad}(\mathfrak{g})$. Its base $G\backslash\backslash\mathfrak{g}^*$ is an affine space; as a set, it can be identified with the quotient $W\backslash\mathfrak{h}^*$ where $\mathfrak{h}$ is a Cartan subalgebra and W its Weyl group. For regular values θ of the map q the fibres $\Omega_\theta = q^{-1}(\theta)$ are single G–orbits; as $\theta \to 0$ they degenerate into the nilpotent cone $\Omega_0 = \mathcal{N}$. From Lefschetz's topological point of view one is interested in the behaviour of cycles Γ_θ on the fibres Ω_θ as $\theta = \theta(t)$ varies along a curve in the regular set and the $\Gamma_{\theta(t)}$ are taken along by continuous deformation, i.e. by isotopic transport in the fibration defined by q over its set of regular values. In homology, the isotopic transport gives the Gauss–Manin transport of homology classes in the fibres; by transport around loops it gives the monodromy representation of the fundamental group of the set of regular values in the homology of the fibre over the base–point. From Picard's analytic point of view, one is interested in the behaviour of integrals

$$I(\theta) = \int_{\Gamma_\theta} \varpi_\theta$$

of holomorphic forms ϖ_θ of top degree $\dim_{\mathbf{C}} \Omega_\theta$ over cycles Γ_θ of this dimension, especially in the asymptotic behaviour of the $I(\theta)$ as θ approaches a singular value and in the ramification of $I(\theta)$ as θ describes a loop in the regular set. Both points of view are of interest for the coadjoint quotient: the topological point of view leads to Springer's theory of Weyl group representations [Springer, 1976, 1978, 1993], the analytic point of view to the character theory of infinite dimensional

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representations of real forms of G [Rossmann, 1984, 1990, 1991]. In this paper we shall be concerned with the topological and geometric aspects of the theory; the analytic aspects, which require very different methods, follow in a sequel.

The classical theory of Picard and Lefschetz is concerned with linear systems on a projective variety, not on an affine variety such as $\mathfrak{g}^*$. This situation can be created by introducing a suitable projective completion $\mathfrak{g}^{*c}$ of $\mathfrak{g}^*$. The family of varieties $\{\Omega_\theta\}$ is viewed as a linear system of dimension equal to the rank of $\mathfrak{g}$, cut out from the graph $\{(\xi,\theta) : \theta = q(\xi)\} \approx \mathfrak{g}^*$ by the linear subspaces $\theta =$ const. It turns out that the closures ${\Omega^c}_\lambda$ of the fibres Ω_λ all have the same part at infinity, say $F = {\Omega^c}_\lambda - \Omega_\lambda$ (isomorphic with the projectivized nilpotent cone Ω_0); in classical terminology, this means that F belongs to the fixed locus of the linear system $\{{\Omega^c}_\lambda\}$. One can therefore require that F remain fixed under the isotopic transport. As a consequence, the monodromy representation can be realized not only in the homology $H_{\cdot}(\Omega)$ with compact supports, but also in the homology with respect to other families of supports. This is essential, both for the topological theory involving Springer's representations and for the analytic theory involving characters.

In overview, the paper is organized as follows. In §1 we discuss some general concepts from Picard–Lefschetz theory and give a construction of *restricted monodromy representations* based on a simple general principle (cf. 1.1). In §2 we introduce the coadjoint quotient, discuss some of its properties, and begin the study of the homology of the standard fibre. This leads to an analysis of the top homology of some rather peculiar varieties, which is carried out in §3 and is summarized in a decomposition theorem (Theorem 3.6). These results are applied to the coadjoint quotient in §4−5, as we now explain in more detail.

Fix a standard fibre Ω, homeomorphic with Ω_λ for regular λ. One can choose such homeomorphisms $p_\lambda : \Omega \to \Omega_\lambda, \lambda \in \mathfrak{h}^*_{\text{reg}}$, which depend continuously on λ and approach a *limit map* $p_0 : \Omega \to \Omega_0 =: \mathcal{N}$ as $\lambda \to 0$. This is the *Springer resolution* of the nilpotent variety $\mathcal{N}$. The fibre $p_0^{-1}(\nu)$ over $\nu \in \mathcal{N}$ is the *Springer variety* $\mathcal{B}^\nu$.

The monodromy representation of the fibration $\mathfrak{g}^*_{\text{reg}} \to W\backslash\mathfrak{h}^*_{\text{reg}}$ can be "restricted" from Ω to $p_0^{-1}(\nu) = \mathcal{B}^\nu$ (in a sense to be clarified) to produce a representation of W on $H_{\cdot}(\mathcal{B}^\nu)$, equivalent to Springer's representation [Springer, 1978]. According to Springer's theory, all irreducible characters $\chi_{\nu,\phi}$ of W are realized in the subspaces of the top homology $H_{2e}(\mathcal{B}^\nu)$ which transform by a given character ϕ of the component group $A_\nu = G_\nu\ /G^o_\nu$ of the stabilizer of ν in G.

Let $\mathfrak{g}_o$ be the real subalgebra of $\mathfrak{g}$ fixed by an involutive automorphism of $\mathfrak{g}$ as real Lie algebra, G_o the corresponding subgroup of G. As a family of supports for the homology on Ω_λ we take the closed subsets contained in strips of the from

$$\|R_{\mathfrak{g}_o}\xi\| \leq \text{const.}$$

where $R_{\mathfrak{g}_o} : \mathfrak{g}^* \to \mathfrak{g}^*_o$ is the natural map, $\mathfrak{g}^*$ being identified with the real dual of $\mathfrak{g}$. The corresponding homology group will be denoted $'H_{\cdot}(\Omega_\lambda)$.

If the family of cycles $\{\Gamma_\lambda\}$ is locally constant for the Gauss–Manin connection, then $\Gamma_\lambda = p_\lambda\Gamma$ for a fixed cycle $\Gamma \in {}'H_{2n}(\Omega)$. As $\lambda \to 0, p_\lambda(\text{supp}\Gamma)$ approaches $p_0(\text{supp}\Gamma)$, which is subset of $\mathcal{N}_o := \mathcal{N} \cap \mathfrak{g}_o^\perp$ stable under G_o. This leads to a filtration of $'H_{2n}(\Omega)$ by closed subsets $A \subset G_o\backslash\mathcal{N}_o$ (i.e. G_o–stable closed subsets of

$\mathcal{N}_o$: $'H_{2n}(\Omega)_A$ consists of the Γ with $p_0(\mathrm{supp}\Gamma) \subset A$. One can describe this situation in a somewhat different way by introducing the inverse image $\mathcal{S} = p_0^{-1}(\mathcal{N}_o)$ of $\mathcal{N}_o$ under the map $\Omega \to \mathcal{N}$, and more generally $\mathcal{S}_A = p_0^{-1}(A)$ for each $A \subset G_o\backslash\mathcal{N}_o$. Then it turns out that $'H_{2n}(\Omega) \approx H_{2n}(\mathcal{S})$ and $'H_{2n}(\Omega)_A \approx H_{2n}(\mathcal{S}_A\ )$, homology with arbitrary supports on $\mathcal{S}$. Thus the monodromy representation of W on $'H_{2n}(\Omega)$ becomes a representation on $H_{2n}(\mathcal{S})$. The closure relation among the A's induce a filtration on $H_{2n}(\mathcal{S})$, whose graded group will be denoted $\mathrm{gr}H_{2n}(\mathcal{S})$. We prove:

Theorem A. *a) $H_{2n}(\mathcal{S}_A)$ is a W–stable subspace of $H_{2n}(\mathcal{S})$ for any closed $A \subset G_o\backslash\mathcal{N}_o$.*
b) There is a natural W–isomorphism

$$\mathrm{gr}H_{2n}(\mathcal{S}) \approx \sum_{\nu} H_{2e}(\mathcal{B}^{\nu})^{A_{o,\nu}}.$$

where ν runs over a set of representatives for $G_o\backslash\mathcal{N}_o$, and $A_{\nu,o}$ denotes the image of $G_{o,\nu}$ in A_ν.

A centrepiece of classical Picard–Lefschetz theory is the Picard–Lefschetz Theorem. It gives a formula for the monodromy of an isolated quadratic singularity; for even fibre dimension it says that the generator acts as a reflection along the vanishing cycle. In §5 we prove an analogous result for the monodromy action of a simple reflection $s \in W$ on $H_{2n}(\mathcal{S})$. To state it, let $\lambda_o \in \mathfrak{h}^*$ be fixed by s and no other reflection. For $\lambda = \lambda_o$ the map $p_\lambda : \Omega \to \Omega_\lambda$ mentioned above becomes a map $p_o : \Omega \to \Omega_o$, the limit map at λ_o. The variety Ω_o contains the orbit O_o of λ_o and we set $\mathcal{S}_o = p_o^{-1}(O_o) \cap \mathcal{S}$. The subvariety $\mathcal{S}_o$ replaces the classical vanishing cycle in the following sense.

Theorem B. *The monodromy action of s in $H_{2n}(\mathcal{S})$ is a reflection along the subspace $H_{2n}(\mathcal{S}_o)$ of $H_{2n}(\mathcal{S})$.*

The result we prove is actually somewhat more precise (cf. Theorem 5.8). The proof uses a reduction to $\mathbf{sl}(2,\mathbf{C})$ where the Picard–Lefschetz Theorem applies in its classical form. This is the basic case, which should be kept in mind throughout; see Example 4.12, especially the pictures.

If Theorem A is applied to the case when $(\mathfrak{g}, \mathfrak{g}_o)$ is of the form $(\mathfrak{g}\times\mathfrak{g}, \mathrm{diag}(\mathfrak{g}))$, we recover Springer's result that every irreducible representation of W is realized on an A_ν– isotypic component of $H_{2e}(\mathcal{B}^\nu)$. In general, the subalgebras of $\mathfrak{g}$ of the type $\mathfrak{g}_o$ fall naturally into pairs $\mathfrak{g}_+$, $\mathfrak{g}_-$ so that $\mathfrak{g}_+ \cap \mathfrak{g}_-$ is a maximal compact subalgebra in either. Fix such a pair $\mathfrak{g}_\pm$ and write $\mathcal{S}_\pm$ for the corresponding varieties $\mathcal{S}$. As a consequence of the Theorem A one finds that

$$H_{2n}(\mathcal{S}_+) \approx H_{2n}(\mathcal{S}_-)$$

as W–modules. This seems a rather remarkable fact, since the varieties $\mathcal{S}_+$ and $\mathcal{S}_-$ look superficially quite different; it is an aspect of the well–known "$K_\mathbf{C} - G_\mathbf{R}$–duality" in the structure theory and representation theory of semisimple groups.

That such a result should exist was pointed out to me by T.A. Springer, who proved a theorem on $H_{2n}(\mathcal{S})$ in an algebraic setting [Springer, 1993], which is in the above sense dual to a result of [Rossmann, 1990]. I thank him for explaining this to me.

1. Some concepts from Picard–Lefschetz theory

1.1 **A general principle**. Let $q : M \to Q$ be a surjective, holomorphic map between complex manifolds which is a topologically trivial fibration, locally over its set of regular values, say $q : M_r \to Q_r$. Picard–Lefschetz theory is concerned with the monodromy of this fibration locally around a critical value θ_o. It is the purpose of this section to make precise the following heuristic principle about this monodromy.

If the deformation of the generic fibre $q^{-1}(\theta_1)$ into the critical fibre $q^{-1}(\theta_o)$ can be described by a limit map $p_0 : q^{-1}(\theta_1) \to q^{-1}(\theta_o)$, *defined at best up to homotopy, then this map is invariant under the monodromy around θ_o.*

This means that after isotopic transport along a small loop around θ_0 in Q_r a fibre $p_0^{-1}(y) \subset q^{-1}(\theta_1)$ of the limit map returns to a neighbourhood of its initial location, but possibly in another position. The change in position corresponds to a transformation of $p_0^{-1}(y)$, defined up to homotopy, which induces a monodromy transformation restricted to cycles in $p_0^{-1}(y)$. These cycles are vanishing cycles in the sense of Lefschetz [1924]: their image under the limit map is the zero cycle. The situation is illustrated in Fig. (4.3) and (5.1).

We shall use this principle only in a very special case, when $q : M \to Q$ becomes a topologically trivial fibration $\tilde{q} : \tilde{M} \to \tilde{Q}$ over a ramified covering $\tilde{Q}$ of Q in which θ_o is covered by a single point λ_o. For technical reasons we shall require that M and $\tilde{M}$ admit fibrewise completions M^c and $\tilde{M}^c$. The complex structure is not required at this point. The precise formulation we shall use can be stated as follows. (The definitions of non–standard terms will be given below.)

1.1.1 **Theorem**. *Let*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\rho} & M \\ \tilde{q}\downarrow & & \downarrow q \\ \tilde{Q} & \xrightarrow{\pi} & Q \end{array}$$

be a commutative diagram of morphisms of real analytic manifolds. Assume there are embeddings $M \subset M^c$ and $\tilde{M} \subset \tilde{M}^c$ as strata semi–analytic sets so that $q, \tilde{q}$, and ρ extend to M^c and $\tilde{M}^c$. Fix $\theta_o \in Q$ and $\lambda_o \in \tilde{Q}$ and assume that

a) $\pi : \tilde{Q} \to Q$ is a ramified covering with $q^{-1}(\theta_o) = \{\lambda_o\}$ as point of total ramification, regular on $\tilde{Q}_r \to Q_r$,
b) $\rho : \tilde{M} \to M$ is proper and induces homeomorphisms $\tilde{q}^{-1}(\lambda) \xrightarrow{\approx} q^{-1}(\theta)$ of the fibres over $\lambda \in \tilde{Q}_r$ and $\theta = \pi(\lambda) \in Q_r$,
c) $\tilde{q} : \tilde{M}^c \to \tilde{Q}$ is proper and its restrictions to the strata of $\tilde{M}^c$ are surjective submersions.

Then
A) $\tilde{q} : \tilde{M} \to \tilde{Q}$ is a topologically trivial fibration over a neighbourhood of λ_o in $\tilde{Q}$,
B) the maps $p_\lambda : \tilde{q}^{-1}(\lambda_o) \to \tilde{q}^{-1}(\lambda) \approx q^{-1}(\pi(\lambda)), \lambda \in \tilde{Q}_r$, defined by a local trivialization have $\rho : \tilde{q}^{-1}(\lambda_o) \to q^{-1}(\theta_o)$ as limit map,
C) for any semi–analytic subset V of $q^{-1}(\theta_o)$, the isotopic monodromy representation of the covering group W of $\tilde{Q}_r \to Q_r$ admits a restriction to $\rho^{-1}(V)$, yielding a homomorphism of the covering group W into the group of homotopy equivalences of $\rho^{-1}(V)$.

1.1.2. The case we have in mind is the coadjoint quotient of a semisimple complex Lie algebra, defined in §2. In that case M and $\tilde{M}$ are smooth complex varieties, M^c and $\tilde{M}^c$ can be taken to be complex varieties as well, and $\tilde{M}^c$ is also smooth. These constructs will be given explicity, but it is clearer to place oneself in the more general situation of the theorem.

In some form, results of this kind go back to the beginnings of Picard–Lefschetz theory. We outline the proof of the version above in (1.2–1.11) in order to introduce the constructions and definitions needed later. The the proofs of various auxiliary lemmas are easy and will be omitted.

1.2 **Trivializations**. Let $\pi : Y \to L$ be any surjective continuous map of topological spaces, assumed to be locally compact and paracompact metric spaces. Recall that $\pi : Y \to L$ is called a *(globally) trivial fibration* if $(Y, L; \pi)$ is homeomorphic with a product $(L \times F, L; \text{proj})$ and is called a *locally trivial fibration* if it is trivial over some neighbourhood of any given point of L. Recall further:

1.2.1 *A locally trivial fibration over a contractible base is globally trivial.*

We shall make use of the following simple observation. A trivialization $Y \approx L \times F$ of π induces homeomorphisms $F \approx \pi^{-1}(\lambda)$ of the fibers of π with a standard fibre F. These homeomorphisms depend on the trivialization, but their homotopy class does not, in the following sense.

1.2.4 **Lemma**. *Let $\pi : Y \to L$ be a continuous map. Suppose π admits a global trivialization $F \times L \xrightarrow{\approx} Y$ and let*

$$p_\lambda : F \to F_\lambda \tag{1}$$

be the corresponding family of homeomorphisms of the standard fibre F with the fibres $F_\lambda = \pi^{-1}(\lambda)$. If L is path–connected, then the homotopy class of the map

$$p_\mu \circ p_\lambda^{-1} : F_\lambda \to F_\mu \tag{2}$$

is independent of the particular trivialization. $\square$

1.3 **Notation and hypotheses**. We now fix a continuous surjective map $q : M \to Q$. We assume that there is an open subset Q_r of Q, $M_r = q^{-1}(Q_r)$ so that $q : M_r \to Q_r$ is a locally trivial fibration. Fix a point in Q, which will be denoted 0, and assume that Q_r and $Q_r \cup \{0\}$ are path–connected. For any $\theta \in Q$ we denote by Ω_θ the fibre $q^{-1}(\theta)$. In particular $\Omega_0 = q^{-1}(0)$.

1.4 **Isotopic transport and isotopic monodromy**. Let $\gamma : \theta = \theta(t)$, $0 \le t \le 1$, be a path in Q_r. A global trivialization of the fibration $M_r \to Q_r$ over γ (which exists, since a locally trivial fibration over an interval is trivial) gives family of homeomorphisms $\Omega_{\theta(0)} \to \Omega_{\theta(t)}$ and in particular a homeomorphism

$$p_\gamma : \Omega_{\theta(0)} \to \Omega_{\theta(1)} \tag{3}$$

which we call *isotopic transport* along γ. The following lemma follows from 1.2.4 with L an interval or a square.

1.4.1 **Lemma**. *The homotopy class of p_γ is uniquely determined by γ and depends only on the homotopy class of γ.* $\square$

For fixed $\theta_1 \in Q_r$, the isotopic transport around loops in Q_r is a homomorphism of $\pi_1(Q_r, \theta_1)$ into the group of homotopy equivalences of Ω_{θ_1}, which we shall call the *isotopic monodromy representation.* It induces a group representation in the usual sense in the homology groups of Ω_{θ_1}.

1.5 **Coverings**. We now suppose that $M_r \xrightarrow{q} Q_r$ becomes globally trivial topologically after passing to a covering. This means that there is a commutative diagram of continuous maps

$$\begin{array}{ccc} \tilde{M}_r & \xrightarrow{\rho} & M_r \\ \tilde{q}\downarrow & & \downarrow q \\ \tilde{Q}_r & \xrightarrow{\pi} & Q_r \end{array} \tag{4}$$

with the following properties.

a) The map $\tilde{q} : \tilde{M}_r \to \tilde{Q}_r$ is a topologically trivial fibration.
b) The map $\pi : \tilde{Q}_r \to Q_r$ is a covering.
c) The map $\rho : \tilde{M}_r \to M_r$ induces a homeomorphism $\tilde{q}^{-1}(\lambda) \xrightarrow{\approx} q^{-1}(\theta)$ if $\theta = \pi(\lambda)$.

Condition (a) means that there is a topological space Ω and a homeomorphism

$$\tilde{Q}_r \times \Omega \xrightarrow{\approx} \tilde{M}_r \tag{5}$$

so that $\tilde{M}_r \to \tilde{Q}_r$ becomes the projection $\tilde{Q}_r \times \Omega \to \tilde{Q}_r$. The space Ω will be called the *standard fibre* and the homeomorphism (5) will be written in the form $(\lambda, z) \to \tilde{p}_\lambda(z)$. For $\theta \in Q$ we write $\Omega_\theta = q^{-1}(\theta)$ as above, and for $\lambda \in \tilde{Q}_r$ we write $\tilde{\Omega}_\lambda = \tilde{q}^{-1}(\lambda)$. Then $\tilde{p}_\lambda$ is a homeomorphism

$$\tilde{p}_\lambda : \Omega \xrightarrow{\approx} \tilde{\Omega}_\lambda. \tag{6}$$

It will be convenient to set $\Omega_\lambda = \Omega_\theta$ if $\theta = \pi(\lambda)$. For $\lambda \in \tilde{Q}_r$, the map $p_\lambda := \rho \circ \tilde{p}_\lambda$ is a homeomorphism

$$p_\lambda : \Omega \to \Omega_\lambda. \tag{7}$$

1.6 If Q_r has a contractible covering space $\tilde{Q}_r \to Q_r$, then the pull–back of the locally trivial fibre space $M_r \to Q_r$ to $\tilde{Q}_r$ becomes globally trivial over $\tilde{Q}_r$, hence in this case one always has a diagram of the type (4). The space Q can here be replaced by any neighbourhood of the critical value $0 \in Q$. We explicitly record the following consequence of 1.4.

1.7 **Lemma**. *Let $\mu, \lambda \in Q_r$. The homotopy class of the map*

$$p_\mu \circ p_\lambda^{-1} : \Omega_\lambda \to \Omega_\mu \tag{8}$$

is independent of the trivialization (5) which defines the maps (7). $\square$

1.8 Let W be the group of covering transformations of $\tilde{Q}_r \to Q_r$. For any $w \in W$, and $\lambda \in Q_r$ define a transformation

$$a_\lambda(w) = p_{w\cdot\lambda}^{-1} \circ p_\lambda : \Omega \to \Omega \quad (w \in W, \lambda \in Q_r). \tag{9}$$

This is essentially the same as the isotopic monodromy: if one identifies Ω with Ω_λ by means of the map (7), then $a_\lambda(w)$ coincides with the isotopic parallel transport around a loop at $\pi(\lambda)$ which lifts to a path with endpoints λ and $w \cdot \lambda$.

1.9 **The limit map**. Fix metrics on Q and on M which define the topology. Both metrics will be written as $d(x, y)$. We use the notation

$$M_\epsilon(V) := \{x \in M : d(x, y) < \epsilon \text{ for some } y \in V\}$$

for the ϵ -neighbourhood of a subset $V \subset M$. We call a neighbourhood U of V *retractible* if there is a deformation retraction $r : U \to V$.

1.9.1 **Assumption and definition**. We assume that there is $p_0 : \Omega \to \Omega_0$ so that

$$\lim_{\pi(\lambda)\to 0} p_\lambda = p_0 \tag{10}$$

uniformly as maps $\Omega \to M$. This means that for all $\epsilon > 0$ there is $\delta > 0$ so that

$$d(\pi(\lambda), 0) \leq \delta \Rightarrow \; d(p_\lambda(x), p_0(x)) \leq \epsilon \tag{11}$$

for all $x \in \Omega$ and all $\lambda \in \tilde{Q}_r$. We shall call p_0 the *limit map*. The limit map p_0 itself depends on the trivialization (5) which defines the maps p_λ, but its homotopy class does not, at least under the rather weak hypotheses of the following lemma.

1.9.2 **Lemma**. *Suppose there is $\epsilon > 0$ so that Ω_0 is a retract of $M_\epsilon(\Omega_0)$ and let $r : M_\epsilon(\Omega_0) \to \Omega_0$ be a retraction. Then there is $\delta > 0$ so that for any $\lambda \in Q_r$ with $d(\pi(\lambda), 0) < \delta$*

$$p_0 \sim r \circ p_\lambda : \Omega \to \Omega_0$$

(homotopic maps). In particular, the homotopy class of p_0 is independent of the trivialization (5) if one takes as standard fibre $\Omega = \Omega_{\lambda_1}$ for a fixed $\lambda_1 \in Q_r$. □

The following lemma is the essential ingredient in the construction of the monodromy representation on of W.

1.9.3 **Lemma**. *Let V be any subset of Ω_0. Assume there is an ϵ–neighbourhood U of V in Ω_0 so that the inclusion $i : p_0^{-1}(V) \to p_0^{-1}(U)$ admits a retraction $r : p_0^{-1}(U) \to p_0^{-1}(V)$. For any $w \in W$ there is $\delta > 0$ so that*

$$a_\lambda(w) p_0^{-1}(V) \subset p_0^{-1}(U) \tag{12}$$

for $\lambda \in \tilde{Q}_r$, $d(\pi(\lambda), 0) < \delta$. The transformations

$$r \circ a_\lambda(w) \circ i : p_0^{-1}(V) \to p_0^{-1}(V) \tag{13}$$

are then defined for such λ and their homotopy class $a_V(w)$ is independent of λ. The map $w \to a_V(w)$ is a homomorphism of W into the group of homotopy equivalences of $p_0^{-1}(V)$. □

We again omit the proof, but note that it is based on the following relation, which we record for reference:

$$a_\lambda(wy) = a_{y\cdot\lambda}(w) a_\lambda(y). \tag{14}$$

We shall paraphrase (12) by saying that the limit map p_0 is *invariant* under the monodromy transformations $a_\lambda(w)$. The homomorphism of W into the group of homotopy equivalence of $p_0^{-1}(V)$ will be called the *restriction* of the isotopic monodromy representation to $p_0^{-1}(V)$.

1.9.4 Concerning the hypothesis in 1.9.3 we recall that any subcomplex of a finite simplicial complex has a retractible neighbourhood. It follows that the hypothesis is satisfied whenever $p_0 : \Omega \to \Omega_0$ can be realized as the restriction of a map $p_0 : \Omega^c \to \Omega_0^c$ of finite simplicial complexes and V as an open subcomplex of Ω_0^c.

1.10 **Simultaneous topological resolutions**. We now suppose that $M \xrightarrow{q} Q$ becomes globally trivial topologically after passing to a ramified covering. This means that (4) extends to a commutative diagram of continuous maps

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\rho} & M \\ \tilde{q}\downarrow & & \downarrow q \\ \tilde{Q} & \xrightarrow{\pi} & Q \end{array} \tag{15}$$

with the following properties, in addition to those listed in connection with (4). First of all we require that (15) extends (4) with $\tilde{M}_r = \tilde{\rho}^{-1}(M_r)$ and $\tilde{Q}_r = \pi^{-1}(Q_r)$. Further we require that

a) The map $\tilde{q} : \tilde{M} \to \tilde{Q}$ is a topologically trivial fibration
b) The map $\pi : \tilde{Q} \to Q$ is continuous, surjective, finite, and $q^{-1}(0) = \{\tilde{0}\}$, a single point.
c) The map $\rho : \tilde{M} \to M$ is proper.

A map $\pi : \tilde{Q} \to Q$ with the properties (b) and which restricts to a covering $q : \tilde{Q}_r \to Q_r$ on open sets with $\tilde{Q}_r \cup \{\tilde{0}\}$ and (hence) $Q_r \cup \{0\}$ arc–connected will be called a *ramified covering* with *point of total ramification* $q^{-1}(0) = \tilde{0}$, *regular* on $\tilde{Q}_r \to Q_r$.We shall refer to a diagram (15) of this kind as a *simultaneous topological resolution* of the map $q : M \to Q$. It is analogous to the simultaneous resolution in algebraic geometry [Brieskorn 1966, Slodowy 1980], except that we require that $\tilde{q} : \tilde{M} \to \tilde{Q}$ be globally trivial and that $q^{-1}(0) = \{\tilde{0}\}$, since we can replace Q by a suitable neighbourhood of 0.
We keep the notation introduced above in connection with (4). Fix a local topological trivialization $\Omega \times \tilde{Q} \xrightarrow{\approx} \tilde{M}$ of $\tilde{q}$ over a neighbourhood of 0. As standard fibre we can now choose

$$\Omega = \tilde{q}^{-1}(\tilde{0}). \tag{16}$$

The maps (6) and (7) are now defined for *all* λ in the neighbourhood of $\tilde{0}$ over which the trivialization is defined. The maps p_λ converge pointwise to p_0 as $\pi(\lambda) \to 0$, and if this limit is uniform then p_0 is the limit map defined by (10). Furthermore, we note that under the choice (16) as standard fibre, the map $p_0 : \Omega \to \Omega_0$ coincides with the resolution map $\rho : \tilde{q}^{-1}(0) \to q^{-1}(0)$.

1.10.1 We shall use Thom's Isotopy Theorem:

Let $f : X \to Y$ be a morphism of real analytic manifolds. Suppose X admits an embedding $X \to X^c$ as a stratum of a stratified semi–analytic set X^c, so that f extends to a proper morphism $f^c : X^c \to Y$. Assume that the restriction of f to each stratum is a surjective submersion. Then each such restriction, and in particular $f : X \to Y$, is locally topologically trivial fibration.

We refer to [Thom, 1969, chap. III] for the proof and for the definition of "stratified semianalytic set".

1.10.2 **Addendum to 1.10.1**. The trivializations have the following localization property.

A trivialization of $f : X \to Y$ can be chosen locally over Y so as to coincide on a compact set $C \subset U$ with a given trivialization of the restriction of f to an open subset U of X.

This follows from the proof of the theorem [loc. cit.]: the trivializations are obtained by integration of vector fields, which can be patched using partitions of unity. It is this localization property which makes possible the localization around the critical set characteristic of Picard–Lefschetz theory.

1.10.3 We now return to theorem 1.1.1. The assertion (A) is a consequence of 1.10.1. The assertion (B) follows from the assumption that $\tilde{q} : \tilde{M}^c \to \tilde{Q}$ is proper: its fibres Ω^c_λ are compact, so the maps $p_\lambda : \Omega^c \to \Omega^c_\lambda$ converge uniformly to $p_0 = \rho|\Omega$, as required in 1.9.1. The assertion (C) follows from lemmas 1.9.2 and 1.9.3 provided V has a retractible ϵ–neighbourhood in M. And this follows from the fact that the compact semi–analytic set Ω^c admits a triangularization into a simplicial complex K containing the semianalytic subset V as a difference $K_1 - K_2$ of subcomplexes $K_1 \supset K_2$. This finishes the outline of the proof of theorem 1.1.1.

1.10.4 We shall need a refinement of the theorem. Namely suppose that the completion $q : M^c \to Q$ is a trivial fibration on the part at infinity $M^c - M$. In that case the standard fibre Ω aquires a part at infinity, say $F = \Omega^c - \Omega$ which remains pointwise fixed by the transformations $a_\lambda(w)$. These transformations then act also in the homology with supports restricted by conditions at infinity, to specified in detail for the coadjoint quotient. We shall now turn to some mostly standard concepts which will be needed later.

1.11 **Gauss–Manin connection and monodromy representation.** The isotopic transport (3) of a locally trivial fibration $M_r \xrightarrow{q} Q_r$ defines a map $(p_\gamma)_* : H_.(\Omega_{\theta(0)}) \to H_.(\Omega_{\theta(1)})$, the *Gauss–Manin transport* along γ. In case q is a C^∞ (holomorphic map) between C^∞ (complex) manifolds, this is the parallel transport with respect to a flat connection in the C^∞ (holomorphic) vector bundle $\theta \to H_.(\Omega_\theta)$ over Q_r. The isotopic monodromy representation of $\pi_1(Q_r, \theta_1)$ induces a representation in $H_.(\Omega_{\theta_1})$ in the usual sense, called *monodromy representation* in homology.

1.12 **Homology with other supports.** The homology above was understood to have compact supports, as is customary. We indicate the adjustments required if one takes for $H_.(\cdot)$ homology with another family of supports. We place ourselves in the situation of Theorem 1.1.1. Let Φ be a family of supports on M. We take $\tilde{\Phi} = \rho^{-1}\Phi$ as family of supports on $\tilde{M}$ and for each $\theta \in Q$ (resp. $\lambda \in \tilde{Q}$) we take $\Phi|\Omega_\theta$ (resp. $\tilde{\Phi}|\tilde{\Omega}_\lambda$) as family of supports on Ω_θ (resp. $\tilde{\Omega}_\lambda$). This applies in particular to $\Omega = \tilde{\Omega}_{\tilde{0}}$, which we take as standard fibre. As a condition on the local trivialization of $\Omega \times \tilde{Q} \xrightarrow{\approx} \tilde{M}$ we require that it induce for each $\lambda \in \tilde{Q}$ a bijection of $(\tilde{\Phi}|\Omega) \times \{\lambda\}$ with $(\tilde{\Phi}|\tilde{\Omega}_\lambda)$ and that it map into $\tilde{\Phi}$ sets of the form $A \times B$ with $A \subset \Omega$ belonging to $\tilde{\Phi}$ and $B \subset \tilde{Q}$ compact. It follows from [Borel–Moore, 1960, 3.5, 4.3] that under these conditions the maps $\tilde{p}_\lambda$ induce maps in homology with the indicated supports which do not change under the type of homotopy used in (2) and are therefore independent of the trivialization.

1.12.1 In case $q : M \to Q$ is the coadjoint quotient map, to be discussed later, M^c can be taken to be a projective variety in which the completions Ω_λ^c of the fibres Ω_λ all have the same part at infinity, say $\Omega_\lambda^c - \Omega_\lambda = F$, in the sense that there is a natural trivialization $M^c - M \approx F \times Q$. In that case it makes sense to require that F stay fixed under the isotopic transport, i.e. that the maps p_λ agree over F. This insures that the corresponding transformations $a_\lambda(w)$ operate in in the homology for any family of supports.

1.12.2 For reference, we record three elementary properties [loc. cit.] which will be used frequently, but only for homology with arbitrary supports here denoted $H_\cdot(\cdot)$.

a) If U is an open subspace of X then the inclusion $U \xrightarrow{j} X$ induces a natural map

$$H_\cdot(X) \xrightarrow{j^*} H_\cdot(U). \tag{17}$$

b) If F is a closed subspace of X, then the inclusion $F \xrightarrow{i} X$ induces a natural map

$$H_\cdot(F) \xrightarrow{i_*} H_\cdot(X). \tag{18}$$

c) If $U = X - F$ with U, F as above, then there exists a long exact sequence

$$\cdots \to H_q(F) \xrightarrow{i_*} H_q(X) \xrightarrow{j^*} H_q(U) \xrightarrow{\partial} H_{q-1}(F) \to \cdots . \tag{19}$$

A model for homology with arbitrary supports for real–analytic varieties which is particularly convenient in the present context may be constructed from subanalytic chains as in [Kashiwara–Shapira, 1990]. We usually have this model in mind, so *chains* or *cycles* will always be understood to be subanalytic.

1.13 **Notes**. The origins of the theory are [Picard–Simart, 1897, Chap. IV] and especially Lefschetz's astonishing monograph [Lefschetz, 1924]. More recent references are [Deligne et Katz, 1970] and [Arnold et al. 1988; chapters I and III]. See also [Brieskorn,1970]. Simultaneous resolutions were used in [Brieskorn,1966]. The simultaneous resolution of the (co)adjoint quotient to be discussed later is due to Grothendieck. It is studied in [Slodowy, 1980].

2. Picard–Lefschetz theory for the coadjoint quotient

2.1 **The coadjoint quotient**. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Consider the quotient map

$$\mathfrak{g}^* \xrightarrow{q} G\backslash\backslash\mathfrak{g}^* \tag{1}$$

with $G = \mathrm{Ad}(\mathfrak{g})$ acting on the complex dual space $\mathfrak{g}^*$ by the coadjoint representation, which will be more natural for what we have in mind than the adjoint representation. The quotient is here taken in the sense of algebraic geometry, as the affine variety associated to the ring $\mathbf{C}[\mathfrak{g}^*]^G$ of G–invariant polynomial functions on $\mathfrak{g}^*$. (The double slash is intended to indicate this; we write the group on the left if it acts on the left.) The ring $\mathbf{C}[\mathfrak{g}^*]^G$ is a polynomial ring on $l = \mathrm{rank}(\mathfrak{g})$ generators, and $G\backslash\backslash\mathfrak{g}^*$ is isomorphic with $\mathbf{C}^l$. More concretely, as a set $G\backslash\backslash\mathfrak{g}^*$ may be identified with the quotient $W\backslash\mathfrak{h}^*$ of the dual of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by the action of the Weyl group W. The fibre of q over the element $W\cdot\lambda \in W\backslash\mathfrak{h}^*$ is the variety

$$\Omega_\lambda = \{\xi \in \mathfrak{g}^* : f(\xi) = f(\lambda) \text{ for all } f \in \mathbf{C}[\mathfrak{g}^*]^G\}.$$

When λ is regular, this is the orbit $G\cdot\lambda$; for $\lambda = 0$ it is the nilpotent variety $\mathcal{N}$ in $\mathfrak{g}^*$. The restriction of q to the set $\mathfrak{g}^*_r$ of regular semisimple elements provides the locally trivial fibration $q : \mathfrak{g}^*_r \to W\backslash\mathfrak{h}^*_r$. We shall be interested in the Picard–Lefschetz theory for the map $\mathfrak{g}^* \xrightarrow{q} G\backslash\backslash\mathfrak{g}^*$ around the most singular fibre $\mathcal{N} = q^{-1}(0)$.

2.2 **The simultaneous resolution**. According to Grothendieck, the quotient map q admits a simultaneous resolution of singularities as follows (cf. [Slodowy, 1980]). Let $\mathcal{B} \approx G/B$ be the flag manifold of $\mathfrak{g}$. For $x \in \mathcal{B}$ denote by $\mathfrak{b}_x$ the Lie algebra of its stabilizer B_x in G, and by $\mathfrak{n}_x$ the nilpotent radical of $\mathfrak{b}_x$. Fix $x_o \in \mathcal{B}$ so that $\mathfrak{b} := \mathfrak{b}_{x_o}$ contains $\mathfrak{h}$ and write $\mathfrak{n} = \mathfrak{n}_{x_o}$. Let

$$\tilde{\mathfrak{g}}^* = \{(x,\xi) : x \in \mathcal{B}, \xi \in (\mathfrak{g}/\mathfrak{n}_x)^*\} \approx G \times_B (\mathfrak{g}/\mathfrak{n})^*.$$

This is a complex manifold. The *simultaneous resolution* is the natural map $\tilde{\mathfrak{g}}^* \xrightarrow{\rho} \mathfrak{g}^*$; it fits into a commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{g}}^* & \xrightarrow{\rho} & \mathfrak{g}^* \\ \tilde{q}\downarrow & & \downarrow q \\ \mathfrak{h}^* & \to & W\backslash\mathfrak{h}^* \end{array} \tag{2}$$

which is basic in what is to follow. It has the properties listed in 1.10. The map $\tilde{q}$ is locally trivial holomorphically and globally trivial topologically. A global trivialization of $\tilde{q}$ may be constructed as follows. The fibre of $\tilde{q}$ over zero is

$$\mathcal{B}^* = \{(x,\nu) : x \in \mathcal{B}, \nu \in (\mathfrak{g}/\mathfrak{b}_x)^*\} \approx G \times_B (\mathfrak{g}/\mathfrak{b})^*$$

which may be identified with the cotangent bundle of $\mathcal{B}$ with projection $\pi : \mathcal{B}^* \to \mathcal{B}$, $\pi(x,\nu) = \nu$. In particular, $\mathcal{B}^*$ is a complex manifold of complex dimension $2n$, $n = \dim_{\mathbf{C}} \mathcal{B}$.

Let U be a compact form of G, chosen so that $U \cap H$ is a maximal torus. Any element of $\mathcal{B}$ is of the form $x = u(x)x_o$ with $u(x) \in U$ unique up to right translation by $U \cap H$.

2.3 **Lemma** . *The fibration $\tilde{\mathfrak{g}}^* \to \mathfrak{h}^*$ admits a real–analytic U–equivariant global trivialization*

$$\tilde{\mathfrak{g}}^* \overset{U}{\approx} \mathcal{B}^* \times \mathfrak{h}^*, \tag{3}$$

or equivalently

$$G \times_B (\mathfrak{g}/\mathfrak{n})^* \overset{U}{\approx} U \times_{(U \cap H)} (\mathfrak{g}/\mathfrak{b})^* \times \mathfrak{h}^* \tag{4}$$

A trivialization map $\mathcal{B}^* \times \mathfrak{h}^* \overset{\approx}{\to} \tilde{\mathfrak{g}}^*$ is given by $(x, \nu, \lambda) \to (x, u(x) \cdot \lambda + \nu)$.

We omit the elementary verification but record that the maps p_λ corresponding to this trivialization as in 1.5, eq.(7) are given by

$$p_\lambda : \mathcal{B}^* \to \Omega_\lambda, (x, \nu) \to u(x) \cdot \lambda + \nu. \tag{5}$$

For regular λ this map is therefore a homeomorphism. One should note that for general λ, this map is not holomorphic and is not equivariant for the action of G (only for U). For $\lambda = 0$, however, it is both:

$$p_0 = \rho : \mathcal{B}^* \to \mathcal{N}, (x, \nu) \to \nu \tag{6}$$

is the restriction of the map ρ in (2), the *Springer resolution* of the nilpotent cone $\mathcal{N}$; it is the moment map for the action of G on the symplectic manifold $\mathcal{B}^*$.

2.3.1 There is a commutative diagram

$$\begin{array}{ccc} \mathcal{B}^* & \overset{p_\lambda}{\to} & \Omega_\lambda \\ \pi \searrow & & \swarrow \pi_\lambda \\ & \mathcal{B} & \end{array}$$

where $\pi_\lambda : \Omega_\lambda \to \mathcal{B}$ is the map $\pi_\lambda(g \cdot \lambda) = g \cdot x_o$. In particular, for regular $\lambda \in \mathfrak{h}^*$ the map $\pi \circ p_\lambda^{-1} = \pi_\lambda$ is G–equivariant and holomorphic, even though p_λ is neither.

2.4 **Completions** . We introduce a completion of the simultaneous resolution (2). We first deal with the coadjoint quotient $\mathfrak{g}^* \to G\backslash\backslash\mathfrak{g}^*$ itself. For any complex vector space V, let $CV = P(V \times \mathbf{C}) = \mathbf{C}^\times\backslash[V \times \mathbf{C} - (0,0)]$ denote its *projective completion* , to be distinguished from $\mathrm{P}\,V = \mathbf{C}^\times\backslash[V - \{0\}]$. We denote by $v/t \in CV$ the class of $(v,t) \in V \times \mathbf{C} - (0,0)$ and we identify V with a subset of CV (the *finite part* of CV) so that $v = v/1$. There is an embedding $\mathrm{P}\,V \subset CV$ as the *hyperplane at infinity* $\{v/0 : v \in V\}$. Note that any linear map $A : V \to W$ extends to $A : CV \to CW$ via $A(v/t) = (Av)/t$.

2.4.1 Instead of the projective completion one could take for CV the *spherical completion* $\mathbf{R}_+^\times\backslash[V \times \mathbf{R} - (0,0)]$. It is a manifold with boundary, homeomorphic with a closed ball in V. The constructions below are then analogous to Milnor's sphere construction [Milnor, 1968], except that the sphere lies at ∞ rather than at ϵ of the singularity. The spherical completion is preferable when the topology of the boundary comes into play, which is simpler for a sphere.

2.4.2 **Definition**. Let

$$\mathfrak{g}^{*c} = \{(\xi/\tau, \theta) \in C\mathfrak{g}^* \times W\backslash\mathfrak{h}^* : q(\xi) = \tau \cdot \theta\}.$$

The variety $\mathfrak{g}^{*c}$ can be viewed as a fibrewise completion of $\mathfrak{g}^*$, with "fibrewise" referring to the fibres Ω_θ of the coadjoint quotient $\mathfrak{g}^* \to W\backslash\mathfrak{h}^*$. The coadjoint quotient extends naturally to a map $\mathfrak{g}^{*c} \xrightarrow{q} W\backslash\mathfrak{h}^*$, also denoted q, and we denote by Ω_θ^c the fibre of this map over $\theta \in W\backslash\mathfrak{h}^*$. We shall analyze it some detail in the following lemma.

2.4.3 **Lemma**. *The natural diagram*

$$\begin{array}{ccc} \mathfrak{g}^* & \stackrel{\subseteq}{\to} & \mathfrak{g}^{*c} \\ q \searrow & & \swarrow q \\ & W\backslash\mathfrak{h}^* & \end{array}$$

is commutative and the map $\mathfrak{g}^* \to \mathfrak{g}^{*c}$ is an embedding onto an open dense submanifold of $\mathfrak{g}^c$ (the finite part of $\mathfrak{g}^{*c}$). Its complement $\mathfrak{g}^{*c} - \mathfrak{g}^*$ (the part at infinity of $\mathfrak{g}^{*c}$) is the direct product $P\mathcal{N} \times W\backslash\mathfrak{h}^*$ and $\Omega_\theta^c - \Omega_\theta = P\mathcal{N} \times \{\theta\}$. ($P\mathcal{N}$ is the projectivized nilpotent cone.)

Proof. The map $\mathfrak{g}^* \stackrel{\subseteq}{\to} \mathfrak{g}^{*c}$ is given by $\xi \to (\xi/1, q(\xi))$ and is one–to–one onto the intersection of $\mathfrak{g}^{*c}$ with $\mathfrak{g}^* \times W\backslash\mathfrak{h}^*$, which is Zariski–open in $\mathrm{C}\mathfrak{g}^* \times \mathfrak{g}$. The map $\mathfrak{g}^c \xrightarrow{q} W\backslash\mathfrak{h}^*$ is induced by the projection of $C\mathfrak{g}^* \times W\backslash\mathfrak{h}^*$ onto $W\backslash\mathfrak{h}^*$ and is surjective. For $\theta \in W\backslash\mathfrak{h}^*$, the fibre in question is

$$\Omega_\theta^c = \{(\xi/\tau, \theta) : \xi/\tau \in C\mathfrak{g}^*, q(\xi) = \tau \cdot \theta\} \tag{7}$$

from which the assertions are evident. $\square$

The following picture emerges. In the projective completion $C\mathfrak{g}^*$ of $\mathfrak{g}^*$ the closures of all fibres Ω_θ intersect at infinity in the projectivized nilpotent cone $P\mathcal{N}$. In $\mathfrak{g}^{*c}$ the fibres get separated, with disjoint copies $P\mathcal{N} \times \{\theta\}$ of $P\mathcal{N}$ at infinity. We now introduce a fibrewise completion of $\tilde{\mathfrak{g}}^*$ in which all fibres become smooth.

2.4.4 **Definition**. Let

$$\tilde{\mathfrak{g}}^{*c} = \{(x, \xi/\tau, \lambda) : x \in \mathcal{B}, \xi/\tau \in C(\mathfrak{g}/\mathfrak{n}_x)^*, \lambda \in \mathfrak{h}^*, q(\tau\lambda) = q(\xi)\}.$$

To analyze $\tilde{\mathfrak{g}}^{*c}$ we introduce two auxiliary varieties:

$$(\mathfrak{g}/\mathfrak{n})^c = \{(\xi/\tau, \theta) \in C(\mathfrak{g}/\mathfrak{n})^* \times W\backslash\mathfrak{h}^* : q(\xi) = \tau \cdot \theta\};$$

$$\mathcal{B}^{*c} = \{(x, \nu/\tau) : x \in \mathcal{B}, \nu/\tau \in C(\mathfrak{g}/\mathfrak{b}_x)^*\}.$$

2.4.5 **Lemma**. *a) There is a natural isomorphism*

$$\tilde{\mathfrak{g}}^{*c} \approx G \times_B (\mathfrak{g}/\mathfrak{n})^c \tag{8}$$

In particular, $\tilde{\mathfrak{g}}^{*c}$ *is smooth and* $\tilde{\mathfrak{g}}^{*c} \to \mathfrak{h}^*$ *is a submersion.*
b) There is a natural commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{g}}^* \subset \tilde{\mathfrak{g}}^{*c} & \overset{\rho}{\to} & \mathfrak{g}^* \subset \mathfrak{g}^{*c} \\ \tilde{q} \downarrow & & \downarrow q \\ \mathfrak{h}^* & \to & W\backslash\mathfrak{h}^* \end{array} \tag{9}$$

The inclusions have open and dense images.
c) The fibration $\tilde{\mathfrak{g}}^{*c} \to \mathfrak{h}^*$ *admits a* U*–equivariant, real–analytic global trivialization given by*

$$\tilde{\mathfrak{g}}^{*c} \overset{U}{\approx} \mathcal{B}^{*c} \times \mathfrak{h}^*, \tag{10}$$

or equivalently

$$G \times_B (\mathfrak{g}/\mathfrak{n})^c \overset{U}{\approx} U \times_T C(\mathfrak{g}/\mathfrak{b})^* \times \mathfrak{h}^*. \tag{11}$$

A trivialization map $\mathcal{B}^{*c}\times\mathfrak{h}^* \overset{\approx}{\to} \tilde{\mathfrak{g}}^{*c}$ is given by $(u{\cdot}x_o, u{\cdot}\nu/\tau, \lambda) \to (u{\cdot}(\tau\lambda{+}\nu)/\tau, \lambda\}$. We omit the simple verifications.

In view of lemmas 2.4.2 and 2.4.4, the constructions of section 1 in the form of Theorem 1.1.1 apply in the present context. In particular, the limit map introduced there is now the Springer resolution (6) and we have the restricted monodromy representations defined in terms of this map as in 1.9.3. We shall now introduce the homology of interest in connection with characters of real semisimple groups.

2.5 **A homology for subalgebras**. From now on we shall consider $\mathfrak{g}$ as *real* Lie algebra and we denote by $\mathfrak{g}^*$ the real dual space of $\mathfrak{g}$, unless specified otherwise. To avoid confusion, we denote the real pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$ by (X, ξ) and keep the notation $\langle X, \xi\rangle$ for the complex pairing. We agree that

$$(X, \xi) = \mathrm{Re}\langle X, \xi\rangle.$$

Let τ be the Cartan involution of $\mathfrak{g}$ whose fixed–point set is the Lie algebra of the compact form U of G. We use the notation (X, Y) for the real Killing form of $\mathfrak{g}$, which should cause no confusion since it agrees with the notation (X, ξ) when $\mathfrak{g}$ is identified with $\mathfrak{g}^*$ by the Killing form, as is sometimes convenient; but generally we distinguish $\mathfrak{g}$ and $\mathfrak{g}^*$ in notation. We shall need an auxiliary Euclidian inner product on $\mathfrak{g}^*$, which we take to be

$$(\xi, \eta)_\tau = -(\xi, \tau\eta).$$

The corresponding Euclidian norm is denoted $\|\xi\|^2$.
Let $\mathfrak{g}_o$ be any real subalgebra of $\mathfrak{g}$ and G_o the corresponding subgroup of $G = \mathrm{Ad}(\mathfrak{g})$. Write $R_{\mathfrak{g}_o} : \mathfrak{g}^* \to \mathfrak{g}_o^*$ for natural projection, $\mathfrak{g}_o^\perp$ for its kernel. As a family of supports on $\mathfrak{g}^*$ we introduce the closed subsets of $\mathfrak{g}^*$ which are contained in strips of the form

$$\|R_{\mathfrak{g}_o}(\xi)\| \le \text{const.}, \quad \|q(\xi)\| \le \text{const.} \tag{12}$$

This condition also defines families of supports on the fibres Ω_θ and $\tilde{\Omega}_\lambda$, in particular on $\mathcal{B}^*$, as explained in 1.12. The corresponding homologies will be denoted $'H(\cdot)$. Note that the second condition in (12) is vacuous on a given Ω_λ.

2.5.1 The homology $'H_*(\Omega_\theta)$ admits realizations by several chain complexes. Here we have in mind subanalytic chains on Ω_θ as in [Kashiwara–Shapira, 1990, §9.2], but subject to the condition (12) on their supports. Another model may be constructed using infinite (but locally finite) singular chains as in [Rossmann 1984, 1990]. A model in classical simplicial homology is also available: consider the completion Ω_θ^c as a simplicial complex $\mathbf{K}$ by a triangularization. Assume that its part at infinity $\Omega_\theta^c - \Omega_\theta$ is a subcomplex $\mathbf{L}$, as is $\mathbf{L}^1 = \Omega_\theta^\infty \cap {\mathfrak{g}_o^\perp}^c$. Put $\mathbf{L}^2 = \mathbf{L} - \mathbf{L}^1$. Then

$$'H_.(\Omega_\theta) = H_.(\mathbf{K} - \mathbf{L}^2; \mathbf{L}^1).$$

The homology on the right is of the type discussed in [Lefschetz, 1965, p.140 ff]. Similar remarks apply to the homology groups $'H_.(\tilde{\Omega}_\lambda)$.

2.6 **The conormal variety** $\mathcal{S}$. The conormal variety of the G_o–action on $\mathcal{B}$ is

$$\mathcal{S} = \{(x,\nu) \in \mathcal{B}^* : \nu \in \mathfrak{g}_o^\perp\}.$$

There are two natural maps

$$\begin{aligned} &\text{a) } \pi : \mathcal{S} \to \mathcal{B}, (x,\nu) \to x, \\ &\text{b) } \rho : \mathcal{S} \to \mathcal{N}(\mathfrak{g}_o^\perp), (x,\nu) \to \nu. \end{aligned} \tag{13}$$

They lead to two quite different views of the variety $\mathcal{S}$ as follows.

a) $\mathcal{S}$ is the union of the conormal bundles of the G_o –orbits on $\mathcal{B}$.
b) $\mathcal{S}$ is the inverse image of $\mathcal{N}(\mathfrak{g}_o^\perp)$ under the Springer resolution $\rho : \mathcal{B}^* \to \mathcal{N}$.

For the interpretation in a) one identifies $\mathcal{B}^*$ with the real cotangent bundle of $\mathcal{B}$ by means of the real pairing $(X,\nu) = Re\langle X,\nu\rangle$ on $(\mathfrak{g}/\mathfrak{b}_x) \times (\mathfrak{g}/\mathfrak{b}_x)^*$. There is another interpretation of $\mathcal{S}$ which is sometimes useful. Namely, define

$$\rho_{\mathfrak{g}_o} : \mathcal{B}^* \to \mathfrak{g}_o^*, \rho_{\mathfrak{g}_o}(x,\nu) = R_{\mathfrak{g}_o}(\nu).$$

This is the moment map of the action of G_o on $\mathcal{B}^*$, considered as the real cotangent bundle of $\mathcal{B}$. Then $\mathcal{S} = \rho_{\mathfrak{g}_o^-1}(0)$, the inverse image of $0 \in \mathfrak{g}_o^*$. The pictures in §4.12 illustrate the situation. The importance of the variety $\mathcal{S}$ in the present context comes from the following fact.

2.6.1 **Proposition**. *The inclusion* $\mathcal{S} \to \mathcal{B}^*$ *induces an isomorphism* $H_.(\mathcal{S}) \xrightarrow{\approx} {'H_.(\mathcal{B}^*)}$ *where* $H_.(\mathcal{S})$ *denotes homology with arbitrary supports.*

Proof. For any $c > 0$ set $V_c = \{(x,\nu) \in \mathcal{B}^* : \|R_{\mathfrak{g}_o}(\nu)\| \le c\}$. Then

$$'H_.(\mathcal{B}^*) = \lim_{c \to 0} H_.(V_c),$$

the inductive limits of the homologies of the V_c with arbitrary supports, cf. [Borel–Moore, 1960, Theorem 3.4]. For any $t > 0$, the map $V_c \to V_{tc}$, $(x, \nu) \to (x, t\nu)$, induces an isomorphism

$$H_.(V_c) \approx H_.(V_{tc}).$$

For sufficiently small $\epsilon > 0$, the closed neighbourhood V_ϵ of $\mathcal{S}$ admits a proper retraction to $\mathcal{S}$: it suffices to choose a triangularization of the projective variety $\mathcal{B}^{*c}$ which contains the closure of $\mathcal{S}$ as a subcomplex and to apply the remark 1.9.4. Thus

$$H_.(V_\epsilon) \approx H_.(\mathcal{S})$$

for such $\epsilon > 0$. Combining these isomorphisms one gets the desired isomorphism $'H_.(\mathcal{B}^*) \approx H_.(\mathcal{S})$. $\square$

2.6.2 The lemma proves in particular the isomorphism of the various models for the homology $'H_.(\mathcal{B}^*)$ mentioned in 2.5.1, since the corresponding isomorphisms hold for $H_.(\mathcal{S})$.

2.6.3 Assume that G_o *has finitely many orbits* on $\mathcal{B}$. It is then clear from (13), (a) that $\mathcal{S}$ is a real–analytic variety of dimension

$$\dim_{\mathbf{R}} \mathcal{S} = \dim_{\mathbf{C}} \mathcal{B}^* = 2n$$

$n = \dim_{\mathbf{C}} \mathcal{B}$ as before. Its homology in dimension $2n = \dim_{\mathbf{R}}(\mathcal{S})$ can be described quite explicitly as follows. Let $\mathcal{S}_{sm}$ the manifold of smooth points of the real analytic variety $\mathcal{S}$. By a component C of $\mathcal{S}$ we shall mean a connected component of $\mathcal{S}_{sm}$. Together with an orientation, it defines a $2n$–chain $[C]$ on $\mathcal{S}$. It is easy to see (and will be verified below in a more general situation) that any $2n$–cycle $\Gamma \in H_{2n}(\mathcal{S})$ a can be uniquely represented in the form

$$\Gamma = \sum_C m_C[C] \quad (m_C \in \mathbf{Z}) \tag{14}$$

where C runs over the connected components of $\mathcal{S}_{sm}$. However, the chains $[C]$ themselves need not be cycles, unless $\mathcal{S}$ is a complex variety.

2.6.4 **Example**. Take $\mathfrak{g}_o = \mathfrak{b}$. The B–orbits on $\mathcal{B}$ are of the form $O_w = Bwx_o, w \in W$. The components of $\mathcal{S}$ are the conormals C_w of the O_w. Recall the maps $p_\lambda : \mathcal{B}^* \to \Omega_\lambda, (x, \nu) \to u(x) \cdot \lambda + \nu$, in (5) which enter into the definition of the monodromy transformations

$$a_\lambda(w) = p_{w\lambda}^{-1} \circ p_\lambda : \mathcal{B}^* \to \mathcal{B}^*$$

(cf.1.8). We consider the basis $[C_w]$, $w \in W$, of $'H_{2n}(\mathcal{B}^*) \approx H_{2n}(\mathcal{S})$.
Let $V_w = B \cap w\bar{N}w^{-1}$ where $\bar{N}$ is the unipotent radical of the Borel subalgebra containing H opposite to B, so that V_w is a set of coset representatives for $B/B \cap wBw^{-1}$. Then $O_w = V_w \cdot wx_o$ and

$$C_w = V_w \cdot (\{wx_o\} \times w\mathfrak{b}^\perp \cap \mathfrak{b}^\perp).$$

We write the decomposition of the cycle $[C_w]$ into the fibres of $C_w \to O_w$ as

$$[C_w] = \bigsqcup_{v \in V_w} v \cdot [\{wx_o\} \times w\mathfrak{b}^\perp \cap \mathfrak{b}^\perp],$$

to be interpreted as a chain (the image of $[V_w \times w\mathfrak{b}^\perp \cap \mathfrak{b}^\perp]$). Under the map p_λ this becomes

$$p_\lambda[C_w] = \bigsqcup_{v \in V_w} [u(vw)\lambda \dot{+} vw\mathfrak{b}^\perp \cap \mathfrak{b}^\perp] \tag{15}$$

where $\dot{+}$ is pointwise addition in $\mathfrak{g}^*$, the dot being added to distinguish it from addition of chains. The element $u(v) \in U/(U \cap H)$ is defined by $G = UB$. For the closed orbit $C_1 = \{x_o\}$ we get

$$p_{w\lambda}(C_1) = w\lambda \dot{+} \mathfrak{b}^\perp.$$

We assume that λ is regular. Then $b \to w \cdot \lambda$ maps B/H bijectively onto $w\lambda \dot{+} \mathfrak{b}^\perp$ and maps $(B \cap wBw^{-1})/H$ onto $(w\lambda \dot{+} w\mathfrak{b}^\perp \cap \mathfrak{b}^\perp)$. Since $B = V_w(B \cap wBw^{-1}) \approx V_w \times (B \cap wBw^{-1})$ we find that $w\lambda \dot{+} \mathfrak{b}^\perp = V_w(w\lambda \dot{+} w\mathfrak{b}^\perp \cap \mathfrak{b}^\perp)$, i.e.

$$p_{w\lambda}[C_1] = \bigsqcup_{v \in V_w} [vw\lambda \dot{+} vw\mathfrak{b}^\perp \cap \mathfrak{b}^\perp]. \tag{16}$$

We note the similarity of (15) and (16). However, these cycles on are not the same. On $\mathcal{B}^*$ they correspond (under p_λ) to the cycles $[C_w]$ and $[a_\lambda(w)C_1]$ respectively. The first lies on $\mathcal{S}$, the second does not (for $w \neq 1$). To realize $[a_\lambda(w)C_1]$ on $\mathcal{S}$ it must be retracted to $\mathcal{S}$ as in the proof of 2.6.1. This cycle lies over O_w, i.e. $\pi(a_\lambda(w)C_1) = \pi \circ p_\lambda^{-1}(Bw \cdot \lambda) = Bw \cdot x_o = O_w$, because $\pi \circ p_\lambda^{-1}$ is G–equivariant (2.3.1). Thus the retraction of $[a_\lambda(w)C_1]$ to $\mathcal{S}$ can take place over the *closure* $\bar{O}_w$ of O_w. It follows from (15) and (16) the result of the retraction is of the form

$$[a_\lambda(w)C_1] = \pm[C_w] + \cdots \tag{17}$$

where the dots indicate a cycle over the topological boundary $\bar{O}_w - O_w$ of O_w in $\bar{O}_w$ and the sign depends on w and on the orientations. (This is also a corollary of Theorem 5.8 below.) It is not easy to find an explicit formula for these cycles. For the case $G = \mathrm{SL}_2(\mathbf{C})$ this is the Picard–Lefschetz formula in its most basic form (cf. 4.12). Some additional information will be given in the next section. The formula (17) shows at least that the cycles $[a_\lambda(w)C_1]$, $w \in W$, form a basis for ${}'H_{2n}(\mathcal{B}^*) \approx H_{2n}(\mathcal{S})$, so that the monodromy representation of W in ${}'H_{2n}(\mathcal{B}^*)$ is equivalent to the regular representation on $\mathbf{Z}[W]$ with $[C_1]$ corresponding to the canonical generator $1 \in \mathbf{Z}[W]$.

2.6.5 While the transformations $a_\lambda(w)$ of $\mathcal{B}^*$ do not leave the subvariety $\mathcal{S}$ invariant, they do leave the zero section $\mathcal{B}$ invariant: $a_\lambda(w)$ operates on $\mathcal{B} \approx U/H \cap U$ by $u \cdot x_o \to uw^{-1}x_o$. This follows directly from the definition (5).

We shall be interested in subalgebras $\mathfrak{g}_o$ for which G_o has finitely many orbits both on $\mathcal{B}$ and on $\mathcal{N}(\mathfrak{g}_o^\perp)$, which is the case when $\mathfrak{g}_o$ is a *symmetric subalgebra* in the sense of the following definition.

2.6.6 **Definition**. A *symmetric subalgebra* of $\mathfrak{g}$ is the fixed–point set $\mathfrak{g}_o = \mathfrak{g}^\sigma$ of an involutive automorphism σ of $\mathfrak{g}$ as *real* Lie algebra.
We shall always assume that σ commutes with the Cartan involution τ fixing the compact real form U of G, as can be arranged by suitable conjugation.

2.6.7 We recall that there are two basic types of involutions:

a) $\mathfrak{g}_{\mathbf{R}}$– case: σ is conjugate linear, $\mathfrak{g}_o = \mathfrak{g}_{\mathbf{R}}$, a real form of $\mathfrak{g}$.
b) $\mathfrak{k}_{\mathbf{C}}$ –case: σ is complex linear, $\mathfrak{g}_o = \mathfrak{k}_{\mathbf{C}}$, the complexification of $\mathfrak{k} := \mathfrak{g}_o^\tau$.

Any pair $(\mathfrak{g}, \sigma)$ can be decomposed into a direct product of pairs of these types. These two types are in a natural duality: $(\mathfrak{g}, \sigma_+)$ is dual to $(\mathfrak{g}, \sigma_-)$ if $\sigma_+\sigma_- = \tau$, where τ is the fixed Cartan involution of $\mathfrak{g}$ commuting with σ_+ and σ_-.

2.7 **Finite orbit structure**. Because of its importance later, we record the following the well–known property of symmetric subalgebras.

$$G_o \text{ has finitely many orbits on } \mathcal{B} \text{ and on } \mathcal{N}(\mathfrak{g}_o^\perp) = \mathcal{N} \cap \mathfrak{g}_o^\perp. \tag{18}$$

For the action of G_o on $\mathcal{B}$ this follows from [Wolf, 1969] in the $\mathfrak{g}_{\mathbf{R}}$–case and from [Rossmann, 1979, Theorem 13] in general. For the action of G_o on $\mathcal{N} \cap \mathfrak{g}_o^\perp$ this follows from [Kostant and Rallis, 1971] in the $\mathfrak{g}_{\mathbf{R}}$–case and in the $\mathfrak{k}_{\mathbf{C}}$–case, and from [Segikuchi, 1987] in general. We shall mainly be interested in the map $\rho : \mathcal{S} \to \mathcal{N}(\mathfrak{g}_o^\perp)$. However, much of the discussion will also apply to the map $\mathcal{S} \to \mathcal{B}$, and the results for this case are also of interest. We therefore formulate the discussion in an abstract setting which applies to both.

3. The top homology of some peculiar varieties

3.1 **Notation**. For any locally compact space X, let $H_.(X)$ denote homology with arbitrary supports; coefficients are taken form $\mathbf{Q}$, unless indicated otherwise. When X is a real–analytic variety, an m–chain on X will be understood to mean a subanalytic m–chain.

3.2 **Lemma**. *Let X be an m–dimensional real analytic variety.*
a) For any closed subset $Z \subset X$ inclusion $Z \stackrel{\subseteq}{\to} X$ induces an injection $H_m(Z) \stackrel{\subseteq}{\to} H_m(X)$.
b) Let X_{sm} the m–dimensional manifold of smooth points of X. The natural map

$$H_m(X) \to H_m(X_{sm})$$

induced by the open embedding $X_{sm} \to X$ is injective. In particular any m–cycle $\Gamma \in H_m(X)$ a can be uniquely represented in the form

$$\Gamma = \sum_C m_C[C] \quad (m_C \in \mathbf{Q}) \tag{1}$$

where C runs over the connected component of X_{sm} and $[C]$ is its fundamental cycle in $H_m(X_{sm})$ corresponding to a fixed orientation of C.

Proof. a) This follows from the exact sequence

$$0 = H_{m+1}(X - Z) \to H_m(Z) \to H_m(X)$$

which is part the homology sequence of the closed subspace $Z \subset X$ (1.19).
b) Consider part of the long–exact sequence of the closed subspace (singular set) $X_{si} = X - X_{sm}$ of X:

$$H_m(X_{si}) \to H_m(X) \to H_m(X_{sm}).$$

Since $\dim X_{si} < m$ we have $H_m(X_{si}) = 0$, hence we get an injection

$$H_m(X) \stackrel{\subseteq}{\to} H_m(X_{sm}). \tag{2}$$

Since X_{sm} is a real manifold of dimension m, the oriented connected components C of X_{sm} form a basis for $H_m(X_{sm})$, which gives the assertion. $\square$

3.3 **Lemma**. *Let G be any real Lie group, $f : X \to Y$ a surjective G–morphism of real analytic G–varieties. Suppose $Y = G \cdot y \approx G/H$ is a homogeneous space and put $F = f^{-1}(y)$. Put $m = \dim X$, and $e = \dim F$. Fix an orientation on Y. For any p, there is a one–to–one correspondence $\gamma \leftrightarrow \Gamma$, denoted $\Gamma = G \cdot \gamma$ and $\gamma = \Gamma \cap F$, between H–invariant $(e-p)$–chains γ on F and G–invariant $(m-p)$–chains Γ on X. This correspondence commutes with the boundary operators. In top degree, it induces an isomorphism in homology,*

$$H_m(X) \approx H_e(F)^A$$

where $A = H/H^o$ is the component group of H.

Proof. On the level of sets such a correspondence is given by $\gamma \to \Gamma = G \cdot \gamma = \bigcup_{gH \in Y} g \cdot \gamma$ with inverse $\Gamma \to \gamma = \Gamma \cap F$. The correspondence of chains is induced by this. Under this correspondence, connected components of X_{sm} correspond to A–orbits of connected components of F_{sm}. This gives $H_m(X_{sm}) \approx H_e(F_{sm})^A$, which restricts to $H_m(X) \approx H_e(F)^A$. $\square$

3.4 **Notation**. a) Let G be any real Lie group, Y a real analytic variety (not necessarily smooth) with a G–action. We assume that G has finitely many orbits on Y. Let $G\backslash Y$ denote the set of orbits of G in Y. We shall identify subsets of $G\backslash Y$ with G–stable subsets of Y.
b) Define a partial order on $G\backslash Y$ by setting

$$O < Q \text{ if } O \subset \bar{Q} - Q$$

where $\bar{Q}$ is the closure of Q in Y. We shall call this the closure order on $G\backslash Y$.
c) For any subset $A \subset G\backslash Y$ define $\partial A \subset G\backslash Y$ by

$$\partial A = \{O \in G\backslash Y : O < Q \text{ for some } Q \in A\}.$$

We also write A' for ∂A, especially if there is risk of confusion with other uses of ∂.
d) For any $A \subset G\backslash Y$, put $A^o = A - \partial A$. Thus A^o consists of the orbits O in A which are not in the closure of any other orbit in A; these are precisely the orbits which are open in A and will be called the leading orbits of A.
e) For any $A \subset G\backslash Y$, write

$$A \supset A' \supset A'' \supset \cdots \supset A^{(k)} \supset A^{(k+1)} \supset \cdots$$

for the chain of subsets on $G\backslash Y$ obtained by repeatedly applying the operation $B \to B' = \partial B$. In particular, we get a filtration on $G\backslash Y$, i.e. a filtration on Y by G–stable subsets, denoted

$$Y \supset Y' \supset Y'' \supset \cdots \supset Y^{(k)} \supset Y^{(k+1)} \supset \cdots .$$

This filtration will be called the closure filtration on $G\backslash Y$.
f) Let $f : X \to Y$ a surjective G–morphism of real analytic G–varieties. For any subset $A \subset G\backslash Y$ let $X_A = f^{-1}(A)$. The closure filtration on $G\backslash Y$ induces a filtration on X,

$$X \supset X' \supset X'' \supset \cdots \supset X^{(k)} \supset X^{(k+1)} \supset \cdots \tag{3}$$

with $X^{(k)} = f^{-1}(Y^{(k)})$.
We now specialize to a very peculiar type of map.

3.5 **Hypotheses.** Let $X \xrightarrow{f} Y$ be a G–morphism of real–analytic G–varieties with the following properties.
a) G has only finitely may orbits on Y.
b) For any $y \in Y$, the component group $A_y = G_y/G_y^o$ of its stabilizer G_y is finite.
c) All $X_O = f^{-1}(O)$ $(O \in G\backslash Y)$ have the same dimension $m = \dim X$.
d) For any $y \in Y$, $(f^{-1}(y))_{si}$ has codimension ≥ 2 in $f^{-1}(y)$.
e) For any $y \in Y$, $H_{e_y - 1}(f^{-1}(y)) = 0$ where $e_y = \dim\, f^{-1}(y)$.

3.6 **Theorem**. a) *The closure filtration on $G\backslash Y$ induces a filtration $H_m(X) \supset H_m(X') \supset \cdots \supset 0$ on $H_m(X)$ whose graded group is*

$$\mathrm{gr}H_m(X) \approx \sum_{O\in G\backslash Y} H_m(X_O). \tag{4}$$

b) For any G–orbit $O = G\cdot y \in G\backslash Y$ one has

$$H_m(X_O) \approx H_{e_y}(f^{-1}(y))^{A_y}. \tag{5}$$

The isomorphism (4) is more precisely described in the following more detailed version of the theorem.

3.6.1 **Theorem**. *For any subset A of $G\backslash Y$ the inclusions $X_{A'} \xrightarrow{i} X_A \xleftarrow{j} X_{A-A'}$ induce an exact sequence*

$$0 \to H_m(X_{A'}) \xrightarrow{i_*} H_m(X_A) \xrightarrow{j^*} H_m(X_{A-A'}) \to 0. \tag{6}$$

and isomorphisms

$$H_m(X_A)/H_m(X_{A'}) \approx H_m(X_{A-A'}) \approx \sum_{O\in A-A'} H_m(X_O).$$

Proof of 3.6.1. a) Let A be *any* subset of $G\backslash Y$ and O a relatively closed orbit in A. The exact sequence 1.12.2(c) of the closed subspace $X_O \subset X_A$ gives an exact sequence

$$H_{m+1}(X_{A-O}) \to H_m(X_O) \to H_m(X_A) \to H_m(X_{A-O}) \to H_{m-1}(X_O). \tag{7}$$

We have $H_{m+1}(X_{A-O}) = 0$ trivially. We show that map $H_m(X_{A-O}) \to H_{m-1}(X_O)$ is zero as well. Every element of $H_m(X_{A-O})$ can be represented by an m–chain Γ on X, contained in $X_{A-O} = X_A - X_O$, with $\partial\Gamma \subset X_O$. As in (1), Γ can be written as a linear combination of oriented m–dimensional components C of $(X_{A-O})_{sm}$, hence $\partial\Gamma$ can be represented as a linear combination of the corresponding $(m-1)$–chains ∂C on X_O. In particular, $\partial\Gamma$ is a G–stable, $(m-1)$–cycle on X_O. Fix $y \in O$, let $F = f^{-1}(y)$, and let $H = G_y$, the stabilizer of y in G. By 3.3, there is a $(e-1)$–chain β on F so that $\partial\Gamma = G\cdot\beta$. Here $e = \dim F$. By hypothesis, $H_{e-1}(F) = 0$, so $\beta = \partial\alpha$ for some e–chain α on F. Let $G_o = \operatorname{supp}\beta$, an $(e-1)$–dimensional closed subanalytic subset of F. Then α determines a class in $H_e(F - G_o)$, because $\partial\alpha = \beta$ has support on G_o. The class of $\beta = \partial\alpha$ in $H_{e-1}(F)$ depends only on the class of α in $H_e(F - G_o)$. Since F_{si} has codimension ≥ 2 in F, we have $H_e(F - G_o) \approx H_e(F_{sm} - F_{sm}\cap G_o)$, hence any element of $H_e(F - G_o)$ can be represented as a linear combination of oriented components of $F_{sm} - F_{sm}\cap G_o$. These components are stable under the connected subgroup H^o of H, because $G_o = \operatorname{supp}\beta$ is stable under H. Hence one can choose an H^o–stable e–chain α on F so that $\beta = \partial\alpha$. Since β is stable under the full group H, and since $A = H/H^o$ is finite, one can can replace α by $|A|^{-1}\sum_{a\in A} a\cdot\alpha$ to obtain an H–stable e–chain α which still satisfies

$\beta = \partial\alpha$. The the G–stable m–chain $G \cdot \alpha$ on X_O which corresponds to α by 3.3 then satisfies $\partial(G \cdot \alpha) = G \cdot (\partial\alpha) = G \cdot \beta = \partial\Gamma$, hence $\partial\Gamma \sim 0$ in $H_m(X_O)$.
Thus (7) reduces to

$$0 \to H_m(X_O) \to H_m(X_A) \to H_m(X_{A-O}) \to 0. \tag{8}$$

This operation may be repeated: if Q is a relatively closed orbit in $A - O$ one gets

$$0 \to H_m(X_Q) \to H_m(X_{A-O}) \to H_m(X_{A-O-Q}) \to 0. \tag{9}$$

The inverse image of $H_m(X_Q) \subset H_m(X_{A-O})$ under the map $H_m(X_A) \to H_m(X_{A-O})$ is $H_m(X_{O\cup Q}) \subset H_m(X_A)$. (We note that $O \cup Q$ is closed in A, since $A - O - Q$ is open in $A - O$ and hence in A.) Thus from (8) and (9) one gets

$$0 \to H_m(X_{O\cup Q}) \to H_m(X_A) \to H_m(X_{A-O-Q}) \to 0. \tag{10}$$

Successively subtracting the orbits in A' from A in such a way that at each step the orbit is closed in the set from which it is subtracted one finds an exact sequence

$$0 \to H_m(X_{A'}) \to H_m(X_A) \to H_m(X_{A-A'}) \to 0. \tag{11}$$

By the definition of A', the orbits O in $A - A'$ are open in $A - A'$, hence $X_{A-A'}$ is a disjoint union of the open subsets X_O, $O \in A - A'$. This implies that

$$H_m(X_{A-A'}) = \sum_{O \in A-A'} H_m(X_O),$$

and proves 3.6.1. $\square$

Proof of theorem 3.6 a) This is a consequence of 3.6.1, together with 3.2(a), which insures that the $H_m(X^{(k)})$ can be considered as subgroups of $H_m(X)$.
c) This follows from 3.3 applied to $f : X_O \to O$. $\square$

3.7 For reference we record some simple observations related to the theorem.
a) In 3.6.1 one can replace A' by any B with $A' \subset B \subset A$ to get

$$0 \to H_m(X_B) \xrightarrow{i_*} H_m(X_A) \xrightarrow{j^*} H_m(X_{A-B}) \to 0. \tag{12}$$

This is clear from the proof and follows from the theorem as stated.
b)In 3.6 (a) one can replace the filtration $Y^{(k)}$ on Y by any G–stable filtration with the property that for all k

$$Y^{(k+1)} - Y^{(k)} \text{ is a union of relatively open orbits.}$$

For example, one can take for $Y^{(k)}$ the union of the orbits of codimension k in X. The theorem remains valid for the corresponding filtration $X^{(k)} = f^{-1}(Y^{(k)})$ of X. The filtration defined in 3.4 is the coarsest one with the above property, hence gives the strongest assertion in part (a) of the theorem.
c) The proof of the theorem requires that the order $|A|$ of the component group defined there be invertible in the coefficient ring of the homology and it is for this reason that the coefficients were taken from $\mathbf{Q}$ rather than from $\mathbf{Z}$. One can naturally formulate a more precise statement by keeping track of these denominators, but we shall not bother to do so. Similar remarks apply elsewhere.

3.8 **Definition**. For any m–chain Γ on X we set $A(\Gamma) = \overline{f(\mathrm{supp}\Gamma)} \subset G\backslash Y$.

3.9 **Lemma**. *Let $\Gamma \in H_m(X)$.*
a) Then there is a unique decomposition

$$\Gamma = \sum_{O \in G\backslash Y} \Gamma_O$$

where Γ_O is a G–invariant m–chain on X_O .
b) $A(\Gamma)$ is the unique minimal closed subset $A \subset G\backslash Y$ so that $\Gamma \in H_m(X_A)$.
c) If $O = G \cdot y \in G\backslash Y$ is a leading orbit in $A(\Gamma)$, then $\Gamma_O \in H_m(X_O)$ is an m–cycle on X_O and $\Gamma_y := \Gamma_O \cap f^{-1}(y) \in H_{e_y}(f^{-1}(y))^{A_y}$ is an e_y–cycle on $f^{-1}(y)$, $e_y = \dim f^{-1}(y)$.

Proof. a) Let $Z = \mathrm{supp}\Gamma$. The disjoint decomposition $X = \bigcup_{O \in G\backslash Y} X_O$ gives a disjoint decomposition $Z = \bigcup_{O \in G\backslash Y} Z_O$ and since $\dim X_O = m$ for all O, this induces a unique decomposition $\Gamma = \sum_{O \in G\backslash Y} \Gamma_O$ where Γ_O is an m–chain on X_O with support C_O. Since any m–cycle Γ on X is a linear combination of connected components of X_{sm} as in (1), it is G–invariant, hence so are the m–chains Γ_O.
b) From its definition, $A(\Gamma)$ is the unique minimal closed subset A of $G\backslash Y$ so that $\mathrm{supp}\Gamma \subset X_A$. This implies the assertion.
c) If O is leading orbit in $A(\Gamma)$, then X_O is open in X_A and Γ_O is the image of Γ under the natural map $H_m(X_A) \to H_m(X_O)$ of 1.12.2 (a). The last assertion follows from 3.3. $\square$

3.10 For an arbitrary orbit O the m–chain Γ_O on X_O need not be a cycle. But if $\Gamma \in H_{2n}(X_A)$, with $A \subset G\backslash Y$ closed, and if O is a leading orbit in A, then Γ_O is a cycle on X_O : it coincides with the image of Γ under the map in the exact sequence (12) with $B = A - O$:

$$0 \to H_m(X_{A-O}) \to H_m(X_A) \to H_m(X_O) \to 0. \tag{13}$$

3.11 **Conormal varieties.** One situation of interest in which the theorem applies concerns the conormal variety of an action with finitely many orbits. The formal definition reads as follows.

3.11.1 **Definition**. *Let G be a Lie group acting on a manifold M. The conormal variety $\mathcal{C}$ of this action is subvariety of the cotangent bundle T^*M defined by*

$$\mathcal{C} = \{(x,\xi) \in T^*M : \langle \xi, X \cdot x\rangle = 0 \textit{ for all } X \in \mathfrak{g}\}.$$

Thus $\mathcal{C}$ is the union of the conormal bundles $\mathcal{C}_O$ of the G–orbits on M.
We observe that the hypotheses of the theorem are satisfied in the following case.

3.11.2 **Lemma**. *Let G be a Lie group acting on a manifold M, $\mathcal{C}$ the conormal variety. Suppose G has only finitely many orbits on M and the isotropy groups have only finitely many connected components. Then the projection $\pi : \mathcal{C} \to M$ satisfies the hypotheses 3.5.*

Proof. Hypotheses a) and b) hold by assumption.
Hypothesis c) follows from the fact that $\pi^{-1}(O) = \mathcal{C}_O$ ($O \in G\backslash M$), the conormal bundle of O.
Hypothesis d) is trivially satisfied since $\pi^{-1}(y) \approx \mathbf{R}^n$.
Hypothesis e) follows from the fact that for homology with arbitrary supports one has

$$H_k(\mathbf{R}^n) = \begin{cases} \mathbf{Q} \text{ for } k = n \\ 0 \text{ otherwise.} \end{cases} \qquad \square$$

4. The homology of the variety $\mathcal{S}$ and representations of Weyl groups

We now return to the variety $\mathcal{S}$ of 2.6.

4.1 **Lemma**. *The two G_o–maps $\mathcal{S} \to \mathcal{B}$ and $\mathcal{S} \to \mathcal{N}(\mathfrak{g}_o^\perp)$ satisfy the hypotheses 3.5.*

Proof. For the map $\mathcal{S} \to \mathcal{B}$ this follows from 3.11.2. It remains to consider the map $\mathcal{S} \to \mathcal{N}(\mathfrak{g}_o^\perp)$. Over any G_o–orbit $O \in H\backslash\mathcal{N}(\mathfrak{g}_o^\perp)$, the map $\mathcal{S} \to \mathcal{N}(\mathfrak{g}_o^\perp)$ restricts to G_o–equivariant fibration

$$\mathcal{B}^\nu \overset{\subseteq}{\to} \mathcal{S}_O \to O \tag{1}$$

where the fibre $\mathcal{B}^\nu$ over $\nu \in O$ is the *Springer variety*

$$\mathcal{B}^\nu = \{x \in \mathcal{B} :\ \nu \in (\mathfrak{g}/\mathfrak{b}_x)^*\}.$$

That hypothesis (c) is satisfied, i.e. that

$$\dim_{\mathbf{R}} \mathcal{S}_O = 2n, \tag{2}$$

is seen as follows. From (1) it follows that for $O = G_o \cdot \nu$ one has

$$\dim_{\mathbf{R}} \mathcal{S}_O = \dim_{\mathbf{R}} G_o \cdot \nu + \dim_{\mathbf{R}} \mathcal{B}^\nu. \tag{3}$$

Results of Spaltenstein and Steinberg imply that

$$\dim_{\mathbf{C}} G \cdot \nu + 2 \dim_{\mathbf{C}} \mathcal{B}^\nu = 2n, \tag{4}$$

cf. [Rossmann, 1990, Lemma 4.4]. Furthermore,

$$\dim_{\mathbf{C}} G \cdot \nu = \dim_{\mathbf{R}} G_o \cdot \nu; \tag{5}$$

this evident when $\mathfrak{g}_o$ is a real form of $\mathfrak{g}$ and follows from a result of Kostant and Rallis [1971, p.770] for any symmetric subalgebra $\mathfrak{g}_o$ of $\mathfrak{g}$. Hence for $O = G_o \cdot \nu$

$$\dim_{\mathbf{R}} \mathcal{S}_O = \dim_{\mathbf{R}} G_o \cdot \nu + \dim_{\mathbf{R}} \mathcal{B}^\nu = \dim_{\mathbf{C}} G \cdot \nu + 2 \dim_{\mathbf{C}} \mathcal{B}^\nu = 2n$$

as required. The hypothesis (d) is clear, since $\mathcal{B}^\nu$ is a *complex* algebraic variety, and (e) follows from the fact that

$$H_k(\mathcal{B}^\nu) = 0 \text{ for odd } k,$$

which is proved in [De Concini, et al. , 1988] for integral homology, and was proved earlier for rational homology by Shoji and by Beynon and Spaltenstein. □

For reference we state explicitly the content of the theorem for the two maps 2.6, eq.(13). For the map $\pi : \mathcal{S} \to \mathcal{B}$ we only point out one consequence.

4.2 **Theorem**. *The dimension of $H_{2n}(\mathcal{S})$ equals to the number of G_o–orbits on $\mathcal{B}$:*

$$H_{2n}(\mathcal{S}) \overset{\mathbf{Q}}{\approx} \mathbf{Q}[G_o \backslash \mathcal{B}]. \tag{6}$$

Proof. By 3.2, any $2n$–chain on $\mathcal{S}$ is uniquely a linear combination of of the chains $[\mathcal{S}_Q]$, $Q \in G_o \backslash \mathcal{B}$, defined by the oriented conormals of the H–orbits $\mathcal{S}$. By 3.6.1 each chains $[\mathcal{S}_Q\ ]$ can be completed to a cycle of the form $[\mathcal{S}_Q] + \cdots$ where the dots denote a linear combination of cycles $[\mathcal{S}_R]$ with $R < Q$. It clear from part (a) of 3.6 that these cycles then from a $\mathbf{Q}$–basis of $H_{2n}(\mathcal{S})$. □

For the map $\rho : \mathcal{S} \to \mathcal{N}(\mathfrak{g}_o^\perp)$ we restate theorem 3.6 in detail.

4.3 **Theorem**. *a) The closure filtration on $H \backslash \mathcal{N}(\mathfrak{h}^\perp)$ induces a filtration on $H_{2n}(\mathcal{S})$ whose graded group is*

$$\mathrm{gr} H_{2n}(\mathcal{S}) \approx \sum_{O \in G_{\mathbf{R}} \backslash \mathcal{N}(\mathfrak{g}_o^\perp)} H_{2n}(\mathcal{S}_O). \tag{7}$$

b) For any G–orbit $O = H \cdot \nu \in H \backslash \mathcal{N}(\mathfrak{g}_o^\perp)$ one has

$$H_{2n}(\mathcal{S}_O) \approx H_{e_\nu}(\mathcal{B}^\nu)^{A_\nu}. \tag{8}$$

Again we add the

4.3.1 **Addendum to 4.3**. *For any subset A of $G_o \backslash \mathcal{N}(\mathfrak{g}_o^\perp)$ the inclusions $\mathcal{S}_{A'} \xrightarrow{i} \mathcal{S}_A \xleftarrow{j} \mathcal{S}_{A-A'}$ induce an exact sequence*

$$0 \to H_{2n}(\mathcal{S}_{A'}) \xrightarrow{i_*} H_{2n}(\mathcal{S}_A) \xrightarrow{j^*} H_{2n}(\mathcal{S}_{A-A'}) \to 0 \tag{9}$$

and isomorphisms

$$H_{2n}(\mathcal{S}_A)/H_{2n}(\mathcal{S}_{A'}) \approx H_{2n}(\mathcal{S}_{A-A'}) \approx \sum_{O \in A-A'} H_{2n}(\mathcal{S}_O). \tag{10}$$

4.4 **Restricted monodromy representations**. Let $V \subset \mathcal{N}(\mathfrak{g}_o^{\perp})$ be a subset of $\mathcal{N}(\mathfrak{g}_o^{\perp})$ which can be realized as a relative subcomplex of some triangulation of the pair of projective varieties $(\mathcal{N}^c, \mathcal{N}^c - \mathcal{N})$. Then the construction of 1.9.3 gives a homomorphism of W into the group of homotopy equivalences of $\mathcal{S}_V = \rho^{-1}(V)$. Thus we have representations of W on all of the homology groups occurring in 4.3. (The comments in 1.10.4 are relevant here.) In particular for $V = \{\nu\}$ we get $\mathcal{S}_V = \mathcal{B}^\nu$, and $H_.(\mathcal{S}_V) = H_.(\mathcal{B}^\nu)$, the usual homology with compact supports. In this case the whole construction is independent of the subalgebra $\mathfrak{g}_o$ which enters into the support condition. Thus we get a representation of W on $H_.(\mathcal{B}^\nu)$ for any $\nu \in \mathcal{N}$; it commutes with the natural representation of the complex component group $A_\nu := G_\nu/G_\nu^o$. In top degree $2e = \dim_{\mathbf{R}} \mathcal{B}^\nu$ these are the *Springer representations* constructed in [Springer, 1978] by other methods.

4.4.1 **Addendum to Theorem 4.3** *All maps in Theorem 4.3 are W–equivariant for the restricted monodromy representations. Furthermore,*

$$H_{2n}(\mathcal{S}_O) \approx H_{2e}(\mathcal{B}^\nu)^{A_{o,\nu}}. \tag{11}$$

Proof. All of the restricted monodromy representation can the be realized by the same operators $a_\lambda(w)$ according to 1.9.3. The W–equivariance of the maps in Theorem 4.3 follows. The isomorphism (11) is given by $\Gamma \to \Gamma \cap \mathcal{B}^\nu$ as in 3.3. The group $G_{o,\nu}/G_{o,\nu}^o$ may here be replaced by its image in the complex component group G_ν/G_ν^o : its representation on $H_{2e}(\mathcal{B}^\nu)$ factors through this map, because $H_{2e}(\mathcal{B}^\nu)$ is spanned by the fundamental cycles of the irreducible components of complex variety $\mathcal{B}^\nu$, and these are cycles fixed by the connected group G_ν^o. □

We recall Springer's fundamental result concerning the representations of W on the $H_{2e}(\mathcal{B}^\nu)$:

4.5 **Theorem** (Springer). *The representation of W on the A_ν–isotypic component $H_{2e}(\mathcal{B}^\nu)_\phi$ of $H_{2e}(\mathcal{B}^\nu)$ transforming according to an irreducible character ϕ of A_ν is an irreducible representation of W. Let $\chi_{\nu,\phi}$ be its character. Then every irreducible character of W is of the form $\chi_{\nu,\phi}$ for a pair (ν, ϕ), which is unique up to conjugacy under G.*

Since it is possible to deduce this theorem from results proved here by a short and instructive argument (also due to Springer, unpublished), we include a proof.

Proof of the theorem. We apply the construction for $(\mathfrak{g}, \mathfrak{g}_o)$ replaced by $(\mathfrak{g} \times \mathfrak{g}, \mathrm{diag}\mathfrak{g})$, with embedded as the diagonal. Theorem 4.3 then gives

$$H_{2n}(\mathcal{S}) \approx \sum_\nu [H_{2e}(\mathcal{B}^\nu) \otimes H_{2e}(\mathcal{B}^\nu)]^{A_\nu} \tag{12}$$

with $A_\nu = G_\nu/G_\nu^o$ acting diagonally and ν running over a set of representatives of the G–orbit in $\mathcal{N} = \mathcal{N}(\mathfrak{g}^*)$. The decomposition is a $(W \times W)$– isomorphism. Since the G–orbits on $\mathcal{B} \times \mathcal{B}$ are in this case in one–to–one correspondence with W, the isomorphism (6), becomes

$$H_{2n}(\mathcal{S}) \approx \mathbf{Q}[W]. \tag{13}$$

This is a $W \times W$–isomorphism for the biregular representation on $\mathbf{Q}[W]$, as follows from the discussion in example 2.6.4. (An analytic proof can be found in [Rossmann, 1991]). Comparison of (12) with the familiar decomposition

$$\mathbf{C}[W] \approx \sum \chi_k \otimes \chi_k,$$

χ_k running over the irreducible characters of W, gives the theorem. □

4.6 For a given ν, not all characters ϕ of A_ν need occur in (12). If those that do are denoted Φ_ν, then the proof shows more precisely that the irreducible characters of W are indexed by G–conjugacy classes of pairs (ν, ϕ), $\nu \in \mathcal{N}$, $\phi \in \Phi_\nu$. There are several other constructions of the Springer representations of W on the $H_{\cdot}(\mathcal{B}^\nu)$. The one of Slodowy [1980 (1)] is similar to the one given here, and we verify in some detail the following comparison lemma, because the proof will exhibit an important localization property of the restricted monodromy representation to a neighbourhood of $\mathcal{B}^\nu$.

4.7 **Lemma**. *The restricted monodromy representation on $H_{\cdot}(\mathcal{B}^\nu)$ defined above is equivalent to the representation defined in* [Slodowy, 1980 (1)].

Proof. Fix $\nu \in \mathcal{N}$. A local trivialization of $\tilde{\mathfrak{g}}^* \xrightarrow{\tilde{q}} \mathfrak{h}*$ around $\mathcal{B}^\nu = \rho^{-1}(\nu)$ can be constructed as follows, cf. [Slodowy, 1980 (1)]. Let $S \subset \mathfrak{g}^*$ be a local transversal slice to the complex orbit $G \cdot \nu$; thus the action map $G \times S \to \mathfrak{g}^*$ is a submersion near ν and one can find a submanifold patch V at the identity in G so that locally around ν one has $V \times S \approx \mathfrak{g}^*$. By G–equivariance of $\tilde{\mathfrak{g}}^* \to \mathfrak{g}^*$ one also has locally $V \times \tilde{S} \approx \tilde{\mathfrak{g}}^*$ where $\tilde{S}$ is the preimage of S. In particular, $\tilde{S}$ is smooth and $\tilde{S} \to \mathfrak{h}^*$ a submersion. This map admits a trivialization $F \times \mathfrak{h}^* \approx \tilde{S}$, locally in a neighbourhood of $\mathcal{B}^\nu$. Hence we get locally $V \times F \times \mathfrak{h}^* \approx V \times \tilde{S} \approx \tilde{\mathfrak{g}}^*$, which gives the local trivialization of $\tilde{\mathfrak{g}}^* \xrightarrow{\tilde{q}} \mathfrak{h}^*$ around $\mathcal{B}^\nu$. We may assume that V is contractible (say a ball), and may then as well be omitted as far as the monodromy is concerned. We are now dealing with $\tilde{S} \xrightarrow{\tilde{q}} \mathfrak{h}^*$ and its local trivialization $F \times \mathfrak{h}^* \approx \tilde{S}$. As before, we obtain from it a homomorphism of W into the group of homotopy equivalences of $\mathcal{B}^\nu \subset \tilde{S}$. This is the construction in [Slodowy, 1980 (1)]. The difference between the two constructions is that one uses a trivialization of $\tilde{\mathfrak{g}}^* \xrightarrow{\tilde{q}} \mathfrak{h}^*$ defined over a whole neighbourhood of 0 in $\mathfrak{h}^*$, the other a trivialization defined only in some neighbourhood of $\mathcal{B}^\nu$ in $\tilde{\mathfrak{g}}^*$ or in $\tilde{S}$. But this is of no consequence as far as the homomorphism of W into the group of homotopy equivalences of $\mathcal{B}^\nu$ is concerned: a given trivialization in a neighbourhood of $\mathcal{B}^\nu$ in $\tilde{\mathfrak{g}}^*$ can always be extended to a trivialization defined everywhere over a neighbourhood of 0 in $\mathfrak{h}^*$ (see 1.10.2). The corresponding transformations $a_\lambda(w)$ defined in 1.9.3 then agree near $\mathcal{B}^\nu$ and therefore induce the same homomorphism of W into the group of homotopy equivalences of $\mathcal{B}^\nu$. $\square$

4.8 The lemma makes the connection between Springer's Theorem 4.5 and the construction in [Slodowy, 1980 (1)], a point left open there. The crux is that the $H_e(\mathcal{B}^\nu)$ appear in the decomposition (7)$-$(8) of $H_{2n}(\mathcal{S})$. The whole theory is naturally just an elaboration of the heuristic principle of 1.1 for the deformation of the Ω_λ into $\Omega_0 = \mathcal{N}$, or rather of $\mathcal{S}$ into $\mathcal{N}(\mathfrak{g}_o^\perp)$, based on closure filtration as in 3.4 (f).

4.9 **Example**. The filtration of the representation of W on $H_{2n}(\mathcal{S})$ induced by the closure order on $H\backslash\mathcal{N}(\mathfrak{g}_o^\perp)$ can be quite intricate combinatorially. The simplest case is $\mathfrak{g} = \mathbf{pgl}(n, \mathbf{C}), \mathfrak{g}_o = \mathbf{pgl}(n, \mathbf{R})$. In this case each term in the decomposition in the decomposition (7) is irreducible: the representation of $W = S_n$ in $H_{2e}(\mathcal{B}^\nu)$ is the irreducible representation of S_n associated to the partition of n given by the size of the Jordan blocks of ν, with $n = 1 + \cdots + 1$ corresponding to the sign representation and $n = n$ to the trivial representation. It can be shown that the representation of S_n in the complete homology $H_.(\mathcal{B}^\nu)$ is in this case induced from the trivial representation of the subgroup of W corresponding to a Levi subgroup of G_o in which ν is the regular nilpotent. (For arbitrary G and ν it can be shown to be induced from a distinguished nilpotent.) We observe that for $G_o \cdot \nu' < G_o \cdot \nu$ in $H\backslash\mathcal{N}(\mathfrak{g}_o^\perp)$ there is a canonical map

$$H_.(\mathcal{B}^\nu) \to H_.(\mathcal{B}^{\nu'}) \tag{14}$$

defined as follows. Choose sufficiently small neighbourhood V of ν' in $\mathcal{N}(\mathfrak{g}_o^\perp)$ so that the inclusion $\mathcal{B}^{\nu'} \subset \rho^{-1}(V)$ induces an isomorphism in homology. Since$G_o \cdot \nu' < G_o \cdot \nu$ one can choose $\nu \in V$ and then gets a map (14) induced by the inclusion $\mathcal{B}^\nu \subset \rho^{-1}(V)$. This map exists in general. For the present example it is the well–known map which is the subject of the so–called Snapper conjecture. (Cf. [Lam, 1977]. This follows from [Hotta and Springer, 1977], Corollary 2.3, as pointed out by the referee.)

4.10 **Duality**. As remarked earlier, the involutions σ of $\mathfrak{g}$ (assumed to commute with the given Cartan involution τ) come naturally in pairs σ_+, σ_- so that $\sigma_+\sigma_- = \tau$. Fix such a pair and write $\mathfrak{g}_+$, $\mathfrak{g}_-$ for the corresponding fixed subalgebras $\mathfrak{g}_o$, and $\mathcal{S}_+, \mathcal{S}_-$ for the conormal varieties $\mathcal{S}$. As a consequence of 4.3 and 4.4.1 we have the

4.11 **Corollary**. $H_{2n}(\mathcal{S}_+) \approx H_{2n}(\mathcal{S}_-)$ *as W–modules.*

Proof. According to a well–known result of Kostant in the $\mathfrak{g}_\mathbf{R}, \mathfrak{k}_\mathbf{C}-$ case, and of [Sekiguchi, 1987] in general, there is a one–to–one correspondence $O_+ \leftrightarrow O_-$ between G_+–orbits in $\mathcal{N}(\mathfrak{g}_+^\perp)$ and G_-– orbits in $\mathcal{N}(\mathfrak{g}_-^\perp)$: $O_+ \leftrightarrow O_-$ it they belong to a common complex orbit $G \cdot \nu$; both kinds of orbits are in one–to–one correspondence with so–called *normal* $\mathbf{sl}_2$–triples. The centralizer G^ν of ν in G admits a semidirect decomposition $G_\nu = G_\phi V_\nu$ where V_ν is a connected unipotent normal subgroup and G_ϕ is the centralizer of the whole $\mathbf{sl}_2$–triple ϕ belonging to ν, a reductive group. ([Collingwood–McGovern, 1993], 3.7.3.) The component group of $G_\pm \cap G_\nu$ is the same as that of the group $G_\pm \cap G_\phi$. Since the latter group is reductive, its component group is the same as that of a maximal compact subgroup ([Wallach, 1988], 2.1.8), which can be taken to be $K \cap G_\phi$ for both $G_+ \cap G_\phi$ and $G_- \cap G_\phi$. Hence the groups $A_{o,\nu}$ in 4.4.1 are the same and the representations of W in the spaces $H_{2n}(\mathcal{S}_O) \approx H_{2n}(\mathcal{B}^\nu)^A_{o,\nu}$ are isomorphic in the two cases. $\square$

4.11.1 We refer to [Kostant and Rallis, 1971] for the theory of normal $\mathbf{sl}_2$–triples and to [Sekiguchi, 1987] for details on the orbit correspondence on $\mathcal{N}$. An exposition of this theory can also be found in [Collingwood and McGovern, 1993]. There is an analogous orbit correspondence on $\mathcal{B}$, due to [Wolf, 1969] in a special case and to [Matsuki, 1979] in general. See also [Mircović et al. , 1992]. The two correspondences should be related through the conormal varieties in $\mathcal{B}^*$, but I am not aware of any results which would the explain the above corollary more directly.

4.12 **Example**. We consider the case $\mathfrak{g} = \mathbf{sl}(2, \mathbf{C})$. This example has some significance for the general theory, because around a generic critical point of $q : \mathfrak{g}^* \to W\backslash\mathfrak{h}^*$, corresponding to the vanishing of a single root, the singularity of q has the same type as the singularity 0 for $\mathbf{sl}(2, \mathbf{C})$, as will be discussed in the next section.
The coadjoint quotient $q : \mathfrak{g}^* \to W\backslash\mathfrak{h}^*$ can be identified with quadratic function $q : \mathbf{C}^3 \to \mathbf{C}$,

$$q(\xi) = \xi_1\xi_2 - \xi_3^2$$

with $\xi = (\xi_1, \xi_2, \xi_3)$, a paradigm of Picard–Lefschetz theory, cf. [Arnold et al. 1988, Chap. I]. The monodromy representation in the homology of the standard fibre $\Omega \approx \mathcal{B}^*$ in degree $2 = \dim_{\mathbf{C}} \Omega$ is given by the classical *Picard–Lefschetz Theorem* [Lefschetz, II.8; V.6–7; Arnold et al. , 1988, Chap. I],

$$s \cdot \Gamma \sim \Gamma + \langle \Gamma, \Delta \rangle \Delta. \tag{15}$$

Here Γ is a cycle with arbitrary support on $\mathcal{B}^*$. The cycle Δ is the classical Picard–Lefschetz *vanishing cycle*; it is the fundamental cycle of the fibre S^2 of the limit map $p_0 : \mathcal{B}^* \to \mathcal{N}$ over the critical point $0 \in \mathcal{N}$, i.e. the fundamental cycle of the zero section in $\mathcal{B}^*$ as cotangent bundle of $\mathcal{B} = \mathbf{P}^1$. The pairing $\langle \Gamma, \Delta \rangle$ is the intersection index. Since $\langle \Delta, \Delta \rangle = -2$, the monodromy action of s can be viewed as a reflection along the vanishing cycle. The operator $\Gamma \to (s-1)\Gamma$ is the classical *variation operator*.
The formula (15) is usually understood as a homology relation, and its specific interpretation as such depends on the family of supports defining the homology. It is actually induced by a proper homotopy on Ω, and therefore holds as an identity of homology classes for any family containing the compact sets.
We now specify three subalgebras $\mathfrak{g}_o$ of $\mathfrak{g}$ and consider the representation of $W =$

$\{1, s\}$ in $'H_2(\Omega) \approx H_2(\mathcal{S})$. In each case we exhibit a homeomorphic image of the variety $\mathcal{S}$ and a basis for $H_2(\mathcal{S})$; this is done in pictures to avoid elaborate explanations. We also write out the action of s on the basis in question.

a) $\mathfrak{g}_o = \mathfrak{b}$, a Borel subalgebra of $\mathfrak{g}$. The cycles Δ and ∇ mentioned above both live in $'H_2(\mathcal{S})$ and form a basis for it. Cf. Fig. (4.1). The transformation formulas for the basis $\{\Delta, \nabla\}$ are

$$s \cdot \Delta = -\Delta, \quad s \cdot \nabla = \nabla + \Delta,$$

in agreement with the Picard–Lefschetz formula (15); but now the cycles Δ and ∇ live in the same homology group.

b) $\mathfrak{g}_o = \mathfrak{k}_{\mathbf{C}} = \mathbf{sl}(2, \mathbf{C})$. The transformation formulas for the basis $\{\Gamma_+, \Gamma_-, \Delta\}$ of $H_2(\mathcal{S})$ are:

$$s \cdot \Delta = -\Delta, \quad s \cdot \Gamma_+ = \Gamma_+ + \Delta, \quad s \cdot \Gamma_- = \Gamma_- + \Delta. \tag{16}$$

Cf. Fig. (4.2). To together with the fundamental cycle Γ_0 of the whole variety $\mathcal{S}$, appropriately oriented, we evidently have the relation

$$\Gamma_0 = \Gamma_+ + \Gamma_- + \Delta. \tag{17}$$

c)$\mathfrak{g}_o = \mathfrak{g}_{\mathbf{R}} = \mathbf{sl}(2, \mathbf{R})$. In this case one can easily visualize the degeneration of $\mathcal{S}$ into its image under the limit map, which here lies in the real subspace $i\mathbf{R}^3$. Cf. Fig. (4.3). $H_2(\mathcal{S})$ has rank three as in (b). We again single out four particular elements of $H_2(\mathcal{S})$; they satisfy the same relation as those in (b):

$$\tilde{\Gamma}_0 = \tilde{\Gamma}_+ + \tilde{\Gamma}_- + \Delta \tag{18}$$

Cf. Fig. (4.4). The transformation formulas for the basis $\{\tilde{\Gamma}_+, \tilde{\Gamma}_-, \Delta\}$ of $H_2(\mathcal{S})$ are also the same as those in (b):

$$s \cdot \Delta = -\Delta, \quad s \cdot \tilde{\Gamma}_+ = \tilde{\Gamma}_+ + \Delta, \quad s \cdot \tilde{\Gamma}_- = \tilde{\Gamma}_- + \Delta. \tag{19}$$

4.12.1 Under the correspondence between cycles and characters mentioned in the introduction, the relation (19) corresponds to the familiar decomposition of principal series characters of SL(2,$\mathbf{R}$). The examples (b) and (c) illustrate the "$K_{\mathbf{C}}, G_{\mathbf{R}}$–duality" in this case.

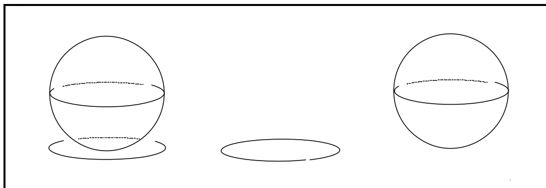

Fig. (4.1). The variety $\mathcal{S}$ and two cycles ∇, Δ.

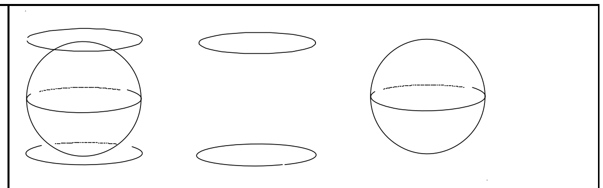

Fig. (4.2). The variety $\mathcal{S}$ and three cycles $\Gamma_{\pm}$, Δ.

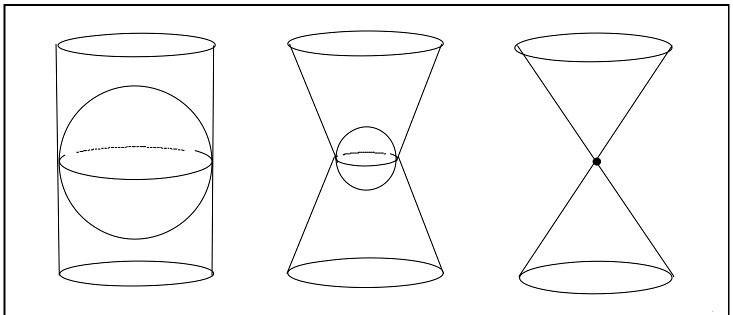

Fig. (4.3). The variety $\mathcal{S}$ and its image under the limit map

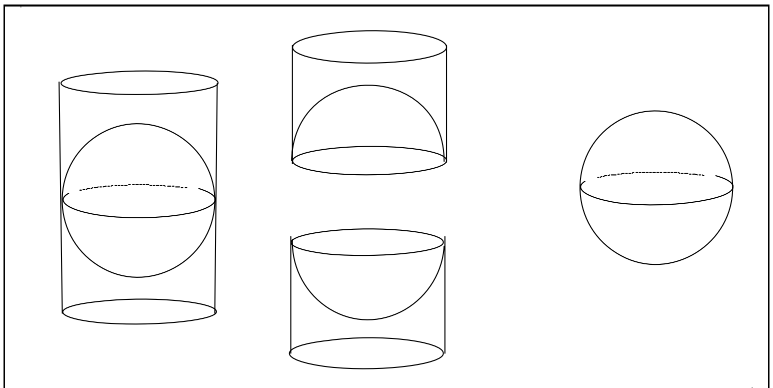

Fig. (4.4). Four cycles on $\mathcal{S} : \tilde{\Gamma}_0 = \tilde{\Gamma}_+ + \tilde{\Gamma}_- + \Delta$.

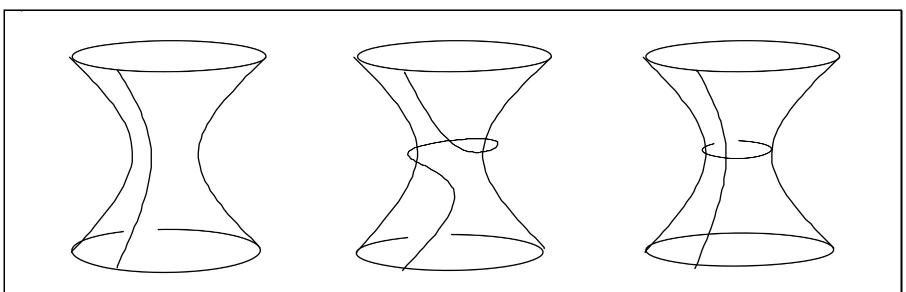

Fig. (5.1). $\Gamma \to s\Gamma \sim \Gamma \pm \langle \Gamma, \Delta \rangle \, \Delta$.

5. A Picard–Lefschetz theorem for simple reflections

5.1 **Preliminaries**. The classical Picard–Lefschetz Theorem concerns the monodromy of an isolated quadratic singularity, which one may take to be given by the function $q(x) = x_0^2 + \cdots + x_m^2$ at $x = 0$ in $\mathbf{C}^{m+1}$. It says that the generator s of $\pi_1(\mathbf{C} - \{0\})$ acts on m–cycles Γ in the standard fibre $\Omega \approx q^{-1}(1)$ according to the homology relation

$$s\Gamma \sim \Gamma + (-1)^{(m+1)(m+2)/2} \langle \Gamma, \Delta \rangle \Delta \tag{1}$$

where Δ is the *vanishing cycle*, represented by the real m–sphere in $q^{-1}(1)$. The pairing $\langle \Gamma, \Delta \rangle$ is the intersection index. For $m = 1$ the situation can be visualized as in Fig. (5.1), which brings out an important point: the cycle $s\Gamma$ on Ω is homotopic to Γ, but this homotopy does not fix the part at infinity and may therefore change a homology class with non–compact supports as for $'H(\Omega)$.

For even m the formula (1) represents a reflection along Δ, since then

$$(-1)^{(m+1)(m+2)/2} = \frac{-2}{\langle \Delta, \Delta \rangle}.$$

The formula (1) applies to the coadjoint quotient of $\mathbf{sl}(2, \mathbf{C})$ with $m = 2$, for Γ in any one of the homology groups $'H_2(\Omega)$ under consideration; this is illustrated in example 4.13. Our Picard–Lefschetz Theorem for a simple reflection s will be derived by a reduction to that case: we prove that s acts on $H_{2n}(\mathcal{S})$ as a reflection along a space of vanishing cycles. Here and generally we understand by a reflection along a subspace S of a vector space V a linear transformation s satisfying $s^2 = 1$ which acts by -1 on S and by $+1$ on V/S. The same terminology applies to modules over rings.

We are concerned with the monodromy of $\mathfrak{g}^* \to W \backslash \mathfrak{h}^*$ around a point where exactly one root vanishes. Let $\alpha \in \mathfrak{h}^*$ be a simple root for $(\mathfrak{g}, \mathfrak{h}, \mathfrak{b}), H_\alpha \in \mathfrak{h}$ the coroot, and $s = s_\alpha$ the reflection along α. Fix an element λ_o of $\mathfrak{h}^*$ orthogonal to H_α, but not orthogonal to any other coroot. When $\lambda = \lambda_o$, we write

$$p_o : \mathcal{B}^* \to \Omega_o. \tag{2}$$

for $p_\lambda : \mathcal{B}^* \to \Omega_\lambda$. It plays the rôle of the limit map at λ_o in the sense of 1.9. We again assume given a subalgebra $\mathfrak{g}_o$ of $\mathfrak{g}$, so that the homology $'H_{2n}(\mathcal{B}^*)$ is defined and isomorphic to the top homology $H_{2n}(\mathcal{S})$ of the conormal variety $\mathcal{S}$ (2.6.1). The subalgebra $\mathfrak{g}_o$ need not be symmetric (2.6.6), but we assume that the corresponding group G_o has finitely many orbits on the flag manifold $\mathcal{B}$.

We define a $2n$–dimensional closed subvariety $\mathcal{S}_o$ of $\mathcal{S}$ which will play the rôle of the vanishing m–sphere in the classical Picard–Lefschetz formula (1).

5.1.1 **Definition**. Let $\mathcal{S}_o = p_o^{-1}(O_o) \cap \mathcal{S}$, where $p_o : \mathcal{B}^* \to \Omega_o$ is the limit map for λ_o, as in (2).

We identify $H_{2n}(\mathcal{S}_o)$ with a subspace of $H_{2n}(\mathcal{S})$. Our Picard –Lefschetz theorem for s then reads:

5.1.2 **Theorem**. *The monodromy action of s in $H_{2n}(\mathcal{S})$ is a reflection along the subspace $H_{2n}(\mathcal{S}_o)$ of $H_{2n}(\mathcal{S})$.*

We shall actually prove a more precise, but also more technical, version of this theorem, stated in 5.8 below. We first explain the simple principle (5.1.3) underlying the theorem; the rest is devoted to the proof of the theorem in its formulation 5.8.

Superseding previous conventions, let $\tilde{Q}$ denote a small disc around λ_o in $\lambda_o + \mathbf{C}H_\alpha \approx \mathbf{C}$ and Q its image in $W\backslash\mathfrak{h}^*$. Then $Q \approx W_s\backslash\tilde{Q}$ where $W_s = \{1, s\}$ and $\tilde{Q} \to Q$ is a branched double covering, whose branch point is the image of λ_o, still denoted λ_o. Let $M \subset \mathfrak{g}^*$ be the connected component of $q^{-1}(Q)$ containing λ_o and $\tilde{M}$ a connected component of its inverse image in $\tilde{\mathfrak{g}}^*$. Then we again have a diagram as in Theorem 1.1.1, the critical fibre under consideration being $q^{-1}(\lambda_o) = \Omega_o$; it contains $O_o := G\cdot\lambda_o$ as unique closed G– orbit. Let $\mathfrak{g}_s$ be the centralizer of λ_o in $\mathfrak{g}$, so that $(\mathfrak{g}_s/\text{centre}) \approx \mathbf{sl}(2, \mathbf{C})$. We write $U_s, \mathcal{B}_s, \mathcal{N}_s$, etc for the items corresponding to $U, \mathcal{B}, \mathcal{N}$, etc when G is replaced by G_s.

5.1.3 **Lemma**. *The map $q : M \to Q$ has λ_o as its only critical value. The critical fibre is Ω_o and the critical locus is O_o. One has $\Omega_o \approx G \times_{G_s} \mathcal{N}_s$ and*

$$M \approx G \times_{G_s} (\mathfrak{g}_s^*/\text{centre}), \tag{3}$$

locally over neighbourhood of λ_o in Q.

We omit the simple verification and note only that the local isomorphism is induced by the map

$$G \times_{G_s} \mathfrak{g}_s^* \to \mathfrak{g}^*, [g, \xi] \to g \cdot \xi.$$

In principle, this lemma this lemma reduces the local monodromy of $\mathfrak{g}^* \to G\backslash\backslash\mathfrak{g}^*$ around λ_o to the coadjoint quotient of $\mathbf{sl}(2,\mathbf{C})$ around 0; it is an instance of the familiar method of descent. The main difficulty is that effect of the splitting (3) on the homology of the standard fibre, i.e. on $'H_{2n}(\mathcal{B}^*) \approx H_{2n}(\mathcal{S})$, is not entirely transparent.

5.2 **Notation**. Let $\mathfrak{p} = \mathfrak{g}_s + \mathfrak{b}$ the parabolic subalgebra of $\mathfrak{g}$ associated to s, P the corresponding subgroup of G, and $\mathcal{P} \approx G/P$ the generalized flag manifold. We write $\mathfrak{p}_y$ for the stabilizer in $\mathfrak{g}$ of $y \in \mathcal{P}, \mathfrak{n}_y$ for its nilpotent radical, and set $\mathfrak{g}_y = \mathfrak{p}_y/\mathfrak{n}_y \approx \mathbf{sl}(2, \mathbf{C}) + (\text{centre})$. We denote by $\mathfrak{p}_{y,o}, \mathfrak{n}_{y,o}$ the intersections with $\mathfrak{g}_o$ and set $\mathfrak{g}_{y,o} = \mathfrak{p}_{y,o}/\mathfrak{n}_{y,o}$. There is a natural map $\pi_s : \mathcal{B} \to \mathcal{P}$. We set $\mathcal{B}_y = \pi_s^{-1}(y)$. If $x \in \pi_s^{-1}(y)$, then $\mathcal{B}_y = P_y \cdot x \approx G_y/B_{y,x}$ where $\mathfrak{b}_{y,x} = \mathfrak{b}_x/\mathfrak{n}_y$ is the image of $\mathfrak{b}_x$ in $\mathfrak{g}_y$. We set

$$\mathcal{P}^* = \{(y, \eta) : y \in \mathcal{P}, \eta \in (\mathfrak{g}/\mathfrak{p}_y)^*\}$$

and note that

$$\mathcal{P}^* \approx G \times_P (\mathfrak{g}/\mathfrak{p})^* \approx U \times_{U_s} (\mathfrak{g}/\mathfrak{p})^*, \tag{4}$$

the cotangent bundle of $\mathcal{P}$.

5.3 **The map** $\mathcal{B}^* \to \mathcal{P}^*$. We shall make use of the particular maps $p_\lambda : \mathcal{B}^* \to \Omega_\lambda$ defined in §2, eq.(5). We shall need a formula for the monodromy transformations $a_\lambda(s) = p_{s\lambda}^{-1} \circ p_\lambda$ of $\mathcal{B}^*$ in terms of the analogous transformations $a_{s,\lambda}(s)$ of $\mathcal{B}_s^*$ defined relative to $(\mathfrak{g}_s, \mathfrak{b}_s, U_s)$.

5.3.1 **Lemma**. *One has*

$$\mathcal{B}^* \approx U \times_{U_s} [\mathcal{B}_s^* \times (\mathfrak{g}/\mathfrak{p})^*] \tag{5}$$

as a U–equivariant fibre bundle over $\mathcal{P} \approx U/U_s$. Under this isomorphism, the map $p_\lambda : \mathcal{B}^* \to \Omega_\lambda$ is given by

$$p_\lambda(u \cdot [(x_s, \nu_s); \eta] = u(u_s(x_s)\lambda + \nu_s + \eta) \tag{6}$$

and the transformation $a_\lambda(s)$ of $\mathcal{B}^*$ by

$$a_\lambda(s) \cdot u \cdot [(x_s, \nu_s); \eta] = u \cdot [a_{s,\lambda}(s)(x_s, \nu_s); \eta] \tag{7}$$

Proof. There is a natural exact sequence

$$0 \to (\mathfrak{g}_s/\mathfrak{b}_s)^* \to (\mathfrak{g}/\mathfrak{b})^* \to (\mathfrak{g}/\mathfrak{p})^* \to 0.$$

We define a splitting of this sequence as follows. Let $\mathfrak{n}(\mathfrak{p})$ be the nilpotent radical of $\mathfrak{p}$. Then there are direct sum decompositions $\mathfrak{p} = \mathfrak{g}_s \oplus \mathfrak{n}(\mathfrak{p})$ and $\mathfrak{b} = \mathfrak{b}_s \oplus \mathfrak{n}(\mathfrak{p})$, which induce the desired splitting

$$(\mathfrak{g}/\mathfrak{b})^* \approx (\mathfrak{g}_s/\mathfrak{b}_s)^* \times (\mathfrak{g}/\mathfrak{p})^*. \tag{8}$$

Then

$$\begin{aligned}\mathcal{B}^* &\approx U \times_{H \cap U} (\mathfrak{g}/\mathfrak{b})^* \\ &\approx U \times_{U_s} [U_s \times_{H \cap U_s} ((\mathfrak{g}_s/\mathfrak{b}_s)^* \times (\mathfrak{g}/\mathfrak{p})^*)] \\ &\approx U \times_{U_s} [\mathcal{B}_s^* \times (\mathfrak{g}/\mathfrak{p})^*],\end{aligned}$$

which gives the isomorphism (5). Explicitly, the isomorphism

$$U \times_{U_s} [\mathcal{B}_s^* \times (\mathfrak{g}/\mathfrak{p})^*] \xrightarrow{\approx} \mathcal{B}^*$$

is given by

$$u \cdot [(x_o, \nu_s); \eta] \to u \cdot (x_o, \nu_s + \eta).$$

The formula (6) follows. To prove (7), fix $u \cdot [(x_o, \nu_s); \eta] \in U \times_{U_s} (\mathcal{B}_s^* \times (\mathfrak{g}/\mathfrak{p})^*)$ and write

$$a_\lambda(s) \cdot u \cdot [(x_o, \nu_s); \eta] = \tilde{u} \cdot [(x_o, \tilde{\nu}_s); \tilde{\eta}]. \tag{9}$$

We solve this equation for the right side in terms of the left as follows. Apply $p_{s\lambda}$ to both sides using $p_{s\lambda} \circ a_\lambda(s) = p_\lambda$ to get

$$u \cdot (\lambda + \nu_s + \eta) = \tilde{u} \cdot (s\lambda + \tilde{\nu}_s + \tilde{\eta}).$$

Put $\tilde{u} = uu_s$ with $u_s \in U_s$ and $\tilde{\nu}_s$, $\tilde{\eta}$ to be determined so that

$$\lambda + \nu_s + \eta = u_s(s\lambda + \tilde{\nu}_s + \tilde{\eta}).$$

Thus specify $u_s \in U_s$ and $\tilde{\nu}_s$ so that

$$\lambda + \nu_s = u_s(s\lambda + \tilde{\nu}_s) \tag{10}$$

and then $\tilde{\eta}$ so that

$$\eta = u_s \cdot \tilde{\eta}$$

The equation (10) means that

$$u_s(x_o, \tilde{\nu}_s) = a_{s,\lambda}(s)(x_o, \nu_s).$$

This gives

$$a_\lambda(s)u[(x_o, \nu_s); \eta] = u[a_{s,\lambda}(s)(x_o, \nu_s); \eta].$$

Any $x_s \in \mathcal{B}_s$ is of the form $x_s = u_s x_o$, so using the U_s–equivariance of $a_{s,\lambda}(s)$ we get

$$\begin{aligned} a_\lambda(s)u[(x_s, \nu_s); \eta] &= a_\lambda(s)uu_s[(x_o, u_s^{-1}\nu_s); u_s^{-1}\eta] \\ &= uu_s[a_{s,\lambda}(s)(x_o, u_s^{-1}\nu_s); u_s^{-1}\eta] \\ &= u[a_{s,\lambda}(s)(x_s, \nu_s); \eta] \end{aligned}$$

as required. $\square$

5.3.2 **Definition**. Define

$$r : \mathcal{B}^* \to \mathcal{P}^* \tag{11}$$

to correspond to the natural map

$$U \times_{U_s} [\mathcal{B}_s^* \times (\mathfrak{g}/\mathfrak{p})^*] \to U \times_{U_s} (\mathfrak{g}/\mathfrak{p})^*$$

under the isomorphisms (4) and (5).

The following lemma explains the appearance of $\mathcal{P}^*$ in the present context.

5.3.3 **Lemma**. There is a commutative diagram of U–equivariant maps

$$\begin{array}{ccc} \mathcal{B}^* & \xrightarrow{p_o} & \Omega_o \\ r \downarrow & & \downarrow \\ O_o & \xrightarrow{\approx} & \mathcal{P}^* \end{array}$$

We omit the verification, which is immediate from the definitions.

For the proof of the theorem it will be convenient to use a slightly different formulation of the above lemmas. For this purpose fix $y \in \mathcal{P}$. For any $x \in \mathcal{B}_y$ we have a sequence

$$0 \to \mathfrak{p}_y/\mathfrak{b}_x \to \mathfrak{g}/\mathfrak{b}_x \to \mathfrak{g}/\mathfrak{p}_y \to 0 \tag{12}$$

hence also

$$0 \to (\mathfrak{g}/\mathfrak{p}_y)^* \to (\mathfrak{g}/\mathfrak{b}_x)^* \to (\mathfrak{p}_y/\mathfrak{b}_x)^* \to 0. \tag{13}$$

We write (13) as

$$0 \to \mathcal{P}_y^* \to \mathcal{B}_x^* \to \mathcal{B}_{y,x}^* \to 0. \tag{14}$$

It has the following interpretation:

$\mathcal{B}_y = \pi_s^{-1}(y) \approx \mathbf{CP}^1$ is the flag manifold of $\mathfrak{g}_y \approx \mathbf{sl}(2, \mathbf{C}) +$ (centre), embedded in $\mathcal{B}$.
$\mathcal{P}_y^*, \mathcal{B}_x^*$, and $\mathcal{B}_{y,x}^*$ are respectively the cotangent spaces at $y \in \mathcal{P}, x \in \mathcal{B}$, and $x \in \mathcal{B}_y$.
$\mathcal{B}_x^* \to \mathcal{B}_{y,x}^*$ is the natural projection.

As in (8), the sequence (14) admits a splitting obtained from a decomposition

$$\mathfrak{p}_y = \mathfrak{b}_{y,x} \oplus \mathfrak{n}_y. \tag{15}$$

As before, $\mathfrak{n}_y$ is the nilpotent radical of $\mathfrak{p}_y$ and the decomposition (15) is orthogonal for the positive definite form on $\mathfrak{g}^*$ corresponding to the involution τ defining U. The decomposition (15) gives

$$\mathcal{B}_x^* \approx \mathcal{B}_{y,x}^* \times \mathcal{P}_y^*. \tag{16}$$

This gives a direct sum decomposition of $\mathcal{B}^*$ as a vector bundle over $\mathcal{P}$, which is equivalent to the isomorphism (5).
The map $r : \mathcal{B}^* \to \mathcal{P}^*$ is the projection onto the second factor in (16); for a given $x \in \mathcal{B}$, this is the τ–orthogonal projection onto the subspace $\mathcal{P}_y^*$ of $\mathcal{B}_x^*$.
In this way, $\mathcal{B}^*$ becomes a fibre–bundle over $\mathcal{P}^*$, with fibres $\mathcal{B}_y^*$ isomorphic to the cotangent bundle of $\mathcal{B}_y \approx \mathbf{CP}^1$. We write this fibre–decomposition of $\mathcal{B}^*$ as

$$\mathcal{B}^* = \bigsqcup_{(y,\eta) \in \mathcal{P}^*} \mathcal{B}_y^* \times \{\eta\}. \tag{17}$$

The equation (17) is more intrinsically interpreted as follows. For $y \in \mathcal{P}$, write $a_{y,\lambda}(s)$ for the transformation of $\mathcal{B}^*_y$ obtained from $a_{s,\lambda}(s)$ by conjugation with an element $u \in U$ satisfying $y = u \cdot y_o$. This is well–defined since $a_{s,\lambda}(s)$ commutes with the action of U_s on $\mathcal{B}^*_s$. Then (7) can be written in the form

$$a_\lambda(s) \cdot [(x_y, \nu_y); \eta_y] = [a_{y,\lambda}(s)(x_y, \nu_y); \eta_y] \tag{18}$$

with $(x_y, \nu_y) \in \mathcal{B}^*_y$ and $\eta_y \in (\mathfrak{g}/\mathfrak{p}_y)^*$. (In the future, we omit the subscripts "y" from the notation when the dependence on y understood from the context.) This means that the transformation $a_\lambda(s)$ of $\mathcal{B}^*$ decomposes compatibly with (17), indicated by

$$a_\lambda(s) = \bigsqcup_{(y,\eta)\in\mathcal{P}^*} a_{y,\lambda}(s) \times 1_\eta. \tag{19}$$

5.4 **The map $\mathcal{S} \to \mathcal{R}$.**We now analyze the restriction to $\mathcal{S}$ of the map $r : \mathcal{B}^* \to \mathcal{P}^*$. Thus we again assume given the subalgebra $\mathfrak{g}_o$ of $\mathfrak{g}$ and assume that G_o has finitely many orbits on $\mathcal{B}$; $\mathcal{S} \subset \mathcal{B}^*$ is the conormal variety of the G_o–action on $\mathcal{B}$. We denote by $\mathcal{R} \subset \mathcal{P}^*$ the conormal variety of the G_o–action on $\mathcal{P}$:

$$\mathcal{R} = \{(y, \eta) \in \mathcal{P}^* : \eta \in (\mathfrak{g}/\mathfrak{g}_o)^*\}.$$

Fix $y \in \mathcal{P}$. For any $x \in \mathcal{B}_y$ we have an exact subsequence of (12),

$$0 \to \mathfrak{p}_{y,o}/\mathfrak{b}_{x,o} \to \mathfrak{g}_o/\mathfrak{b}_{x,o} \to \mathfrak{g}/\mathfrak{p}_{y,o} \to 0. \tag{20}$$

hence also of (13),

$$0 \to \left(\frac{\mathfrak{g}/\mathfrak{p}_y}{\mathfrak{g}_o/\mathfrak{p}_{y,o}}\right)^* \to \left(\frac{\mathfrak{g}/\mathfrak{b}_x}{\mathfrak{g}_o/\mathfrak{b}_{x,o}}\right)^* \to \left(\frac{\mathfrak{p}_y/\mathfrak{b}_x}{\mathfrak{p}_{y,o}/\mathfrak{b}_{x,o}}\right)^* \to 0. \tag{21}$$

Analogous to (14), we write (21) as

$$0 \to \mathcal{R}_y \to \mathcal{S}_x \to \mathcal{S}_{y,x} \to 0. \tag{22}$$

The interpretation is now as follows.

$\mathcal{S}_y$ is the conormal variety of the $G_{y,o}$–action on $\mathcal{B}_y$.
$\mathcal{R}_y, \mathcal{S}_x$, and $\mathcal{S}_{y,x}$ are the conormals to $G_o \cdot y, G_o \cdot x$, and $G_{y,o} \cdot x$ at $y \in \mathcal{P}, x \in \mathcal{B}$, and $x \in \mathcal{B}_y$.
$\mathcal{S}_x \to \mathcal{S}_{y,x}$ is the natural projection.

Thus $\mathcal{R}_y$, $\mathcal{S}_x$, and $\mathcal{S}_{y,x}$ are, respectively, fibres of $\mathcal{R} \to \mathcal{P}$, $\mathcal{S} \to \mathcal{B}$, and $\mathcal{S}_y \to \mathcal{B}_y$. As in (16), the τ–orthogonal projection in (15) gives a decomposition

$$\mathcal{S}_x \approx \mathcal{S}_{y,x} \times \mathcal{R}_y. \tag{23}$$

The restriction of $r : \mathcal{B}^* \to \mathcal{P}^*$ is obtained from the projection onto the second factor in (23), hence gives a surjection

$$r : \mathcal{S} \to \mathcal{R}. \tag{24}$$

The fibres of (24) are isomorphic to the conormal varieties $\mathcal{S}_y$ of the $G_{y,o}$–actions on the $\mathcal{B}_y \approx \mathbf{CP}^1$. Like (17), we write this fibre–decomposition of $\mathcal{S}$ as

$$\mathcal{S} = \bigsqcup_{(y,\eta)\in\mathcal{R}} \mathcal{S}_y \times \{\eta\}. \tag{25}$$

The fibre decomposition (25) applies also to $2n$–chains on $\mathcal{S}$, as we shall now discuss in a more general context.

5.5 **Fiber decompositions of chains**. Let $f : X \to Y$ be a real analytic map. Let $C \subset X$ and $D \subset Y$ be oriented analytic submanifolds so that $f : C \to D$ is a locally trivial fibration with fibres $C_y = C \cap f^{-1}(y)$ of dimension d. These fibres are then also analytic submanifolds. Thus C, D, and C_y define subanalytic chains, denoted by the same letters. We write

$$C = \bigsqcup_{y\in D} C_y \tag{26}$$

This construction and notation carries over to subanalytic chains C, D provided f restricts to a locally trivial fibration $f : \mathrm{supp}(C) \to \mathrm{supp}(D)$ and we use the same notation (26) for chains.

5.5.1 **Lemma**. *The boundary operator satisfies the relation*

$$\partial C = \bigsqcup_{y\in D} \partial C_y + C' \tag{27}$$

where C' is a subanalytic chain with support in the preimage $f^{-1}(\partial D)$ of the topological boundary ∂D of D.

Proof. This follows directly from the definition of the boundary operator, cf. [Kashiwara–Shapira, 1990, §9.2] $\square$

5.6 **Fiber decompositions of cycles on $\mathcal{S}$**. The map (24) is not G_o–equivariant, but it is compatible with the map $\pi_s : \mathcal{B} \to \mathcal{P}$, which is. Therefore r maps the part of $\mathcal{S}$ over a G_o–orbit in $\mathcal{P}$ to the conormal of this orbit in $\mathcal{R}$. We shall decompose r according to the G_o–orbits on $\mathcal{P}$.

5.6.1 **Notation**. a)We apply the notation 3.4 to the action of G_o on $\mathcal{P}$. In order to avoid confusion with the notation O for G_o–orbits on $\mathcal{N}(\mathfrak{g}_o^\perp)$ used previously, we shall denote G_o–orbits on $\mathcal{P}$ by O_ω where

ω runs over some index set for $G_o\backslash\mathcal{P}$, at times identified with $G_o\backslash\mathcal{P}$ itself. We apply the notation introduced in 3.4 also to the indices ω; for example $\omega' < \omega$ denotes the closure order of 3.4(a).
b)The part of $\mathcal{R}$ over O_ω, denoted $\mathcal{R}_\omega$, is the conormal bundle of O_ω. Write $\mathcal{R}_{\omega,\epsilon}$ for the connected components of the smooth part of $\mathcal{R}$ which are contained in $\mathcal{R}_\omega$.
c)For any $2n$–chain Γ on $\mathcal{S}$ we denote by $\mathrm{S}(\Gamma) \subset G_o\backslash\mathcal{P}$ the image of $\mathrm{supp}(\Gamma) \subset \mathcal{S}$ under the natural map $\mathcal{S} \to \mathcal{B} \to \mathcal{P}$. We recall that the leading orbits O_ω in $\mathrm{S}(\Gamma)$ those which are not contained in the closure of any other orbit in $\mathrm{S}(\Gamma)$; the same terminology applies to the indices ω.
Any $2n$–cycle Γ on $\mathcal{S}$ admits a decomposition

$$\Gamma = \sum_{\omega,\epsilon} \bigsqcup_{(y,\eta)\in\mathcal{R}_{\omega,\epsilon}} \Gamma_y \times \{\eta\} \tag{28}$$

where Γ_y is a 2–chain on the conormal variety $\mathcal{S}_y$ of the $G_{y,o}$– action on $\mathcal{B}_y \approx \mathbf{CP}^1$. (Cf. Lemma 3.2; to interpret the fibre decomposition (28) in accordance with (26) one may have to decompose the $\mathcal{S}_y$ further into the connected components of its set of smooth points.)
By (19) we have

$$a_\lambda(s)\Gamma = \sum_{\omega,\epsilon} \bigsqcup_{(y,\eta)\in\mathcal{R}_{\omega,\epsilon}} (a_{y,\lambda}(s)\Gamma_y) \times \{\eta\} \tag{29}$$

This is an equation of chains on $\mathcal{B}^*$, not of homology classes. In general the Γ_y are not even cycles; but when ω is a leading element of $\mathrm{S}(\Gamma)$, then Γ_y for $y \in O_\omega$ is a cycle on $\mathcal{S}_y$, cf. 3.10.

5.7 **The homology of the variety $\mathcal{S}_o$**. Recall that $\mathcal{S}_o = p_o^{-1}(O_o)\cap \mathcal{S}$.

5.7.1 **Lemma**. *a) Under the decomposition (25), one has*

$$\mathcal{S}_o = \bigsqcup_{(y,\eta)\in\mathcal{R}} \mathcal{B}_y \times \{\eta\} \tag{30}$$

where $\mathcal{B}_y$ is embedded as the zero–section in $\mathcal{B}^*_y$.
b) The restriction $r : \mathcal{S}_o \to \mathcal{R}$ is surjective with fibres $\mathcal{B}_y \approx \mathbf{CP}^1$.
c) There is a natural isomorphism $H_{2n-2}(\mathcal{R}) \xrightarrow{\approx} H_{2n}(\mathcal{S}_o), \gamma \to \Delta$, given by

$$\Delta = \bigsqcup_{(y,\eta)\in\gamma} \Delta_y \times \{\eta\} \tag{31}$$

where Δ_y is the fundamental cycle of $\mathcal{B}_y$.
d) The closure filtration on $G_o\backslash\mathcal{P}$ induces a filtration on $H_{2n}(\mathcal{S}_o)$ whose graded group is

$$\mathrm{gr}H_{2n}(\mathcal{S}_o) \approx \mathbf{Q}[G_o\backslash\mathcal{P}]. \tag{32}$$

Proof. (a) Under the isomorphism $\mathcal{B}^* \approx U \times_{U_s} [\mathcal{B}_s^* \times (\mathfrak{g}/\mathfrak{p})^*]$ in (5) one has

$$p_o^{-1}(O_o) \approx U \times_{U_s} [\mathcal{B}_s \times (\mathfrak{g}/\mathfrak{p})^*].$$

By formula (6),

$$p_o(u \cdot [(x_s, \nu_s); \eta]) = u(u_s(x_s)\lambda_o + \nu_s + \eta). \tag{33}$$

In this notation, the subset $O_o \subset \Omega_o$ is given by $\nu_s = 0$. The assertion (a) follows.
b) This is clear.
c) This is a consequence of (b) and the Gysin sequence for $\mathcal{S}_o \to \mathcal{R}$.
d) It follows from 3.11.2 that the hypotheses 3.5 hold for the G_o–map $\mathcal{R} \to \mathcal{P}$. As in 4.2, this gives

$$\mathrm{gr}H_{2n}(\mathcal{R}) \approx \mathbf{Q}[G_o\backslash\mathcal{P}].$$

The assertion then follows from (c). $\square$

5.7.2 The fibre decomposition (30) of $\mathcal{S}_o$ over $\mathcal{R}$ can also be written as a fibre decomposition over $\mathcal{P}$:

$$\mathcal{S}_o = \bigsqcup_{y\in\mathcal{P}\mathcal{B}_y\times\mathcal{R}_y}. \tag{34}$$

This means that $\mathcal{S}_o = \pi_s^*\mathcal{R}$, the pull–back of $\mathcal{R}$ by $\pi_s : \mathcal{B} \to \mathcal{P}$. Over each G_o–orbit O_ω the map $\mathcal{S}_o \to \mathcal{P}$ is a locally trivial fibration. The assertion (d) of the lemma means that there is a basis $\{\Delta_\omega : \omega \in G_o\backslash\mathcal{P}\}$ for $H_{2n}\{\mathcal{S}_o\}$ so that

$$\Delta_\omega = \bigsqcup_{y\in O_\omega} [\mathcal{B}_y \times \mathcal{R}_y] + \cdots \tag{35}$$

where the dots indicate a cycle over ∂O_ω and $[\mathcal{B}_y \times \mathcal{R}_y]$ is the chain defined by an orientation on $\mathcal{B}_y \times \mathcal{R}_y$. We call any such basis compatible with (32).

5.8 **Theorem.** *Let $\{\Delta_\omega : \omega \in G_o\backslash\mathcal{P}\}$ be any basis for $H_{2n}(\mathcal{S}_o)$ compatible with (32). Then there are unique elements $\check{\Delta}_\omega \in H^{2n}(\mathcal{S}) :=$ $\mathrm{Hom}_{\mathbf{Q}}(H_{2n}(\mathcal{S}), \mathbf{Q})$ so that for any $\Gamma \in H_{2n}(\mathcal{S})$,*

$$s \cdot \Gamma = \Gamma + \sum_{\omega \in S(\Gamma)} \langle \Gamma, \check{\Delta}_\omega \rangle \Delta_\omega. \tag{36}$$

They satisfy

$$\langle \Delta_\omega, \check{\Delta}_{\omega'} \rangle = -2\delta_{\omega\omega'}. \tag{37}$$

In particular

$$s \cdot \Gamma = -\Gamma \text{ if and only if } \Gamma \in H_{2n}(\mathcal{S}_o). \tag{38}$$

Furthermore, if ω is a leading element of $S(\Gamma)$, then $\langle \Gamma, \check{\Delta}_\omega \rangle$ is the intersection index $\langle \Gamma_y, \Delta_y \rangle$ for any $y \in O_\omega$. Here Γ_y is the 2–cycle on $\mathcal{B}_y^*$ defined by (28), Δ_y the fundamental cycle of of $\mathcal{B}_y \approx \mathbf{CP}^1$, and the intersection index is taken as cycles on $\mathcal{B}_y^*$.

Proof. We first consider the restriction of the transformation $a_\lambda(s)$ to $\mathcal{S}_o$. From (19) and (30) we get that

$$a_\lambda(s)|\mathcal{S}_o = \bigsqcup_{(y,\eta)\in\mathcal{R}_\omega} (a_{y,\lambda}(s)|\mathcal{B}_y) \times \{\eta\}.$$

By 2.6.5, the transformation $a_{y,\lambda}(s)$ maps $\mathcal{B}_y$ into itself, reversing the orientation. It follows from (31) that

$$a_\lambda(s)\Delta = -\Delta \tag{39}$$

for any $2n$–cycle $\Delta \in H_{2n}(\mathcal{S}_o)$.
Now let $\Gamma \in H_{2n}(\mathcal{S})$ be an arbitrary $2n$–cycle on $\mathcal{S}$, represented as a fibre union as in (28), so that the formula (29) applies. If ω is a leading element in S(Γ), then Γ_y is a 2–cycle on $\mathcal{S}_y$ and the Picard–Lefschetz theorem (1), applied to the coadjoint quotient of $\mathfrak{g}_y/(\text{centre}) \approx \mathbf{sl}(2,\mathbf{C})$ gives

$$a_{y,\lambda}(s)\Gamma_y = \Gamma_y + \langle \Gamma_y, \Delta_y \rangle \Delta_y + \partial C_y \tag{40}$$

for some boundary ∂C_y. One finds that

$$a_\lambda(s)\Gamma = \sum_{\omega,\epsilon} \bigsqcup_{(y,\eta)\in\mathcal{R}_{\omega,\epsilon}} (\Gamma_y + \langle \Gamma_y, \Delta_y \rangle \Delta_y + \partial C_y) \times \{\eta\} + C' \tag{41}$$

with the sum extending over the leading ω in S(Γ) only, and C a chain over $\partial S(\Gamma)$. In view of (27), there is a homology

$$\sum_{\omega,\epsilon} \bigsqcup_{(y,\eta)\in\mathcal{R}_{\omega,\epsilon}} \partial C_y \sim C''$$

where C'' is another chain over $\partial S(\Gamma)$. Thus

$$a_\lambda(s)\Gamma \sim \sum_{\omega,\epsilon} \bigsqcup_{(y,\eta)\in\mathcal{R}_{\omega,\epsilon}} (\Gamma_y + \langle \Gamma_y, \Delta_y\rangle \Delta_y) + \cdots . \tag{42}$$

where the dots indicate a chain over $\partial S(\Gamma)$. Modulo such chains, the unions of the terms involving the Δ_y may be completed to cycles on $\mathcal{S}_o$ and then expressed in terms of the basis $\{\Delta_\omega\}$. Thus (42) can be written in the form

$$a_\lambda(s)\Gamma \sim \Gamma + \sum_\omega \langle \Gamma, \check{\Delta}_\omega\rangle \Delta_\omega + \Gamma' \tag{43}$$

where ω runs over the leading terms in $\mathrm{S}(\Gamma)$, and Γ' is a chain, necessarily a cycle, with $\mathrm{S}(\Gamma') \subset \partial S(\Gamma)$; we have set $\langle \Gamma,\, \check{\Delta}_\omega\rangle = \langle \Gamma_y,\, \Delta_y\rangle$ for any $y \in O_\omega$.
From (43) it follows first of all that $s\Gamma \neq -\Gamma$ in $H_{2n}(\mathcal{S})$ unless $\Gamma \in H_{2n}(\mathcal{S}_o)$. Thus $H_{2n}(\mathcal{S}_o)$ is precisely the (-1)–eigenspace of the involution s on $H_{2n}(\mathcal{S})$, i.e.

$$(s-1)H_{2n}(\mathcal{S}) = H_{2n}(\mathcal{S}_o).$$

Hence the cycle Γ' in (43) must be a linear combination of basis elements $\Delta_{\omega'}$ with $\omega' \in \partial S(\Gamma)$, so that one can write

$$a_\lambda(s)\Gamma = \Gamma + \sum_\omega \langle \Gamma, \check{\Delta}_\omega\rangle \Delta_\omega$$

where the sum is now extended over all $\omega \in S(\Gamma)$ and the coefficients $\langle \Gamma,\, \check{\Delta}_\omega\ \rangle$ are integers depending linearly on Γ. This gives the formula (36). The relation (37) is a consequence of (38), which was established during the proof, together with the rest. □

It is sometimes convenient to have a description of $\mathcal{S}_o$ directly in terms of the G_o–orbits on $\mathcal{B}$. We denote these by O_v with v running over some index set for $G_o\backslash\mathcal{B}$, and use the conventions of 5.6.1(a) for these as well.

5.8.1 **Lemma**. *$\mathcal{S}_o$ is the closure of the union of the conormal bundles of the G_o–orbits O_v on $\mathcal{B}$ satisfying*

$$O_v \textit{ is open in } \pi_s^{-1}\pi_s(O_v). \tag{44}$$

Proof. The part of $\mathcal{S}_o$ over O_v is

$$\bigsqcup_{y\in\pi_s(O_v)} (\mathcal{B}_y \cap O_v) \times \mathcal{R}_y$$

This has dimension $2n$ if and only if $\mathcal{B}_y \cap O_v$ has dimension 2, i.e. $\pi_s^{-1}(y) \cap O_v$ is open in $\pi_s^{-1}(y)$. This happens if and only if O_v is open in $\pi_s^{-1}\pi_s(O_v)$. □

5.9 **Examples**. a) We take $\mathfrak{g}_o = \mathfrak{b}$, as in 2.6.4 and again write $O_w = Bwx_o$ ($w \in W$) for this B–orbit on $\mathcal{B}$, C_w for the conormal bundle of O_w, and $[C_w]$ for the corresponding cycle. These form a basis for $H_{2n}(\mathcal{S})$. Let $s \in W$ be a simple reflection as above. Then O_w is open in $\pi_s^{-1}\pi_s(O_w)$ if and only if BwB is open in $(BwB)(BsB)$. It follows from 5.8.1 and well–known facts about the Bruhat order that $\mathcal{S}_o$ is the union of the C_w satisfying $ws < w$. One can then take the cycles $\{[C_w] : ws < w\}$ for the basis $\{\Delta_\omega\}$ in the theorem. On the other hand, if $ws > w$, then the Δ_ω corresponding to the (unique) leading orbit $O_\omega = \pi_s(O_w)$ in S($[C_w]$) is $\Delta_\omega = [C_{ws}]$. The relevant Picard–Lefschetz formula (40) for $\mathfrak{g}_y \approx \mathbf{sl}(2, \mathbf{C}) +$ (centre) reads

$$s_y \cdot [C_w]_y \sim [C_w]_y + [C_{ws}]_y.$$

Hence the theorem gives in this case the relations

$$s \cdot [C_w] = -[C_w], \quad \text{if } ws < w \tag{45}$$

$$= [C_w] + [C_{ws}] + \sum_{w' < w, w's < sw} m_{ww'}[C_{w'}], \quad \text{if } ws > w \tag{46}$$

for certain coefficients $m_{ww'}$, which may depend on s.

We remark that similar formulas hold whenever $\mathfrak{g}_o$ is a complex subalgebra of $\mathfrak{g}$, always under the assumption that G_o has finitely many orbits O_v on $\mathcal{B}$, which can then replace the O_w above. This holds in particular when $\mathfrak{g}_o$ is a complex symmetric subalgebra of $\mathfrak{g}$ (the $\mathfrak{k}_{\mathbf{C}}$–case of 2.2.7). For such $\mathfrak{g}_o$, the closure order on $G_o\backslash\mathcal{B}$, which enters into (45)–(46), is studied in [Richardson– Springer, 1989, 1992].

b) The formula (36) has implications for the action of s in the representation of W on $H_{2e}(\mathcal{B}^\nu)$ inside of which the irreducible representation $H_{2e}(\mathcal{B}^\nu)_\phi$ of W is realized (cf. 4.5).

To avoid confusion with the conormal variety $\mathcal{S}$ of the diag(G)–action on $\mathcal{B} \times \mathcal{B}$, used in the proof of 4.5, we now denote by $\mathcal{C}$ the conormal variety of the B–action on $\mathcal{B}$:

$$\mathcal{C} = \{(x, \nu) \in \mathcal{B}^* : x \in \mathcal{B}, \nu \in (\mathfrak{g}/\mathfrak{b})^*\}.$$

We observe that $\mathcal{C}$ is embedded in $\mathcal{S}$ as the fibre over x_o in the projection $\mathcal{S} \to \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ on the second factor. For any G–stable subset A of $\mathcal{N}$ we write $\mathcal{C}_A$ for the preimage of $A \cap \mathfrak{b}^\perp$ under the natural map $\mathcal{C} \to \mathfrak{b}^\perp = (\mathfrak{g}/\mathfrak{b})^*$. Under the embedding $\mathcal{C} \subset \mathcal{S}$, we have

$$\mathcal{C}_A = \mathcal{S}_A \cap \mathcal{C}$$

where $\mathcal{S}_A$ is the inverse image of diag(A) under the natural map $\mathcal{S} \to$ diag($\mathcal{N}$). Now fix $\nu \in \mathcal{N}$ and set $O = G \cdot \nu$. As a consequence of Theorem

4.3, one has W–isomorphisms (for the restricted monodromy representations)

$$H_{2e}(\mathcal{B}^\nu) \approx H_{2n}(\mathcal{C}_{\bar{O}})/H_{2n}(\mathcal{C}_{\partial O}) \approx H_{2n}(\mathcal{C}_O). \tag{47}$$

The space in the middle gives a realization of $H_{2e}(\mathcal{B}^\nu)$ as a subquotient of $H_{2n}(\mathcal{C})$, to which the formulas (45)–(46) apply, as we shall now explain. First we describe the isomorphisms (47) more explicitly. The closure of $\mathcal{C}_O$ in $\mathcal{C}$ is the union of the closures of conormals C_w of certain B–orbits O_w on $\mathcal{B}$, namely of those for which C_w has an open intersection with $\mathcal{C}_O$. These C_w then cut out the irreducible components of the variety $\mathcal{B}^\nu$, which in turn give a basis for $H_{2e}(\mathcal{B}^\nu)$.
Now fix a simple reflection s as earlier. The fundamental cycles of the conormals just mentioned form part of the basis $\{[C_w]\}$ of $H_{2n}(\mathcal{C})$ which figures in (45)– (46). Write simply $\{[C]\}$ for the corresponding basis of $H_{2e}(\mathcal{B}^\nu)$; the C's are then just the components of $\mathcal{B}^\nu$. Now consider only the part of the formulas (45)–(46) which involves these C_w's and replace the condition "$ws < w$" by the condition

$$C \text{ is open in } \pi_s^{-1}\pi_s(C) \tag{48}$$

which is equivalent, by (44). One arrives at the following result.

$$s \cdot [C] = -[C], \quad \text{if (48) holds} \tag{49}$$

$$= [C] + \sum_{C'} m_{CC'}[C'], \quad \text{otherwise} \tag{50}$$

for certain coefficients $m_{CC'}$, which may depend on s. The sum in (50) goes over components C' of $\mathcal{B}^\nu$ with $\pi_s(C') \subset \pi_s(C)$ and for which (48) holds with C replaced by C'.

5.9.1 **Remarks**. a)The formulas (49)–(50) are those of [Hotta, 1983], where a geometric description of the coefficients $m_{CC'}$ is given. In the present context a description of the $m_{CC'}$ can be given in terms of the corresponding cycles C_w, $C_{w'}$ on $\mathcal{S}$ as follows. The coefficient $m_{CC'}$ is the multiplicity with which the chain $a_\lambda(s)C_w$ contains the chain the chain $C_{w'}$ when retracted to $\mathcal{S}$ as in the proof of 2.6.1. It may described as the degree of a local transversal projection onto C' or, equivalently, as an intersection number with a local transversal. (In the simplicial model $H_\cdot(\mathbf{K} - \mathbf{L}^2; \mathbf{L}^1)$ mentioned in 2.5.1, this is a consequence of [Lefschetz, 1965], Theorem III, p.178, for example, provided the triangulation is chosen compatibly with the representation 2.6.3(14) of the cycles Γ.) Unfortunately, this does not give an explicit algorithm.
b)The formulas (45)–(46) give somewhat more information than stated in (49)–(50) when combined with a description of those w's in the former which correspond to the C's in the latter. In type A_n, such a description is available through the Robinson–Schensted algorithm ([Spaltenstein, 1982] p.142). A generalization of this algorithm to the other classical

groups can be found in [Barbasch and Vogan, 1982]; it should give the correspondence $w \to C$ at least when ν is special (as defined there) and the C's are replaced by A^ν–orbits of components.

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