Nilpotent orbital integrals in a real semisimple Lie algebra and representations of Weyl groups

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Dedicated to Jacques Dixmier on his 65th birthday

Introduction. Let $\mathfrak{g}_{\mathbb{R}}$ be a semisimple real Lie algebra, $G_{\mathbb{R}} = \mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$, $\mathfrak{h}_{\mathbb{R}}$ a real Cartan subalgebra. For $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, let μ_λ denote the canonical invariant measure on the $G_{\mathbb{R}}$-orbit of λ in $\mathfrak{g}_{\mathbb{R}}^*$. A well-known theorem of Harish-Chandra [8] says that

$$\lim_{\lambda \to 0} \varpi_\lambda \mu_\lambda = \kappa \mu_0, \quad \varpi := \prod_{\alpha \epsilon \Delta^+} \partial_\alpha;$$

the limit is taken through regular λ, and κ is a constant, which is non-zero if and only if $\mathfrak{h}_{\mathbb{R}}$ is fundamental. The differential operator ϖ transforms according to sgn under the Weyl group W of $(\mathfrak{g},\mathfrak{h})$. This is the irreducible character associated to $\{0\}$ under Springer's correspondence between nilpotent orbits in $\mathfrak{g}^*$ and irreducible characters of W.

This paper deals with the problem of finding an analogous formula for arbitrary nilpotent $G_{\mathbb{R}}$-orbits. The problem is solved in theorem 5.3 only under an additional hypothesis. The correspondence between real nilpotent orbits and certain representations of W is given in theorem 3.3, based on the theory for complex groups developed in [16], which is recalled in §2. The general framework for the study of Fourier transforms of orbital contour-integrals is given in §1.
Barbasch and Vogan [3] solved the problem for complex orbits in the classical groups and formulated the result as a conjecture for complex orbits in general [4]. Their conjecture was proved in [10] and in [16]. For $G_{\mathbb{R}} = \mathrm{U}(p,q)$ the solution was given by Barbasch and Vogan in [5].
I thank Michèle Vergne for correcting a mistake in the proof theorem 4.1.

1. Coherent families of contours and invariant eigendistributions

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{g}_{\mathbb{R}}$ a real form of $\mathfrak{g}$. Let $G := \mathrm{Ad}(\mathfrak{g})$, $G_{\mathbb{R}} := \mathrm{Ad}(\mathfrak{g}_{\mathbb{R}})$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let W be the Weyl group of $(\mathfrak{g},\mathfrak{h})$.

We recall some definitions and results from [15], [16]. We shall be interested in integrals of the form

$$\frac{1}{(-2\pi \mathrm{i})^n n!} \int_{\Gamma(\lambda)} \varphi \sigma_\lambda^n \tag{1}$$

where $\lambda \in \mathfrak{h}^*_{\mathrm{reg}}$ is a regular element in the dual of the complex Cartan subalgebra $\mathfrak{h}$,

$$\Gamma(\lambda) \subset \Omega_\lambda := G \cdot \lambda \subset \mathfrak{g}^*$$

is an $2n$-cycle with $2n = \dim_{\mathbb{C}} \Omega_\lambda$, σ_λ^n is the n-th exterior power of the canonical holomorphic 2-form on Ω_λ,

$$\sigma_\lambda(x \cdot \xi,\, y \cdot \xi) := \xi([x,y]), \quad \xi \in G \cdot \lambda, \;\; x, y \in \mathfrak{g},$$

and

$$\varphi(\xi) := \int_{\mathfrak{g}_{\mathbb{R}}} f(x) e^{\xi(x)} dx, \quad f \in \mathrm{C}^\infty_{\mathrm{c}}(\mathfrak{g}_{\mathbb{R}})$$

is the Fourier transform of a compactly supported C^∞-function on $\mathfrak{g}_{\mathbb{R}}$. We note that φ is an entire holomorphic function on $\mathfrak{g}^*$ satisfying for all N an estimate of the form

$$|\,\varphi(\xi)|\,| \leq \frac{A e^{B||\mathbb{R}\mathrm{e}\,\xi||}}{1 + \|\xi\|^{\mathrm{N}}}. \tag{2}$$

The cycles $\Gamma(\lambda)$ must be restricted so that (2) guarantees the convergence of (1): one considers locally finite sums of singular m-simplices on Ω_λ, $\gamma = \sum c_k \sigma_{\mathrm{k}}$, $c_k \in \mathbb{Z}$,which satisfy

$$\text{(a)} \|\mathrm{Re}\, \sigma_k\| \leq C \text{ for all } k, \qquad \text{(b)} \sum |\, c_k|\,|\, \max \frac{1}{1 + \|\sigma_k\|^{\mathrm{N}}} \;<\; \infty \text{ for some } N;$$

$\partial\gamma$ is required to have the same properties. Using such m-chains γ one defines a homology group $'H_m(\Omega_\lambda)$ as usual. An oriented, closed, m-dimensional, real submanifold of Ω_λ which admits a triangulation satisfying (a) and (b) defines an element of $'H_m(\Omega_\lambda)$ independent of the triangulation. The integral (1) depends only on the class of $\Gamma(\lambda)$ in $'H_{2n}(\Omega_\lambda)$ so that we shall take $\Gamma(\lambda) \in' H_{2n}(\Omega_\lambda)$. Let $\mathcal{B} := \{$Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}\}$ be the flag manifold of $\mathfrak{g}$,

$$\mathcal{B}^* := \{(\mathfrak{b}, \nu) \;|\; \mathfrak{b} \in \mathcal{B},\; \nu \in \mathfrak{b}^\perp \subset \mathfrak{g}^*\},$$

the cotangent bundle of $\mathcal{B}$, and

$$\mathcal{S} := \{(\mathfrak{b}, \nu) \,|\; \mathfrak{b} \in \mathcal{B},\, \nu \in \mathrm{i}\mathfrak{b}^\perp_{\mathbb{R}} := \mathfrak{b}^\perp \cap \mathrm{i}\mathfrak{g}^*_{\mathbb{R}}\}$$

the conormal variety (union of conormal bundles) of the $G_{\mathbb{R}}$-orbits on $\mathcal{B}$. For $\lambda \in \mathfrak{h}^*_{\mathrm{reg}}$, define a map

$$p_\lambda : \ \mathcal{B}^* \to \Omega_\lambda, \quad p_\lambda(u \cdot (\mathfrak{b}_1, \nu)) := u \cdot (\lambda + \nu), \quad u \in U, \ \nu \in \mathfrak{b}_1^\perp. \tag{3}$$

$\mathfrak{b}_1 \in \mathcal{B}$ is a fixed base-point so that $\mathfrak{b}_1 \supset \mathfrak{h}$; $U \subset G$ is a compact real form for which $\mathfrak{h} \cap u$ is a maximal torus in the Lie algebra u of U. The map p_λ is a U-equivariant, real-analytic bijection. By [15], this map induces an isomorphism

$$p_\lambda : H_{2n}(S) \overset{\approx}{\to} {}'H_{2n}(\Omega_\lambda) \tag{4}$$

$H_{2n}(\mathcal{S})$ is the (Borel-Moore) homology with arbitrary supports.

The image of a contour $\Gamma(\lambda)$ on Ω_λ under the inverse map p_λ^{-1}: $\Omega_\lambda \to \mathcal{B}^*$ does generally not lie in $\mathcal{S}$. Rather it lies in a homology group ${}'H_{2n}(\mathcal{B}^*)$ defined by the same conditions (a), (b) above, with $\|\mathrm{Re}(\mathfrak{b},\nu)\| := \|\mathrm{Re}\nu\|$and $\|(\mathfrak{b},\nu)\| := \|\nu\|$ for $(\mathfrak{b},\nu) \in \mathcal{B}^*$. $H_{2n}(\mathcal{S})$ is free of rank equal to the number of $G_{\mathbb{R}}$-orbits in $\mathcal{B}$ [15]. The isomorphism (4) asserts that a class in ${}'H_{2n}(\mathcal{B}^*)$ has a representative in $\mathcal{S}$, which defines a class in $H_{2n}(\mathcal{S})$. $\mathcal{S}$ is the union of the conormal bundles on the $G_{\mathbb{R}}$-orbits on $\mathcal{B}$, which are smooth (but not closed) real analytic submanifolds of $\mathcal{B}^*$ of dimension

$$\dim_{\mathbb{R}} \mathcal{S} = \dim_{\mathrm{C}} \mathcal{B}^* = 2 \dim_{\mathrm{C}} \mathcal{B} = 2n.$$

This means that a class in $H_{2n}(\mathcal{S})$ is represented by a finite $\mathbb{Z}$-linear combination of the chains defined by a triangulation of these submanifolds. It should be noted however that arbitrary chains constructed in this way are generally not closed in the sense of homology. In particular, the chains constructed from the conormal bundles themselves in this way are generally not cycles.

An element $\Gamma(\lambda) \in {}'H_{2n}(\Omega_\lambda)$ will be referred to as a *contour* on Ω_λ; a *coherent family of contours* is a family of the form

$$\Gamma(\lambda) = p_\lambda \Gamma, \quad \Gamma \in H_{2n}(\mathcal{S}) \text{ fixed.}$$

$\Gamma(\lambda)$ is considered a function of $\lambda \in \mathfrak{h}^*_{\mathrm{reg}}$. For fixed $\Gamma(\lambda) \in H_{2n}(\Omega_\lambda)$, the integral (1), considered as a function of $\mathrm{f} \in \mathrm{C}^\infty_{\mathrm{c}}(\mathfrak{g}^*_{\mathbb{R}})$, defines a distribution θ on $\mathfrak{g}_{\mathbb{R}}$, which is easily seen to be an invariant eigendistribution [16]. When $\Gamma(\lambda) = \mathrm{p}_\lambda\Gamma$, then $\theta = \theta(\Gamma,\lambda)$ is called a *coherent family of invariant eigendistributions* on $\mathfrak{g}^*_{\mathbb{R}}$; these form a $\mathbb{Z}$-module denoted $\mathrm{CH}(\mathfrak{g}_{\mathbb{R}})$, isomorphic with $H_{2n}(\mathcal{S})$ by the map $H_{2n}(\mathcal{S}) \to \mathrm{CH}(\mathfrak{g}_{\mathbb{R}})$, $\Gamma \mapsto \theta(\Gamma)$ defined by the integral (1). Write this as

$$\theta(\Gamma, \lambda) = \frac{1}{(-2\pi \mathrm{i})^n n!} \int_{\xi \in p_\lambda \Gamma} p^\xi \sigma_\lambda^n \tag{5}$$

the integral being understood in the distribution sense as explained above. We set $\mathrm{CH}(\mathfrak{g}_{\mathbb{R}},\mathbb{C}) := \mathrm{CH}(\mathfrak{g}_{\mathbb{R}}) \otimes \mathbb{C}$. We introduce the following notation.

$\Delta :=$ the root system of $(\mathfrak{g},\mathfrak{h})$,

$\Delta^+ :=$ a system of positive roots for Δ,
$W :=$ the Weyl group of $(\mathfrak{g},\mathfrak{h})$,
$\pi := \prod_{\alpha\in\Delta^+}\alpha$,

The subscipt "c" in the following symbols indicates conjugation by an element $c\in G$: $\mathfrak{h}_c := c\cdot\mathfrak{h}$, $\mathfrak{b}_c := c\cdot\mathfrak{b}_1$, Δ_c, Δ_c^+, W_c, π_c, λ_c. If $\mathfrak{h}_{c,\mathbb{R}} := \mathfrak{h}_c\cap\mathfrak{g}_{\mathbb{R}}$ is a real Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$ we let

$W_{c,\mathbb{R}} :=$Norm$_{G_{\mathbb{R}}}(\mathfrak{h}_{c,\mathbb{R}})$/Cent$_{G_{\mathbb{R}}}(\mathfrak{h}_{c,\mathbb{R}})$,
$\Delta_{c,\mathbb{R}}$, $\Delta_{c,\mathrm{I}}$:=real, imaginary roots in Δ_c,
$W(\Delta_{c,\mathbb{R}})$:=the Weyl group of $\Delta_{c,\mathbb{R}}$,
$\epsilon_{c,\mathbb{R}} :=$ sgn $\prod_{\alpha\epsilon\Delta^+_{c,\mathbb{R}}}\ \alpha$.

Note that $w\cdot(\epsilon_{c,\mathbb{R}}\pi_c) := \mathrm{sgn}_{c,\mathrm{I}}(w)(\epsilon_{c,\mathbb{R}}\pi_c)$ for $w\in W_{c,\mathbb{R}}$ where $\mathrm{sgn}_{c,\mathrm{I}}(w)=\pm1$ may be defined by this equation.

A real Cartan subalgebra decomposes as $\mathfrak{h}_c = \mathfrak{t}_c+\mathfrak{a}_c$ so that the roots of $\mathfrak{h}_c$ are imaginary on $\mathfrak{t}_{c,\mathbb{R}}$ and real on $\mathrm{a}_{c,\mathbb{R}}$. Introduce a partial order on the (conjugacy classes of) real Cartan subalgebras by stipulating that $\mathfrak{h}_c < \mathfrak{h}_{c'}$ if $\mathfrak{t}_c\subset\mathfrak{t}_{c'}$ strictly, after suitable conjugation by $G_{\mathbb{R}}$. When $\mu\in\mathfrak{h}^*_{\mathrm{reg}}$ and $\mu_c\in\mathrm{i}\mathfrak{g}^*_{\mathbb{R}}$, then the real orbit $G_{\mathbb{R}}\cdot\mu_c$ defines a contour $[G_{\mathbb{R}}\cdot\mu_c]\in{}'H_{2n}(\Omega_\mu)$: the orientation on $G_{\mathbb{R}}\cdot\mu_c$ is specified by the form $(-\mathrm{i}\sigma_\mu)^n$, which is real-valued and non-vanishing on $G_{\mathbb{R}}\cdot\mu_c$.

1.1 Lemma. *Let* $\mu\in\mathfrak{h}^*_{reg}$, $c\in G$, $\mu_c\in i\mathfrak{g}^*_{\mathbb{R}}$. *For* $w\in W(\Delta_{c,\mathbb{R}})$

$$p^{-1}_{w\mu}\circ p_\mu[G_{\mathbb{R}}\cdot\mu_c] = [G_{\mathbb{R}}\cdot\mu_c].$$

Proof. Let $M=$Cent$_G(\mathfrak{a})$, $\mathfrak{p}=\mathfrak{m}+\mathfrak{b}_1$. Then

$$G_{\mathbb{R}}\cdot\mu_c = (U\cap G_{\mathbb{R}})\cdot(M_{c,\mathbb{R}}\cdot\mu_c+\mathrm{i}\mathfrak{p}^\perp_{c,\mathbb{R}}).$$

On $G_{\mathbb{R}}\cdot\mu_c$ the map $p^{-1}_{w\mu}\circ p_\mu$ is given by $k\cdot(u_M\cdot(\mu_c+\nu_M)+\nu_N)\mapsto k\cdot(u_M\cdot(w_c\mu_c+\nu_M)+\nu_N)$. Here $k\in U\cap G_{\mathbb{R}}$, $u_{\mathrm{M}}\in U\cap M$, $\nu_M\in\mathfrak{b}_1^\perp\cap\mathfrak{m}$, $\nu_N\in\mathrm{i}\mathfrak{p}^\perp_{c,\mathbb{R}}$, and $u_M\cdot(\mu_c+\nu_M) = m_c\cdot\mu_c$ with $m_c\in M_{c,\mathbb{R}}$. Since M_c fixes $w_c\mu_c-\mu_c$, $u_M\cdot(\mu_c+\nu_M)=m_c\cdot\mu_c$ gives $u_M\cdot(w_c\mu_c+\nu_N) = m_cw_c\cdot\mu_M$. Thus on $G_{\mathbb{R}}\cdot\mu_c$ the map $p^{-1}_{w\mu}\circ p_\mu$ is given by $k\cdot(m_c\cdot\mu_c+\nu_N)\mapsto k\cdot(m_cw_c\cdot\mu_c+\nu_N)$. This transformation maps $G_{\mathbb{R}}\cdot\mu_c = G_{\mathbb{R}}w_c\mu_c$ into itself. The map $M_{c,\mathbb{R}}\cdot\mu_c\to M_{c,\mathbb{R}}w_c\mu_c$ preserves the orientations induced by the restrictions of $\mathrm{i}\sigma_\mu$ (because $\mathrm{sgn}_{\mathrm{I}}(\mathrm{w})=1$), hence the transformation $p^{-1}_{w\mu}\circ p_\mu$ of $G_{\mathbb{R}}\mu_c$ preserves orientation as well. Q.E.D.

1.2 Remark. It may be shown that for *any* $w\in W_{c,\mathbb{R}}$

$$p^{-1}_{w\mu}\circ p[_\mu[G_{\mathbb{R}}\cdot\mu_c] = \mathrm{sgn}_{\mathrm{I}}(w)[G_{\mathbb{R}}\cdot\mu_c]+\cdots$$

where the dots indicate a $\mathbb{Z}$-linear combination of contours $[G_{\mathbb{R}}\cdot\mu_{c}{}']$ with $\mathfrak{h}_{c}{}'<\mathfrak{h}_c$.

The contours $[G_{\mathbb{R}}\cdot\mu_c]$ fit into coherent families as follows.

1.3 Theorem. *(a) Let S be a $G_{\mathbb{R}}$-orbit on $\mathcal{B}$. There is a unique coherent family $\Gamma(S,\cdot)$ so that $\Gamma(S,\mu)=[G_{\mathbb{R}}\cdot\mu_c]$ whenever $c\in G$ and $\mu\in\mathfrak{h}^*_{reg}$ satisfying the following conditions:*

(1) $\mu_c\in\mathrm{i}\mathfrak{g}^*_{\mathbb{R}}$, (2) $\mathfrak{b}_c\in S$, (3) $\mu_c(H_{\alpha_c})>0$ if α_c is an imaginary root of $\mathfrak{h}_c$ on $\mathfrak{b}_c$.

(b) The coherent families of eigendistributions $\theta(S,\cdot)$ corresponding to the $\Gamma(S,\cdot)$, S running over the $G_{\mathbb{R}}$-orbits on $\mathcal{B}$, form a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module $CH(\mathfrak{g}_{\mathbb{R}})$.

(c) Let $(\mathfrak{h}_c)_{\mathbb{R}}$, $c\in G$, be a real Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Let S be a $G_{\mathbb{R}}$-orbit on $\mathcal{B}$ so that $\mathfrak{b}_c\in S$. Then

$$\theta(S,\lambda)\equiv\frac{1}{\epsilon_{c,\mathbb{R}}\pi_c}\sum_{w\epsilon W_{c,\mathbb{R}}} sgn_{c,I}(w)e^{w^{-1}\lambda_c} \tag{6}$$

on $(\mathfrak{h}_c)_{\mathbb{R}}$ and vanishes on a real Cartan subalgebra $(\mathfrak{h}_{c'})_{\mathbb{R}}$ unless $\mathfrak{h}_{c'}\leq\mathfrak{h}_c$.

Proof. (a) Fix S and a pair (c,μ) with the indicated properties. Let $\Gamma=\Gamma(c,\mu,S)\in{}'H_{2n}(\mathcal{B}^*)$ be defined by

$$\Gamma=p_\lambda\circ p_\mu^{-1}[G_{\mathbb{R}}\cdot c\mu]. \tag{7}$$

We show that Γis independent of (c,μ). Let (c',μ') be another such pair. In view of (2) we may assume that $\mathfrak{b}_c=\mathfrak{b}_{c'}=:\mathfrak{b}_S$. In view of (1), $\mathfrak{h}_c$ and $\mathfrak{h}_{c'}$ are both conjugation-stable Cartan subalgebras in $\mathfrak{b}_{\mathrm{S}}$. It is well-known that such Cartan subalgebras are $G_{\mathbb{R}}$-conjugate. We may therefore assume that $\mathfrak{h}_c=\mathfrak{h}_{c'}=:\mathfrak{h}_S$. Thus $c=c'h$ for some $h\in\mathrm{Cent}_G(\mathfrak{h})$. We may as well assume $c=c'$. Observe: there is w$\in W(\Delta_{c,\mathbb{R}})$ so that

$$c\cdot\mu \textit{ and } wc\cdot\mu' \textit{ lie in the same connected component of } (\mathfrak{h}_{S,\mathbb{R}})^*_{\mathrm{reg}}. \tag{8}$$

Reason: such a connected component is a chamber cut out by the real and imaginary roots of $\mathfrak{h}_S$; in view of (3), only reflections in real roots are needed to transform the chamber containing $c\cdot\mu$ into the chamber containing $c\mu'$. Fix w$\in W(\Delta_{c,\mathbb{R}})$ so that $c\cdot\mu$ and $wc\cdot\mu'$ lie in the same connected component of $(\mathfrak{h}_{S,\mathbb{R}})^*_{\mathrm{reg}}$. There is a continuous curve $\mu(t)$, $0\leq t\leq 1$, in $\mathfrak{h}^*$ so that $\mu(0)=\mu$, $\mu(1)=c^{-1}wc\mu'$, $c\cdot\mu(w)\in(\mathfrak{h}_{S,\mathbb{R}})^*_{\mathrm{reg}}$, $0\leq t\leq 1$. Consider the following one-parameter family of cycles on $\mathcal{B}^*$: $p^{-1}_{\mu(t)}[G_{\mathbb{R}}\cdot c\mu(t)]$, $0\leq t\leq 1$. This is a homotopy, which satisfies the conditions (a) and (b) defining ${}'H_{2n}(\mathcal{B}^*)$ uniformly in t, hence represents the same class in ${}'H_{2n}(\mathcal{B}^*)$. Thus $p_\mu^{-1}[G_{\mathbb{R}}\cdot c\mu]=p_{c^{-1}wc\mu'}[G_{\mathbb{R}}\cdot c\mu']$. By Lemma 1.2, $p^{-1}_{c^{-1}wc\mu'}[G_{\mathbb{R}}\cdot c\mu']=p_{\mu'}[G_{\mathbb{R}}\cdot c\mu']$. This proves that Γ is independent of (c,μ). As remarked in 1.1, the class $\Gamma\in{}'H_{2n}(\mathcal{B}^*)$ has a representative which lies on $\mathcal{S}$ and defines an element $\Gamma_S\in H_{2n}(\mathcal{S})$. Let $\Gamma(S,\lambda)=p_\lambda\Gamma_S$. This is the required coherent family.

(b) Let S be a $G_{\mathbb{R}}$-orbit on $\mathcal{B}$, $\mathcal{S}_S$ the part of $\mathcal{S}$ over S, i.e. $\mathcal{S}_{\mathrm{S}}:=\{(\mathfrak{b},\nu)\in\mathcal{S}\mid\mathfrak{b}\in S\}$, (the conormal bundle of the $G_{\mathbb{R}}$-orbit S). The element $\Gamma_S\in H_{2n}(\mathcal{S})$ corresponding to $\Gamma(S,\cdot)$ is of the form $\Gamma_S=[\mathcal{S}_S]+\cdots$ where $[\mathcal{S}_S]$ is the

$2n$-chain on $\mathcal{S}$ defined by an orientation on $\mathcal{S}_S$ and the dots indicate a $\mathbb{Z}$-linear combination of $[\mathcal{S}_{S'}]$'s with $S' \subset \partial S := \bar{S} - S$ ([15]). This implies that the Γ_S form a $\mathbb{Z}$-basis for $H_{2n}(\mathcal{S})$, hence (b).
(c) For $(\mathfrak{h}_c)_\mathbb{R}$ of compact type, this follows from [13]. The general case is proved by a familiar reduction thereto using parabolic subalgebras. ([22], Part I, §3.7. The positive constant in Theorem 32 of [22] becomes $c = 1$ with the normalization of forms used here. This may be seen as in §2 of [15].) Q.E.D.

1.4 Remarks. We mention some supplementary facts about coherent families.
(a) By a well-known result of Harish-Chandra, any invariant eigendistribution θ with regular infinitesimal character $\lambda \in \mathfrak{h}^*_{\text{reg}}$ is given by a locally L^1-function, whose restriction to a real Cartan subalgebra $\mathfrak{h}_{c,\mathbb{R}}$ is given by a formula

$$\theta \equiv \frac{1}{\epsilon_{c,\mathbb{R}} \pi_c} \sum_{y \epsilon W_c} m_{c,y} e^{y^{-1}\lambda_c} \quad \text{on } (\mathfrak{h}_c)_\mathbb{R},$$

where the $m_{c,\mathrm{y}}$ are $\mathbb{C}$-valued, locally constant functions on $\mathfrak{h}_{\mathbb{R},\text{reg}}$. A family $\theta(\cdot)$ of the above type is *coherent* (i.e. $\in \mathrm{CH}(\mathfrak{g}_\mathbb{R},\mathbb{C})$) if and only if the $m_{c,y}$ are independent of λ. (Compare the definition of "coherent familiy of invariant eigendistributions" given in [17].) In particular, a coherent family $\theta(\cdot)$ extends to a holomorphic function on all of $\mathfrak{h}^*$ with values in the space of distributions on $\mathfrak{g}_{\mathbb{R},\text{ reg}}$.
(b) A family $\Gamma(\lambda) \in {}'H_{2n}(\Omega_\lambda)$ of contours parametrized by $\lambda \in \mathfrak{h}^*_{\text{reg}}$ is *coherent* in the sense defined above if and only if the integral (1) is a holomorphic function of $\lambda \in \mathfrak{h}^*_{\text{reg}}$ (for arbitrary φ).
(c) For any $\lambda_\text{o} \in \mathfrak{h}^*_{\text{reg}}$ the map $\theta(\cdot) \mapsto \theta(\lambda_\text{o})$ induces a linear isomorphism from $\mathrm{C}H(\mathfrak{g}_\mathbb{R},\mathbb{C})$ to the space of invariant eigendistributions with infinitesimal character λ_o.

2. Representations of Weyl groups

We require some constructions from [16]. Recall the bijection $p_\lambda : \mathcal{B}^* \to \Omega_\lambda$, $\lambda \in \mathfrak{h}^*_{\text{reg}}$, defined in (3). For any $w \in W$, define a transformation $a_\lambda(w)$ of $\mathcal{B}^*$ by

$$a_\lambda(w) := p^{-1}_{w\lambda} \circ p_\lambda \colon \mathcal{B}^* \to \mathcal{B}^*. \tag{9}$$

Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}^*$ and $p\colon \mathcal{B}^* \to \mathcal{N}, (\mathfrak{b}, \nu) \mapsto \nu,$ the *Springer map*. For any subset V of $\mathcal{N}$, let $\mathcal{B}^*(V) := p^{-1}(V) \subset \mathcal{B}^*$. Choose $\epsilon > 0$ and let $U = \{\nu \in \mathcal{N} \mid \|\nu - \nu'\| < \epsilon \text{ for some } \nu' \in V\}$. Assume that for sufficiently small $\epsilon > 0$ the inclusion $i : \mathcal{B}^*(V) \to \mathcal{B}^*(U)$ admits a proper homotopy inverse in the sense that there is a continuous map $j\colon \mathcal{B}^*(U) \to \mathcal{B}^*(V)$ so that $j \circ i \sim 1$ on $\mathcal{B}^*(V)$, $i \circ j \sim 1$ on $\mathcal{B}^*(U)$ with "$\sim$" meaning "properly homotopic". Then for $\lambda \in \mathfrak{h}^*_{\text{reg}}$ in some ball around 0, the map $a(w) := j \circ a_\lambda(w) \circ i$ induces a transformation of $\mathcal{B}^*(V)$ whose proper homotopy class is independent of λ. This gives a representation of W on $H_*(\mathcal{B}^*(V))$. Applied to $V = \{\nu\}$, $\nu \in \mathcal{N}$, the construction produces a representation of W on the homology of $\mathcal{B}^*(\nu)$;

$\mathcal{B}^*(\nu)$ may be identified with $\mathcal{B}^\nu := \{\mathfrak{b} \in \mathcal{B} \mid \nu \in \mathfrak{b}^\perp\}$. The component group $A_\nu := G_\nu/(G_\nu)_\circ$ of the stabilizer G_ν of ν in G acts naturally on $H_*(\mathcal{B}^\nu)$ by W-automorphisms, so that one has a representation of $W \times A_\nu$ thereon. In highest degree $2e := 2\dim_{\mathbb{C}} \mathcal{B}^\nu$, this representation decomposes as

$$H_{2e}(\mathcal{B}^\nu) \approx \sum_{\varphi \epsilon \Phi_\nu} \chi_{\nu,\varphi} \otimes \varphi$$

where Φ_ν is a set of irreducible characters of A_ν and $\chi_{\nu,\varphi}$ an irreducible character of W. Every irreducible character of W is of this form $\chi_{\nu,\varphi}$, and two pairs (ν, φ), (ν', φ') correspond to the same character if and only if they are conjugate by G. These results are due to Springer [19], [20] with a different construction of the representations. The construction outlined here is given in [16].

3. Decomposition of the coherent continuation representation

We apply the construction of representations of W outlined in §2 with $V = \mathcal{N}_{\mathbb{R}} := \mathcal{N} \cap i\mathfrak{g}_{\mathbb{R}}^*$. (As explained in [16], it follows from general facts in algebraic topology that $V = \mathcal{N}_{\mathbb{R}}$ satisfies the conditions required for the construction.) Note that for $V = \mathcal{N}_{\mathbb{R}}$, $\mathcal{B}^*(V) = \mathcal{S}$, so that we get a representation of W on $H_*(\mathcal{S})$. In top degree $2n = \dim_{\mathbb{R}} \mathcal{S}$, we get a representation of W on on $H_{2n}(\mathcal{S})$. Essentially by definition:

3.1 Lemma. *The isomorphism $H_{2n}(\mathcal{S}) \to \mathrm{CH}(\mathfrak{g}_{\mathbb{R}})$, $\Gamma \to \theta(\Gamma)$, satisfies $\theta(w\Gamma,\lambda) := \theta(\Gamma, w^{-1}\lambda)$.*

Proof. Let $\Gamma \in H_{2n}(\mathcal{S})$. By definition (9) of the action of W on $H_{2n}(\mathcal{S})$, $w \cdot \Gamma \sim p_{w\lambda}^{-1} \circ p_\lambda(\Gamma)$ for any $\lambda \in \mathfrak{h}^*_{\mathrm{reg}}$ in some ball around 0. Here "$\sim$" means "properly homotopic". Thus $p_{w\lambda} w\Gamma \sim p_\lambda \Gamma$ which gives $\theta(w\Gamma, w) = \theta(\Gamma,\lambda)$ as required. Q.E.D.

Thus under the isomorphism $H_{2n}(\mathcal{S}) \to \mathrm{CH}(\mathfrak{g}_{\mathbb{R}})$, $\Gamma \to \theta(\Gamma,\cdot)$, the representation of W on $H_{2n}(\mathcal{S})$ becomes the *coherent continuation representation* of W on the $\mathbb{Z}$-module $\mathrm{C}H(\mathfrak{g}_{\mathbb{R}})$ of coherent families $\theta(\cdot)$ of invariant eigendistribution given by $(w \cdot \theta)(\lambda) = \theta(w^{-1} \cdot \lambda)$.

3.2 Corollary. *As representation of W,*

$$H_{2n}(\mathcal{S}) \approx \sum_c \mathrm{Ind}_{W_{c,\mathbb{R}}}^W \mathrm{sgn}_{c,\mathrm{I}} \tag{10}$$

where $c \in G$ is chosen so that $\mathfrak{h}_{c,\mathbb{R}}$ runs over a complete set of representatives for the $G_{\mathbb{R}}$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_{\mathbb{R}}$, and $W_{c,\mathbb{R}}$ is considered as a subgroup of W via c.

Proof. This follows from the lemma and theorem 1.4(c). Q.E.D.

3.3 Theorem. *The representation of W on $H_{2n}(\mathcal{S})$ decomposes as*

$$H_{2n}(\mathcal{S}) \approx \sum_{\nu,\varphi} m(\nu,\varphi)\chi_{\nu,\varphi}$$

where ν runs over a set of representatives for the nilpotent $G_{\mathbb{R}}$-orbits, φ over the set of irreducible representations of A_ν in Φ_ν which contain a vector fixed by $A_{\nu,\mathbb{R}}$, and $m(\nu,\varphi)$ is the dimension of the space of these vectors.

The proof of theorem 3.3 requires some preliminary lemmas. Consider the Springer map p: $\mathcal{S} \to \mathcal{N}_{\mathbb{R}}$. Let $\mathcal{N}_{\mathbb{R}} = \bigcup_O O$ be the decomposition into $G_{\mathbb{R}}$-orbits, and correspondingly $\mathcal{S} = \bigcup_O \mathcal{S}(O)$
with $\mathcal{S}(O) := p^{-1}(O)$.

3.4 Lemma. *For all O, $\dim_{\mathbb{R}} \mathcal{S}(O) = 2n$. Furthermore, the restriction of $\mathcal{S} \to \mathcal{N}_{\mathbb{R}}$ to $\mathcal{S}(O)$ is a $G_{\mathbb{R}}$-equivariant fibration with fibre$\mathcal{B}^\nu$ over $\nu \in O$:*

$$\mathcal{B}^\nu \hookrightarrow \mathcal{S}(O) \to O. \tag{11}$$

Proof. The second assertion is evident. The first is a consequence of results of Spaltenstein and Steinberg, as follows. According to [18], the irreducible components of the complex variety $\mathcal{B}^\nu$ have the same dimension, say $\dim_{\mathrm{C}} \mathcal{B}^\nu = e$. According to [21], the components of $\mathcal{B}^\nu \times \mathcal{B}^\nu$are the intersections of the components of the variety $\mathcal{Z} := \{(\mathfrak{b}, \mathfrak{b}') \mid \nu \in \mathfrak{b}^\perp \cap (\mathfrak{b}')^\perp\}$ with the fibre $\mathcal{B}^\nu \times \mathcal{B}^\nu$ of the map $\mathcal{Z} \to \mathcal{N}$. Furthermore, the components of $\mathcal{Z}$ all have the same $\mathbb{C}$-dimension, namely $2n$, because they are the closures of the conormal bundles of the G-orbits on $\mathcal{B} \times \mathcal{B}$ (under the diagonal action). From the fibration $\mathcal{B}^\nu \times \mathcal{B}^\nu \hookrightarrow \mathcal{Z}(\mathcal{O}) \to \mathcal{O}$ induced by $\mathcal{Z} \to \mathcal{N}$ over a fixed G-orbit $\mathcal{O} = G \cdot \nu$, one sees that $\dim_{\mathrm{C}} \mathcal{O} + 2\dim_{\mathrm{C}} \mathcal{B}^\nu = 2n$. For a $G_{\mathbb{R}}$-orbit $O = G_{\mathbb{R}} \cdot \nu$, $\nu \in \mathcal{N}_{\mathbb{R}}$, the fibration $\mathcal{B}^\nu \hookrightarrow \mathcal{S}(O) \to O$ mentioned above now gives $\dim_{\mathbb{R}} \mathcal{S}(O) = \dim_{\mathbb{R}} O + \dim_{\mathbb{R}} \mathcal{B}^\nu = \dim_{\mathrm{C}} O_{\mathrm{C}} + 2\dim_{\mathrm{C}} \mathcal{B}^\nu = 2n$ as required. Q.E.D.

For $\nu \in \mathcal{N}_{\mathbb{R}}$ set $A_{\nu,\mathbb{R}} := G_{\nu,\mathbb{R}}/G_{\nu,\mathbb{R}} \cap (G_\nu)_{\mathrm{o}}$ a subgroup of $A_\nu := G_\nu/(G_\nu)_{\mathrm{o}}$.

3.5 Lemma. *For any $G_{\mathbb{R}}$-orbit $O = G_{\mathbb{R}} \cdot \nu$ on $\mathcal{N}_{\mathbb{R}}$,*

$$H_{2n}(\mathcal{S}(O)) \approx H_{2e}(\mathcal{B}^\nu)^{A_{\nu,\mathbb{R}}},$$

where $e = \dim_C \mathcal{B}^\nu$ and the right side denotes the $A_{\nu,\mathbb{R}}$-invariants under the natural action of A_ν on $H_{2e}(\mathcal{B}^\nu)$.

Proof. The fibration (11), $\mathcal{B}^\nu \hookrightarrow \mathcal{S}(O) \to O$, leads to an isomorphism

$$H_{2n}(\mathcal{S}(O)) \approx H_{2e}(\mathcal{B}^\nu)^{\pi_1(O,\nu)}, \tag{12}$$

which may be described as follows. A continuous path from ν to ν' in O gives an isomorphism $H_*(\mathcal{B}^\nu) \to H_*(\mathcal{B}^{\nu'})$ by trivialization of the fibration (11) along the path. On the invariants of the monodromy action of $\pi_1(O,\nu)$ in $H_*(\mathcal{B}^\nu)$, this isomorphism is independent of the path. An element $\Gamma \in H_{2n}(\mathcal{S}(O))$ decomposes into its fibres $\Gamma_\nu \in H_{2\mathrm{e}}(\mathcal{B}^\nu)^{\pi_1(O,\nu)}$ under $\mathcal{S}(O) \to O$:

$\Gamma_\nu \hookrightarrow \Gamma \to O$. The Γ_ν and $\Gamma_{\nu'}$ belonging to ν and ν' are related by the isomorphism mentioned. $\Gamma \leftrightarrow \Gamma_\nu$ gives the isomorphism (12). An element $[\gamma]$ of $\pi_1(O,\nu)$ may be realized in the form $\gamma(\mathrm{t}) = g(t)\cdot\nu$, where $g(t)$ $(0 \leq \mathrm{t} \leq 1)$ is a continuous path in $G_{\mathbb{R}}$ with $g(0)=1$, $g(1) \in G_{\nu,\mathbb{R}}$. The monodromy action of $[\gamma]$ on $H_*(\mathcal{B}^\nu)$ coincides with the natural action of $g(1)$. If $g(1) \in (G_\nu)_{\mathrm{o}}$, then $g(1)$ acts trivially, hence the action of $\pi_1(O,\nu)$ on $H_*(\mathcal{B}^\nu)$ factors though the homomorphism $\pi_1(O,\nu) \to A_{\nu,\mathbb{R}}$, $[\gamma] \to [g(1)]$. Q.E.D.

Fix a $G_{\mathbb{R}}$-orbit $O = G_{\mathbb{R}}\cdot\nu$ on $\mathcal{N}_{\mathbb{R}}$. Let $\bar{O}$ be its closure, $\partial O := \bar{O} - O$ its topological boundary.

3.6 Lemma. *The natural maps*
$$0 \to H_{2n}(\mathcal{S}(\partial O)) \to H_{2n}(\mathcal{S}(\bar{O})) \to H_{2n}(\mathcal{S}(O)) \to 0$$
form an exact sequence of W-modules.

Proof. The couple $\partial O \subset \bar{O}$ of closed $\mathbb{R}$-subvarietes of $\mathcal{N}_{\mathbb{R}}$ gives a long-exact sequence of W-modules

$$\begin{aligned}\cdots \quad &\to H_{i+1}(\mathcal{S}(O)) \to H_i(\mathcal{S}(\partial O)) \to H_i(\mathcal{S}(\bar{O})) \to \\ &\to H_i(\mathcal{S}(O)) \to H_{i-1}(\mathcal{S}(\partial O)) \to \quad \cdots .\end{aligned}$$

In top degree $i = 2n$ this gives

$$0 \to H_{2n}(\mathcal{S}(\partial O)) \to H_{2n}(\mathcal{S}(\bar{O})) \to H_{2n}(\mathcal{S}(O)).$$

The surjectivity of the map on the right follows from the fact that $\bar{O}$ and ∂O admit compatible CW-decompositions [7] and standard facts about the homology of CW-complexes ([12], Theorem 4.1). Q.E.D.

Proof of theorem 3.1. The map $H_{2n}(\mathcal{S}(\bar{O})) \to H_{2n}(\mathcal{S})$ is injective, as one sees from the exact sequence associated to the inclusion $\mathcal{S}(\bar{O}) \subset \mathcal{S}$. $H_{2n}(\mathcal{S}(\bar{O}))$ may therefore be considered a W-submodule of $H_{2n}(\mathcal{S})$, and $H_{2n}(\mathcal{S}(O))$ a W-subquotient of $H_{2n}(\mathcal{S})$:

$$H_{2n}(\mathcal{S}(O)) \approx H_{2n}(\mathcal{S}(\bar{O}))/\sum_{O'<O} H_{2n}(\mathcal{S}(O))$$

where $O' < O$ means $O' \subset \partial O$. The filtration of $\mathcal{N}_{\mathbb{R}}$ according to the dimension of the $G_{\mathbb{R}}$-orbits, $\mathcal{N}_{i,\mathbb{R}} := \bigcup_{\dim_{\mathbb{R}} O \leq i} O$ induces a filtration of $\mathcal{S}, \mathcal{S}_i := \bigcup_{\dim_{\mathbb{R}} O \leq i} \mathcal{S}(O)$, and hence of $H_{2n}(\mathcal{S})$:

$$H_{2n}(\mathcal{S}_{\mathrm{i}}) = \sum_{\dim_{\mathbb{R}} O \leq i} H_{2n}(\mathcal{S}(\bar{O})).$$

The corresponding graded group is then

$$\mathrm{gr}H_{2n}(\mathcal{S}) = \sum_{O} H_{2n}(\mathcal{S}(O)) \tag{13}$$

From Lemma 3.5

$$H_{2n}(\mathcal{S}(O)) \approx H_{2e}(\mathcal{B}^\nu)^{A_\nu(\mathbb{R})} \tag{14}$$

and from Springer's theory (§2)

$$H_{2\mathrm{e}}(\mathcal{B}^\nu) \approx \sum_{\varphi \epsilon \Phi_\nu} \chi_{\nu,\varphi} \otimes \varphi$$

as representation of $W \times A_\nu$. Putting these pieces together one gets the decomposition of theorem 3.3, since $\mathrm{gr}H_{2n}(\mathcal{S}) \approx H_{2n}(\mathcal{S})$ as representations of W. Q.E.D.

We point out the following consequence of theorem 3.3 and corollary 3.2.

3.7 Corollary. *Let $\nu \in \mathcal{N}_\mathbb{R}$ and $G \cdot \nu \cap \mathrm{i}\mathfrak{g}_\mathbb{R} = G_\mathbb{R} \cdot \nu_1 \cup \cdots \cup G_\mathbb{R} \cdot \nu_m$ the decomposition of $G \cdot \nu \cap \mathrm{i}\mathfrak{g}_\mathbb{R}$ into distinct $G_\mathbb{R}$-orbits. Then*

$$m = \sum_c [\mathrm{Ind}_{W_{c,\mathbb{R}}}^W \mathrm{sgn}_{c,\mathrm{I}} : \chi_{\nu,1}].$$

Proof. By Theorem 3.3, $m = [H_{2n}(\mathcal{S}){:}\chi_{\nu,1}]$;by cCrollary 3.2
$[H_{2n}(\mathcal{S}){:}\chi_{\nu,1}] = \sum_c [\mathrm{Ind}_{W_{c,\mathbb{R}}}^W \mathrm{sgn}_{c,\mathrm{I}} : \chi_{\nu,1}]$.

Q.E.D.

3.1 Remark. For $G_\mathbb{R} = U(p,q)$, Barbasch and Vogan [5] prove the decomposition of the coherent continuation representation (defined in terms of characters) according to nilpotent $G_\mathbb{R}$-orbits by direct verification based on (10). (In that case A_ν is trivial for all $\nu \in \mathcal{N}$.)

4. Nilpotent orbital integrals

We return to the integral (5), $\theta(\Gamma,\lambda) = \frac{1}{(-2\pi\mathrm{i})^n n!}\int_{\xi\in \mathrm{p}_\lambda\Gamma} e^\xi \sigma_\lambda^n$ with $\Gamma \in H_{2n}(\mathcal{S})$. Recall the natural isomorphism (13), $\mathrm{gr}H_{2n}(\mathcal{S}) \approx \sum_O H_{2n}(\mathcal{S}(O))$ and (14), $H_{2n}(\mathcal{S}(O)) \approx H_{2e}(\mathcal{B}^\nu)^{A_\nu(\mathbb{R})}$. The inclusion $\mathcal{B}^\nu \hookrightarrow \mathcal{B}$ induces a W-injection

$$H_{2e}(\mathcal{B}^\nu)^{A_\nu} \hookrightarrow H_{2e}(\mathcal{B}) \tag{15}$$

([16], Lemma 3.1.) The character of W on $H_{2\mathrm{e}}(\mathcal{B}^\nu)^{A_\nu}$ is $\chi_\nu := \chi_{\nu,1}$. According to Borel [6], the cohomology ring of $\mathcal{B}$ can be described as follows. For $\lambda \in \mathfrak{h}^*$, let τ_λ denote the U-invariant 2-form on $\mathcal{B}$ which at the base point $\mathfrak{b}_1$ is given by $\tau_\lambda([x\cdot\mathfrak{b}_1, y\cdot\mathfrak{b}_1]) := \lambda([x,y])$ for $x, y \in \mathfrak{u}$. Let I^+ denote the ideal in the ring $\mathbb{C}[\mathfrak{h}]$ of polynomial function on $\mathfrak{h}$ generated by the W-invariants without constant term. There is a unique isomorphism of rings

$$\mathbb{C}[\mathfrak{h}]/I^+ \to H^*(\mathcal{B}), \quad [\lambda] \mapsto [\frac{1}{-2\pi\mathrm{i}}\tau_\lambda]. \tag{16}$$

The dual space of $\mathbb{C}[\mathfrak{h}]/\mathrm{I}^+$ is the space $\mathcal{H}(\mathfrak{h}^*)$ of W-harmonic polynomials on $\mathfrak{h}^*$, and the transpose of the isomorphism (16) is a W-isomorphism

$$H_*(\mathcal{B}) \to \mathcal{H}(\mathfrak{h}^*),\ \gamma \mapsto c_\gamma, \quad c_\gamma(\lambda) := \frac{1}{(-2\pi\mathrm{i})^e}\int_\gamma \tau_\lambda^e \tag{17}$$

given by for $\gamma \in H_{2e}(\mathcal{B})$. For $\nu \in \mathcal{N}$, let $\mathcal{H}_\nu(\mathfrak{h}^*)$ denote the space of W-harmonic polynomials on $\mathfrak{h}^*$ which are homogeneous of degree $e = \dim_{\mathbb{C}} \mathcal{B}^\nu$ and transform according to the irreducible character χ_ν of W. From (13), (14), (15), and (17), together with

$$H_{2n}(\mathcal{S}(O)) \approx H_{2n}(\mathcal{S}(\bar{O}))/\sum_{O'<O} H_{2n}(\mathcal{S}(O)),$$

there results a map $H_{2n}(\mathcal{S}(\bar{O})) \to \mathcal{H}_\nu(\mathfrak{h}^*)$, $\Gamma \to c_\Gamma$. We introduce the following notation.

$O := G_{\mathbb{R}} \cdot \nu$, a $G_{\mathbb{R}}$-orbit on $\mathcal{N}_{\mathbb{R}} \subset \mathrm{i}\mathfrak{g}^*_{\mathbb{R}}$;
$2d := \dim_{\mathbb{R}} O = \dim_{\mathbb{C}} G \cdot \nu$;
$e := \dim_{\mathbb{C}} \mathcal{B}^\nu$;
σ_ν :=the canonical holomorphic 2-form on $G \cdot \nu$;
μ_ν :=the canonical measure on O, i.e. $\mu_\nu(\varphi) = \frac{1}{(-2\pi \mathrm{i})^d} \int_{\xi \in O} \varphi \sigma_\nu^d$;
θ_ν :=the Fourier transform of μ_ν, i.e. $\theta_\nu(f) := \mu_\nu(\varphi)$ where
$\varphi(\xi) := \int_{\mathfrak{g}_{\mathbb{R}}} f(x) e^{\xi(x)} dx$.

With this notation:

4.1 Theorem. *Let $\Gamma \in H_{2n}(\mathcal{S}(\bar{O}))$, $\theta(\Gamma, \cdot)$ the corresponding coherent family of invariant eigendistributions. Then*

$$\theta(\Gamma,\lambda) = c_\Gamma(\lambda)\theta_\nu + o(\|\lambda\|^e). \tag{18}$$

Proof. Let $\mathcal{O} = G \cdot \nu$ be the complex orbit containing $O = G_{\mathbb{R}} \cdot \nu$, $\mathcal{B}^*(\mathcal{O})$ the inverse image of $\mathcal{O}$ under $p : \mathcal{B}^* \to \mathcal{N}$. Since $\mathcal{B}^*(\mathcal{O}) \subset \mathcal{B} \times \mathcal{O}$, we may consider $\tau_\lambda + \sigma_\nu$ as 2-form on $\mathcal{B}^*(\mathcal{O})$. By [16], this form agrees on $\mathcal{B}^*(\mathcal{O})$ with the pull-back $\mathrm{p}_\lambda^* \sigma_\lambda$ of the form σ_λ on Ω_λ by the map p_λ: $\Omega_\lambda \to \mathcal{B}^*$. Note that \ $d + e = n$ because of the fibration (11), $\mathcal{B}^\nu \hookrightarrow \mathcal{S}(O) \to O$, of Lemma 3.4. Let Γ be a $2n$-cycle on $\mathcal{S}(O)$. As noted in the proof of lemma 3.5, the fibration (11) induces a fibre-decomposition

$$\Gamma_\nu \hookrightarrow \Gamma \to O, \tag{19}$$

with Γ_ν a $2e$-cycle on $\mathcal{B}^\nu$. Now compute the integral (1) of §1 as a repeated integral according to (19):

$$\begin{aligned}
\frac{1}{(-2\pi \mathrm{i})^n n!} \int_{\mathrm{p}_\lambda \Gamma} \varphi \sigma_\lambda^n &= \frac{1}{(-2\pi \mathrm{i})^n n!} \int_\Gamma (\varphi \circ p_\lambda)(\tau_\lambda + \sigma_\nu)^n \\
&= \frac{1}{(-2\pi \mathrm{i})^n n!} \int_\Gamma \sum_{r+s=n} (\varphi \circ p_\lambda) \ \frac{n!}{r!s!} \tau_\lambda^r \sigma_\nu^s \\
&= \frac{1}{(-2\pi \mathrm{i})^n e! d!} \int_{\mu \in O} \Big\{ \int_{\mathfrak{b} \in \Gamma_\mu} \varphi(p_\lambda(\mathfrak{b}, \mu)) \tau_\lambda^e \Big\} \sigma_\nu^d + o(\|\lambda\|^e).
\end{aligned}$$

Write $(\mathfrak{b},\mu) = u\cdot(\mathfrak{b}_1,\mu')$ with $u\in U$, and $\mu'\in\mathfrak{b}_1^{\perp}$. Then

$$\begin{aligned}\varphi(p_\lambda(\mathfrak{b},\mu)) &= \varphi(u\cdot(\lambda+\mu')) \\ &= \varphi(u\cdot\mu') + o(\|u\cdot\lambda\|) \\ &= \varphi(\mu) + \mathrm{o}(\|\lambda\|).\end{aligned}$$

Thus the integral is

$$= \frac{1}{(-2\pi \mathrm{i})^d d!}\int_{\mu\in O}\varphi(\mu)\Big\{\frac{1}{(-2\pi\mathrm{i})^e e!}\int_{\Gamma_\mu}\tau_\lambda^e\Big\}\sigma_\nu^d \ + \mathrm{o}(\|\lambda\|^e).$$

The expression in parentheses is independent of μ and equals $c_\Gamma(\lambda)$. Thus the integral is

$$= c_\Gamma(\lambda)\ \frac{1}{(-2\pi\mathrm{i})^d d!}\int_{\mu\in O}\varphi(\mu)\sigma_\nu^d \ + o(\|\lambda\|^e),$$

which is the desired formula when Γ is a $2n$-cycle on $\mathcal{S}(O)$. Since $\partial\bar{O} = \bigcup O'$ with $e' > e$, the same formula holds for $\Gamma\in H_{2n}(\mathcal{S}(\bar{O}))$, with c_Γ depending only on the image of Γ under the map $H_{2n}(\mathcal{S}(\bar{O}))\to H_{2n}(\mathcal{S}(O))$. Q.E.D.

5. A limit formula

We fix a real Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$, which we may as well assume to be $\mathfrak{h}_\mathbb{R}$ for the previously fixed $\mathfrak{h}$. It will be convenient to use another parametization of the Γ(S,$\cdot$) with S containing a fixed-point of $\mathfrak{h}$, as follows. For $y\in W$, let $\mathfrak{b}_y := y\cdot\mathfrak{b}_1$, $S_y := G_\mathbb{R}\cdot\mathfrak{b}_\mathrm{y}$. Then $\Gamma(S_y,\cdot)$ is the unique coherent family of contours satisfying $\Gamma(S_y,\mu) = [G_\mathbb{R}\cdot y\mu]$ if $y\mu(H_\alpha) > 0$ for all $\alpha\in(y\Delta^+)\cap\Delta_\mathrm{I}$. Let $C_\mathrm{y} := \{\mu\in\mathfrak{h}^*_{\mathbb{R},\mathrm{reg}} \mid \mu(H_\alpha)>0 \text{ for } \alpha\in(\mathrm{y}\Delta^+)\cap\Delta_\mathrm{I}\}$ so that $\Gamma(S_y,\cdot)$ is the unique coherent family satisfying $\Gamma(S_y, y^{-1}\mu) = [G_\mathbb{R}\cdot\mu]$ if $\mu\in C_y$. Define $\Gamma(C,\cdot) := y^{-1}\Gamma(S_y,\cdot)$ if $C = C_\mathrm{y}$. Γ(C,$\cdot$) is a coherent family depending only on the chamber C for Δ_I. As (C,y) runs over a set of representatives of the $W_\mathbb{R}$-orbits of pairs satisfying $C_y = C$, the corresponding families $y\cdot\Gamma(C,\cdot)$ run through $\{\Gamma(S,\cdot) \mid \mathfrak{h}\subset\mathfrak{b} \text{ for some } \mathfrak{b}\in\mathrm{S}\}$. Write $\theta(C,\cdot)$ for the coherent family of invariant eigendistributions corresponding to $\Gamma(C,\cdot)$. The chamber C for Δ_I will remain fixed from now on. Let $\mathcal{N}_\mathrm{C} := \bigcap_{\mu\in C}\Big(\mathcal{N}\cap\overline{G_\mathbb{R}\cdot\mathbb{R}_+^\times\mu}\Big)$. Let $O = G_\mathbb{R}\cdot\nu$ be a nilpotent orbit in $\mathrm{i}\mathfrak{g}^*_\mathbb{R}$, $\mathcal{O} = G\cdot\nu$ the complex orbit containing O. We shall repeatedly refer to the following *hypothesis* (O):

$$O = \mathcal{O}\cap\mathcal{N}_\mathrm{C}. \qquad (O)$$

5.1 Remarks. (a) It seems likely that $\mathcal{N}\cap\overline{G_\mathbb{R}\cdot\mathbb{R}_+^\times\mu}$ is in fact the same for all $\mu\in$C.
(b) It follows from the theory of SL$_2$-triples ([1] Proposition 3.1) that for any nilpotent orbit $O = G_\mathbb{R}\cdot\nu$ in $\mathrm{i}\mathfrak{g}^*_\mathbb{R}$ there is an elliptic (not necessarily regular) element $\mu\in\mathrm{i}\mathfrak{g}^*_\mathbb{R}$ so that $O\subset\overline{G_\mathbb{R}\cdot\mathbb{R}_+^\times\mu}$.

(c) It is known that the hypothesis (O) is satisfied for nilpotent orbits of dimension $2n$ with C a chamber in the fundamental Cartan subalgebra [14]. It is trivally satisfied at the opposite extreme, $O = \{)\}$, and for all O when $\mathfrak{g}_{\mathbb{R}}$ is complex.

For any $\xi \in \mathrm{i}\mathfrak{g}_{\mathbb{R}}^*$, let μ_ξ be the canonical measure on $G_{\mathbb{R}} \cdot \xi$ and θ_ξ its Fourier transform (defined as in §4 for $\xi = \nu$). In particular $\theta(C,\, \mu) = \theta_\mu$ if $\mu \in C$. A polynomial $p \in \mathbb{C}[\mathfrak{h}]$ on $\mathfrak{h}$ may be considered a differential operator on $\mathfrak{h}^*$, denoted $p(\partial)$ or $p(\partial_\lambda)$ with λ indicating the variable of differentiation.

5.2 Lemma. *Let $p \in \mathbb{C}[\mathfrak{h}]$, $\theta \in \mathrm{CH}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h})$. Then $\lim_{\lambda \to 0} p(\partial_\lambda)\theta(\lambda)$ exits as a distribution on $\mathfrak{g}_{\mathbb{R},reg}$ and defines a W-invariant pairing on $\mathbb{C}[\mathfrak{h}] \otimes \mathrm{CH}(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h})$.*

Proof. This is clear, since $\theta(\cdot)$ extends to a holomorphic function on all of $\mathfrak{h}^*$ with values in the space of distributions on $\mathfrak{g}_{\mathbb{R},\mathrm{reg}}$ (1.4(a)). Q.E.D.

5.3 Theorem. *Assume O satisfies hypothesis (O). Let p be any polynomial on $\mathfrak{h}$, homogeneous of degree e and transforming according to χ_ν by W. Then*

$$\lim_{\lambda \to 0(C)} p(\partial_\lambda)\mu_\lambda = \kappa \mu_\nu$$

for some constant $\kappa = \kappa(C, p, \nu)$. The constant $\kappa \neq 0$ for some p if and only if χ_ν occurs in the W-module generated by $\theta(C,\cdot)$.

Proof. Let $\theta_\nu(C, \cdot)$ be the component of $\theta(C, \cdot)$ transforming according to χ_ν,

$$\theta_\nu(C, \lambda) = \frac{\deg \chi_\nu}{|\, W\, |} \sum_{y \in W} \chi_\nu(y)\theta(C, y^{-1}\lambda),$$

$\Gamma_\nu(C) \in H_{2n}(\mathcal{S})$ the element correspondig to $\theta_\nu(C, \cdot) \in \mathrm{C}H(\mathfrak{g}_{\mathbb{R}})$. According to Theorem 3.3, the subspace of $H_{2n}(\mathcal{S})_\nu$ of $H_{2n}(\mathcal{S})$ of type χ_ν can be described as follows. Let $\mathcal{O} := G \cdot \nu$ the complex orbit containing $O = G_{\mathbb{R}} \cdot \nu$, and write $\mathcal{O} \cap \mathcal{N}_{\mathbb{R}} = \bigcup O'$ with $O' = G_{\mathbb{R}} \cdot \nu'$. (We choose $\nu' = \nu$ for $O' = O$.) Then

$$H_{2n}(\mathcal{S})_\nu \approx \sum_{\nu'} H_{2n}(\mathcal{S}(O'))^{A_{\nu'}} \approx \sum_{\nu'} H_{2e}(\mathcal{B}^{\nu'})^{A_{\nu'}}. \tag{20}$$

applied to $\Gamma_\nu(\mathrm{C}) \in \sum_{\nu'} H_{2n}(\mathcal{S}(O'))^{A'}$, Theorem 4.1 gives

$$\theta_\nu(C, \lambda) = \sum_{\nu'} c_{\nu'}(\lambda)\theta_{\nu'} + o(\|\lambda\|^e). \tag{21}$$

We record that

$$\begin{array}{l} c_{\nu'}(\cdot) \in \mathcal{H}(\mathfrak{h}^*) \textit{ represents the component of } \Gamma_\nu(C) \\ \textit{in } H_{2e}(\mathcal{B}_{\nu'})^{A_{\nu'}} \textit{ according to the decomposition } (20). \end{array} \tag{22}$$

Suppose $p \in \mathbb{C}[\mathfrak{h}]$ is homogeneous of degee p and transforms by χ_ν. Then (21) and Lemma 5.2 give

$$\lim_{\lambda \to 0} p(\partial_\lambda)\theta(C, \lambda) = \sum_{\nu'} (p, c_{\nu'})\theta_{\nu'} \tag{23}$$

where $(p, c) := p(\partial)c(0)$ is the natural pairing. Take the limit in (23) from within C and apply the inverse Fourier transform to find that $\lim_{\lambda \to 0(C)} p(\partial_\lambda)\mu_\lambda = \sum_{\nu'} (p,c_{\nu'})\mu_{\nu'}$. The left side of this equation has support in $\mathcal{N} \cap \overline{G_{\mathbb{R}} \cdot \mathbb{R}_+^{\times}\lambda}$ for any fixed $\lambda \in$C, hence in $\mathcal{N}_{\mathrm{C}}$. The hypothesis (O) implies that $(p, c_{\nu'}) = 0$ for $\nu' \neq \nu$. Since this holds for all p

$$c_{\nu'} = 0 \text{ for } \nu' \neq \nu, \tag{24}$$

i.e.

$$\theta_\nu(\mathrm{C},\lambda) = c_\nu(\lambda)\theta_\nu + \mathrm{o}(\|\lambda\|^e). \tag{25}$$

This gives the desired formula$\lim_{\lambda \to 0(C)} p(\partial_\lambda)\mu_\lambda = (p, c_\nu)\mu_\nu$ and it follows from (22) and (24) that $(p, c_\nu) \neq 0$ for some p if and only if $\theta_\nu(C, \cdot) \neq 0$. Q.E.D.

It would be desirable to have for each O an explicit formula for a corresponding polynomial p. Under an additional hypothesis, this can be done as follows.

5.4 Corollary. *Assume O satisfies hypothesis (O) and assume the restriction of χ_ν to $W_{\mathbb{R}}$ contains the character* $\mathrm{sgn_I}$ *of $W_{\mathbb{R}}$ with multiplicity one. Let p_ν be the (up to scalars unique) W-harmonic polynomial on $\mathfrak{h}$ which is homogeneous of degee $e = \dim_C \mathcal{B}^\nu$, transforms by χ_ν under W, and transforms by* $\mathrm{sgn_I}$ *under $W_{\mathbb{R}}$. Then*

$$\lim_{\lambda \to 0(C)} p_\nu(\partial_\lambda)\mu_\lambda = \kappa\mu_\nu$$

with $\kappa \neq 0$.

Proof. As in Theorem 1.4 (c), write

$$\theta(C, \lambda) \equiv \frac{1}{\epsilon_{\mathbb{R}}\pi} \sum_{w \in W_{\mathbb{R}}} \mathrm{sgn_I(w)} e^{w^{-1}\lambda} \quad \text{on } \mathfrak{h}_{\mathbb{R}}.$$

$(\mathrm{sgn_I}(w))_{w \in W_{\mathbb{R}}}$ can be considered as an element in $\mathrm{Ind}_{W_{\mathbb{R}}}^{W} \mathrm{sgn_I}$; its component in the the irreducible subspace of type χ_ν is non-zero if and only if the restriction of χ_ν to $W_{\mathbb{R}}$ contains the character $\mathrm{sgn_I}$ of $W_{\mathbb{R}}$. This is the case, by assumption. Let $m = (m(w))_{w \in W}$ be the (up to scalars unique) non-zero element of $\mathrm{Ind}_{W_{\mathbb{R}}}^{W}$ transforming according to χ_ν under W and according to $\mathrm{sgn_I}$ under $W_{\mathbb{R}}$. We assume m normalized so that $\sum_{w \in W} m(w)w^{-1}$ operates as the projection on the space of such elements. Then

$$\theta_\nu(C, \lambda) \equiv \frac{1}{\epsilon_{\mathbb{R}}\pi} \sum_{w \in W} m(w) e^{w^{-1}\lambda} \quad \text{on } \mathfrak{h}_{\mathbb{R}}. \tag{26}$$

By what has just been said, the expression on the right is non-zero. Expand the exponentials in (26):

$$\theta_\nu(C, \lambda) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{\epsilon_{\mathbb{R}}\pi} \sum_{w \in W} m(w) w^{-1} \lambda^k. \tag{27}$$

Compare (27) with (25):

$$\theta_\nu(\mathrm{C},\lambda) = c_\nu(\lambda)\theta_\nu + \mathrm{o}(\|\lambda\|^{\mathrm{e}}), \quad c_\nu(\cdot) \neq 0.$$

One finds that

$$\sum_{w\in W} m(w)w^{-1}\lambda^k = 0 \text{ for } k < e$$
$$\frac{1}{\epsilon_{\mathbb{R}}\pi}\sum_{w\in W} m(w)w^{-1}\lambda^e \equiv c_\nu(\lambda)\theta_\nu.$$

Since $m(yw) = \mathrm{sgn}_{\mathrm{I}}(y)m(w)$ for $y \in W$, also $c_\nu(y\lambda) = \mathrm{sgn}_{\mathrm{I}}(y)\, c_\nu(y^{-1}\lambda)$. Hence $c_\nu(\cdot)$ is the (up to scalars unique) W-harmonic polynomial on $\mathfrak{h}^*$ which is homogeneous of degee e, transforms by χ_ν under W, and transforms by $\mathrm{sgn}_{\mathrm{I}}$ under $W_{\mathbb{R}}$. Hence $(p_\nu, c_\nu) \neq 0$. Q.E.D.

The "multiplicity-one" hypothesis, when it holds, may be verified with the help of the following criterion.

5.5 Lemma. *Let G be a locally compact group, σ an automorphism of G, H a σ-stable, compact subgroup of G. Assume that for g in some set of H:H double-coset representatives (and hence for all $g \in G$), $g^\sigma = h_g g^{-1} k_g$ with $h_g, k_g \in H$. Let ρ: $H \to \mathbb{C}^x$ be a one-dimensional representation of H so that $\rho(h^\sigma) = \rho(h)$ for all $h \in H$ and $\rho(h_g) = \rho(h_g)$ for all $g \in G$. Then $\mathrm{Ind}_H^K\rho$ decomposes with multiplicity one.*

Proof. The algebra of continuous G-endomorphisms of $\mathrm{Ind}_H^G\rho$ contains as a dense subalgebra the convolution algebra of compactly supported, continuous, $\mathbb{C}$-valued functions φ on G satisfying

$$\varphi(hgk) = \rho(h)\varphi(g)\rho(k)^{-1} \tag{28}$$

for all $g \in G$, $h, k \in H$. If suffices to show that this convolution algebra is abelian. Let $g^\tau = (g^\sigma)^{-1}$. If $\varphi(g)$ satisfies (28), so does $\varphi^\tau(g) := \varphi(^\tau)$. Since τ is an anti-automorphism of G,

$$(\varphi^*\psi)^\tau = \psi^{\tau *}\varphi^\tau. \tag{29}$$

On the other hand, a function φ as in (28) satisfies $\varphi^\tau = \varphi$:

$$\varphi^\tau(g) = \rho(k_g^{-1})\varphi(g)\rho(h_g) = \varphi(g).$$

Applied to the functions in (29), this gives $\varphi^*\psi = \psi^*\varphi$ as required. Q.E.D.

5.6 Remark. This kind of argument goes back at least to Gelfand [11]. (See also [9] Ch.X, Theorem 4.1.) The following result is due to E. Neher (unpublished).

5.7 Lemma. *Assume $\mathfrak{h}_{\mathbb{R}}$ is a Cartan subalgebra of compact type in $\mathfrak{g}_{\mathbb{R}}$. Then any left (or right) coset of $W_{\mathbb{R}}$ in W has a representative $s \in W$ satisfying $s^2 = 1$.*

Proof. Any element of W is a product of reflections, say

$$\cdots s_\alpha \mathrm{s}_\beta \cdots . \tag{30}$$

Generally

$$s_\alpha s_\beta = s_\gamma s_\alpha, \text{ where } \gamma = s_\alpha \beta. \tag{31}$$

If α is compact and β is non-compact, then $\mathrm{s}_\alpha\beta$ is non-compact. So (31) may be used to bring all the compact roots in (30) to the left (or to the right). The coset representative may therefore be chosen of the form (30) with non-compact roots only. It suffices to verify the following statement.

If α, β are non-orthogonal, non–compact roots, then $s_\alpha\beta$ is compact. (32)

Assuming this, any representative of the form (30) with a minimal number of reflections must consist of non-compact orthogonal reflections: otherwise (32) and (31) could be used to reduce the numbers of reflections without changing the coset. –The statement (32) concerns only the subalgebra of $\mathfrak{g}$ generated by root-vectors for α, β and its real form obtained by intersection with $\mathfrak{g}_\mathbb{R}$. This algebra has rank two and (32) may be verified by inspection of the rank-two root systems. Q.E.D.

5.8 Corollary. *Assume $\mathfrak{h}_\mathbb{R}$ is a Cartan subalgebra of compact type in $\mathfrak{g}_\mathbb{R}$. Then* $\mathrm{Ind}_{W_\mathbb{R}}^W \mathrm{sgn}_\mathrm{I}$ *decomposes with multiplicity one.*

Proof. One can take σ =identity in lemma 5.5. Q.E.D.

5.9 Example: Harish-Candra's Limit Formula [8]. Take $\nu = 0$. Then $\chi_\nu = \mathrm{sgn}$. Let $\varpi = \prod_{\alpha\in\Delta^+} \partial_\alpha$, with $\partial_\alpha\varphi = \mathrm{d}\varphi(\alpha)$ as operator on $\mathfrak{h}^*$. ϖ transforms according to $\chi_0 = \mathrm{sgn}$. Theorem 5.3 becomes $\lim_{\lambda\to 0} \varpi_\lambda \mu_\lambda = \kappa\mu_0$. The constant κ is non-zero if and only if $\mathfrak{h}_\mathbb{R}$ is fundamental.

The last assertion is seen as follows.

$$[\mathrm{sgn}|_{W_\mathbb{R}} : \mathrm{sgn}_\mathrm{I}] = \begin{cases} 1 & \text{if } \mathfrak{h} \text{ is fundamental} \\ 0 & \text{otherwise} \end{cases}$$

It follows that $\chi = \mathrm{sgn}$ occurs exactly once in $H_{2\mathrm{n}}(\mathcal{S}) \approx \sum_c \mathrm{Ind}_{W_{c,\mathbb{R}}}^W \mathrm{sgn}_{c,\mathrm{I}}$ namely in the summand for which $\mathfrak{h}_c$ is fundamental. Hence $\chi = \mathrm{sgn}$ occurs in the W-module generated by $\theta(\mathrm{C},\cdot)$ if only if C lies in the fundamental Cartan subalgebra. ("If" because of the formula for $\theta(\mathrm{C},\cdot)$ in theorem 1.4 (c).)

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