Invariant eigendistributions on a semisimple Lie algebra and homology classes on the conormal variety II: representations of Weyl groups

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Contents

1. Introduction.

This second part contains applications of the Integral Formula, as promised in the introduction to the first part. The main results I consider to be Theorems (5.1) and (7.1) , even though neither is hard to prove. Apart of being of some interest in themselves, they imply a number of results which are generally considered hard, if I am not mistaken, and certainly occupy considerable space in the literature. Among the results that follow from these theorems one might mention a formula for the global character of a (g_o, K_o)-module with regular integral infinitesismal character as an integral over the characteristic cycle of the corresponding $(\mathcal{D}, K)$-module (for g_o complex); a formula for the harmonic polynomials studied by Jantzen [1977], Joseph [1980], King [1979], Vogan [1978], and others through an interpretation of these polynomials as cohomology classes on the flag manifold; a proof of a conjecture of Joseph [1984]; and various other things.

The core of the paper is logically self-contained, except for reference to the Integral Formula and some basic facts about flag manifolds and conormal varieties, but factually indebted to many sources, as I shall point out where appropriate. Some peripheral results, however, rely on theorems not proved here (and beyond what might reasonably be called "basic facts"). In section 6, for example, I use a theorem of Kashiwara and of Tanisaki [1984, 1985], and in section 11 I quote a result of Hotta [1984]. On the whole, I have made some effort to keep the paper as elementary and self-contained as possible without becoming repetitious.

As mentioned, some of the results presented here among the applications of the Integral Formula are known; I included them when I felt that the present approach sheds some additional light thereon. An example is the Kazhdan-Lusztig [1980] completeness theorem for the Weyl group representation on the top homology of the conormal variety, which here appears in a very simple and explicit form as a consequence of Theorem 5.1. Another example is a formula for measures on nilpotent orbits proved by Barbasch and Vogan [1982, 1983] for special orbits and by Hotta and Kashiwara [1984] in general. Here this formula falls out as a byproduct of Theorem 7.1. I am indebted to Michèle Vergne for correcting a

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mistake in my proof of that theorem. The table of contents may serve as a further guide to the topics treated.

Some related papers, which have appeared or come to my attention since this paper was written, are cited in the supplementary references. In particular, the recent work of Joseph [1989] and Vergne [1989] provides an interesting alternative approach to some of the questions discussed in sections 7 and 10.

2. Construction of Weyl group representations

We keep the notation introduced in part I, except that (until further notice) g can be any complex, semisimple Lie algebra (not necessarily $\approx g_o \times g_o$), b_1 any Borel subalgebra of g containing the Cartan subalgebra h. For *any* (not necessarily regular) $\lambda \in h^*$ we set

$$\Omega_\lambda = \{\xi \in g^* : p(\xi) = p(\lambda) \text{ for all } G\text{-invariant polynomials } p \text{ on } g^*\}.$$

The map

$$p_\lambda = p_{b_1,\lambda} : \mathcal{B}^* \to \Omega_\lambda, u \cdot (b_1, \nu) \to u \cdot (\lambda + \nu).$$

with $u \in U$ (compact form of G), $\nu \in {b_1}^\perp$ is then well-defined and surjective for *all* $\lambda \in h^*$.

Assume now λ regular. Then p_λ is bijective and for any $w \in W$ we may define a transformation $a_\lambda(w) = a_{b_1,\lambda}(w)$ of $\mathcal{B}^*$ by

$$a_\lambda(w) = {p_{w\lambda}}^{-1} \circ p_\lambda : \mathcal{B}^* \to \mathcal{B}^*.$$

It is evident that

$$a_\lambda(wy) = a_{y\lambda}(w) a_\lambda(y). \tag{1}$$

If one could set $\lambda = 0$ in this equation, one would get an action of W on $\mathcal{B}^*$, which would leave the map $p_o : \mathcal{B}^* \to \Omega_0$ invariant, so that W would permute the fibers of this map. This is of course only trivially possible, as the fibers are generically single points: the map p_0 is the *Springer map*, which is a desingularization $\pi : \mathcal{B}^* \to \mathcal{N}$ of the nilpotent cone $\Omega_o = \mathcal{N}$ in g^* [Steinberg 1976]. Borrowing an idea of Kazhdan and Lusztig [1980] we look for what one might call a (proper) *homotopy action* of W on $\mathcal{B}^*$, meaning a homomorphism of W into the group of proper homotopy equivalences of $\mathcal{B}^*$, rather than a genuine action. But we shall not use the Kazhdan-Lusztig construction (which in fact they could not prove to give a homotopy action); instead we use the $a_\lambda(w)$. (We shall prove in an appendix that our construction agrees with Kazhdan-Lusztig's, proving incidentally that their construction gives a homotopy action after all. Of course, what we are ultimately interested in is a representation of W in homology, and that they did obtain -at least for the top-dimensional homology. Several other constructions of (variants of) this representation of W are known, first of all Springer's original construction [Springer 1976,1978], another construction of Lusztig [1981], and others.)

We write out explicitly the definition of $a_\lambda(w)$:

$$a_\lambda(w)(u \cdot (b_1, \nu)) = u' \cdot (b_1, \nu') \text{ where } u' \cdot (w\lambda + \nu') = u(\lambda + \nu).$$

Thus

$$u' \cdot \nu' - u \cdot \nu = -u' \cdot w\lambda + u \cdot \lambda$$

hence

$$|u' \cdot \nu' - u \cdot \nu| \leq \text{const.}|\lambda|.$$

Thus for λ close to 0 (but regular) the transformations $a_\lambda(w)$ leave the Springer map $\pi : \mathcal{B}^* \to \mathcal{N}$ approximately invariant in the sense that

$$|\pi(a_\lambda(w)(b,\nu)) - \pi(b,\nu))| \leq \text{const.}|\lambda|. \tag{2}$$

For any subset V of $\mathcal{N}$ let

$$\mathcal{B}^*(V) = \{(b,\nu) \in \mathcal{B}^* : \nu \in V\},$$

the inverse image of V in $\mathcal{B}^*$. We wish to constuct a proper homotopy action of W on $\mathcal{B}^*(V)$. This requires a regularity condition on V. Namely, for fixed $\epsilon > 0$, let U the be the ϵ-neighbourhood of V in $\mathcal{N}$,

$$U = \{\nu \in \mathcal{N} : |\nu - \nu'| < \epsilon \text{ for some } \nu' \in V\},$$

and

$$\mathcal{B}^*(U) = \{(b,\nu) \in \mathcal{B}^* : |\nu - \nu'| < \epsilon \text{ for some } \nu' \in V\} \tag{3}$$

its inverse image in $\mathcal{B}^*$. We require that

for sufficiently small $\epsilon > 0$, the inclusion i : $\mathcal{B}^*(V) \to \mathcal{B}^*(U)$ should admit a proper homotopy inverse $p : \mathcal{B}^*(U) \to \mathcal{B}^*(V)$, i.e.

$$p \circ i \sim 1 \text{ on } \mathcal{B}^*(V), \text{ and } i \circ p \sim 1 \text{ on } \mathcal{B}^*(U) \tag{4}$$

with "$\sim$" meaning "properly homotopic".

We observe that this condition is satisfied in either the following two cases:

(a) V is a finite subpolyhedron of a triangulation of $\mathcal{N}$, (5)
(b) V is a constructible subset of $\mathcal{N}$, stable under scalar multiplications. (6)

The first condition may be explained as follows. Let $\bar{g}^* = P(g^* \oplus \mathbf{C})$ be the projective completion of g^*, $\bar{\mathcal{N}}$ the closure of $\mathcal{N}$ in $\bar{g}^*$, $\bar{\mathcal{B}}^*$ the closure of $\mathcal{B}^* \subset \mathcal{B} \times \mathcal{N}$ in $\mathcal{B} \times \bar{\mathcal{N}}$. As a projective algebraic variety, $\bar{\mathcal{N}}$ admits a triangulation [Hironaka 1975], hence (5) makes sense.

That (5) implies (4) is seen as follows. Assume (5). Then $\mathcal{B}^*(V)$ is also a finite subpolyhedron of a triangulation of $\mathcal{B}^*$. It is an elementary fact that a finite subpolyhedron has a retractable neighbourhood [Alexandroff-Hopf, 1935, sections 6.1-2]. In particular $\mathcal{B}^*(V)$ has a retractible neighbourhood W in $\mathcal{B}^*$. Since π is a proper map, there is a neighbourhood U of V in $\mathcal{N}$ so that $\pi^{-1}(U) \subset W$, which gives (4).

That the condition (6) also implies (4) is seen as follows. One may suppose that V excludes 0, as a neighbourhood of 0 may be treated separately. Because of the assumption that V is stable under scalar multiplications and excludes 0, one may then replace V by $V \cap \{|\nu| = 1\}$ in order to prove (4). Since V is also constructible, one may apply (5).

From now on we shall assume that all sets V under consideration satisfy either (5) or (6), so that (4) applies. (This may well be unnecessarily restrictive, but is sufficient for the applications we have in mind.) It is clear from (2) and (3) that

$$a_\lambda(w)\mathcal{B}^*(V) \subset \mathcal{B}^*(U)$$

for $\lambda \in h^*$ sufficiently close to 0. Thus the transformation

$$a_{\lambda,V}(w) = p \circ a_\lambda(w) \circ i \tag{7}$$

of $\mathcal{B}^*(V)$ is defined for all regular λ in a small ball about 0 in h^*. Since these λ form a connected set, the proper homotopy class $a_V(w)$ of $a_{\lambda,V}(w)$ is independent of λ, and the equation (1) implies that

$$a_V(wy) = a_V(w)a_V(y)$$

so that *the $a_V(w)$ give a proper homotopy action of W on $\mathcal{B}^*(V)$.*

As a consequence we have a representation of W in the homology with integral coefficients, denoted $H_*(\mathcal{B}^*(V), \mathbf{Z})$. As in Kazhdan-Lusztig [1980], "homology" may here be understood either as "Borel-Moore homology" or as relative homology $H_*(\bar{\mathcal{B}}^*(V), \partial\mathcal{B}^*(V); \mathbf{Z})$ of the finite polyhedron $\bar{\mathcal{B}}^*(V)$ with respect to its finite subpolyhedron $\partial\mathcal{B}^*(V)$ (see the explanations in connection with (5)). The coefficient ring $\mathbf{Z}$ will be omitted from the notation when understood or unimportant.

For the further analysis of the representations of W in these $H_*(\mathcal{B}^*(V))$ we follow Kazhdan-Lusztig [1980]. Suppose $X \subset Y$ are closed subvarieties of $\mathcal{N}$, and put $U = Y - X$. There is a long exact sequence

$$\cdots \to H_{i+1}(\mathcal{B}^*(U)) \to H_i(\mathcal{B}^*(X)) \to H_i(\mathcal{B}^*(Y)) \to H_i(\mathcal{B}^*(U)) \to H_{i-1}(\mathcal{B}^*(X)) \to \cdots \tag{8}$$

In top degree, when $i = 2m$, $m = \dim_{\mathbf{C}}\mathcal{B}^*(Y)$, $\quad H_{2m+1}(\mathcal{B}^*(U)) = 0$, trivially, and the boundary map $H_{2m}(\mathcal{B}^*(U)) \to H_{2m-1}(\mathcal{B}^*(X))$ is $= 0$ because the topological boundary of a complex variety has real codimension at least 2. So (8) leads to the short exact sequence

$$0 \to H_{2m}(\mathcal{B}^*(X)) \to H_{2m}(\mathcal{B}^*(Y)) \to H_{2m}(\mathcal{B}^*(U)) \to 0 \tag{9}$$

The maps (7) and (9) are W-maps.

3. Specialization

In the above construction, choose $V = \{\nu\}$, a single point. Then

$$\mathcal{B}^*(\nu) = \{(b, \nu) | \nu \in b^\perp\},$$

which may be identified with

$$\mathcal{B}^\nu = \{b \in \mathcal{B} | \nu \in b^\perp\}.$$

This is the fixed point set of (the one parameter group generated by) ν, when $g*$ is identified with g. We thus have a representation of W in $H_*(\mathcal{B}^\nu)$.

The component group $A(\nu)$ of the stabilizer of ν in G acts on $H_*(\mathcal{B}^\nu)$. This action commutes with the action of W, because the elements of $A(\nu)$ have representatives in U (the compact form of G) and the operators $a_\lambda(w) = {p_{w\lambda}}^{-1} \circ p_\lambda$ commute with the action of U on $\mathcal{B}^*$, the p_λ being U-equivariant.

The Weyl group W acts on the flag-manifold $\mathcal{B} \approx U/T$ by

$$a(w) \cdot u \cdot b_1 = u \cdot w^{-1} b_1 (u \in U).$$

In the context of Springer's construction, the analogue of the following lemma is the *Specialization Theorem* of Hotta-Springer [1977].

3.1 Lemma. *The inclusion $\mathcal{B}^\nu \to \mathcal{B}$ induces a W-map $H_*(\mathcal{B}^\nu) \to H_*(\mathcal{B})$. This map factors through the projection $H_*(\mathcal{B}^\nu) \to H_*(\mathcal{B}^\nu)^{A(\nu)}$ onto the $A(\nu)$-invariants.*

Proof. It suffices to show that the homotopy equivalence of $\mathcal{B}^\nu$ constuct above has a representative $\mathcal{B}^\nu \to \mathcal{B}^\nu$, denoted $a^\nu(w)$, so that the inclusion $i : \mathcal{B}^\nu \to \mathcal{B}$ satisfies

$$i \circ a^\nu(w) \sim a(w) \circ i.$$

Choose a neighbourhood V of $\mathcal{B}^\nu$ for which the inclusion $k : \mathcal{B}^\nu \to V$ has a homotopy inverse $q : V \to \mathcal{B}^\nu$:

$$q \circ k \sim 1 \text{ on } \mathcal{B}^\nu, \quad k \circ q \sim 1 \text{ on } V.$$

Let $U = \{(b, \nu') : b \in V, |\nu'| \leq R\}$ for some fixed $R > |\nu|$. Define maps

$$j : \mathcal{B}^\nu \to U, j(b) = (k(b), \nu) = (b, \nu), \quad p : V \to \mathcal{B}^\nu, p(b, \nu') = q(b).$$

We show that

$$p \circ j \sim 1 \text{ on } \mathcal{B}^\nu, \quad j \circ p \sim 1 \text{ on } U. \tag{1}$$

The first relation is clear since

$$p \circ j(b) = q \circ k(b) \text{ and } q \circ k \sim 1 \text{ on } \mathcal{B}^\nu.$$

To see the second relation, use $k \circ q \sim 1$ to choose a homotopy $q_s : V \to V, 0 \leq s \leq 1$, from $q_0 = 1_V$ to $q_1 = k \circ q$. Perform successively the following homotopies of maps $U \to U$:

(1) $(b, \nu') \to (b, s\nu')$, s going from 1 to 0,
(2) $(b, \nu') \to (q_s(b), 0)$, s going from 0 to 1,
(3) $(b, \nu') \to (k \circ q(b), s\nu)$, s going from 0 to 1.

This gives a homotopy from the identity $1_U : (b, \nu') \to (b, \nu')$ to $p \circ j : (b, \nu') \to (k \circ q(b), \nu)$ as required.

For λ close to 0, $a_\lambda(w) j(\mathcal{B}^\nu)$ stays in the neighbourhood U of $j(\mathcal{B}^\nu)$. The relation (1) shows that the p, j can take the place of the p, i in the definition (2.7) of $a^\nu(w)$. For such λ the map $p \circ a_\lambda(w) \circ j : \mathcal{B}^\nu \to \mathcal{B}^\nu$ therefore represents the homotopy class $a^\nu(w)$:

$$a^\nu(w) \sim p \circ a_\lambda(w) \circ j : \mathcal{B}^\nu \to \mathcal{B}^\nu. \tag{2}$$

For $0 \leq s \leq 1$, define of maps

$$j_s : \mathcal{B}^\nu \to U, j_s(b) = (b, s\nu), \quad p_s : U \to V, p_s(b, \nu') = q_s(b)$$

with q_s as above. Consider the homotopy of maps

$$p_s \circ a_\lambda(w) \circ j_s : \mathcal{B}^\nu \to V, \quad 0 \leq s \leq 1. \tag{3}$$

For s = 0 we get

$$p_o \circ a_\lambda(w) \circ j_o(b) = q_o \circ a_\lambda(w)(b, 0) = a(w)b$$

because $a_\lambda(w)$ and $a(w)$ coincide on $\mathcal{B}$ considered as the zero section in $\mathcal{B}^*$: this is clear from the definition of $a_\lambda(w)$. So for $s = 0$, (3) reduces to $a(w) \circ i$. On the other hand, for $s = 1$ we get

$$p_1 \circ a_\lambda(w) \circ j_1(b) = p \circ a_\lambda(w)(b, \nu)) = p \circ a_\lambda(w) \circ j(b)) = a^\nu(b)b$$

by (2). So for $s = 1$, (3) reduces to $a^\nu(w) \circ i$. Hence (3) provides the desired homotopy $i \circ a^\nu(w) \sim a(w) \circ i$.

To see the second assertion of the lemma one only has to note that the action of the stabilizer of ν in U, which induces the action of $A(\nu)$ in $H_*(\mathcal{B}^\nu)$, becomes trivial in $H_*(\mathcal{B})$ as U is connected. This proves the lemma.

3.2 Corollary. *In top dimension the W-map*

$$H_{2e(\nu)}(\mathcal{B}^\nu)^{A(\nu)} \to H_{2e(\nu)}(\mathcal{B}), \qquad e(\nu) = \dim_{\mathbf{C}} \mathcal{B}^\nu,$$

is an isomorphism onto its image.

Proof. This is because the representation of W on $H_{2e(\nu)}(\mathcal{B}^\nu)^{A(\nu)}$ is irreducible, as we shall see in (4.1).

4. Springer Theory

We shall now apply the above construction with G_o replaced by $G \approx G_o \times G_o$. $\mathcal{B}$ will again denote the flag manifold of g, $\mathcal{B}_o$ that of G_o, so that $\mathcal{B} = \mathcal{B}_o \times \mathcal{B}_o$. $\mathcal{N} = \mathcal{N}_o \times \mathcal{N}_o$ is the nilpotent cone in g^*, $\mathcal{N}_o$ that in g_o^*. We again set $z_1 = b_o \times b_o$ and $z_w = w^{-1} z_1$. For V we now take

$$\mathcal{N} \cap k^\perp = \{(\nu, -\nu) | \nu \in \mathcal{N}_o\} \approx \mathcal{N}_o.$$

Then $\mathcal{B}^*(V)$ becomes

$$\mathcal{Z} = \{(b, b'; \nu, -\nu) | \nu \in b^\perp \cap b'^\perp\}$$

the conormal variety of the K-orbits on $\mathcal{B}$. Our construction therefore gives a representation of $W \approx W_o \times W_o$ on $H_*(\mathcal{Z}, \mathbf{Z})$.

We may identify $\mathcal{N}_o$ with the subset $\mathcal{N} \cap k^\perp$ of $\mathcal{N}$. For a subset V of $\mathcal{N}_o$ we write $\mathcal{Z}(V)$ for $\mathcal{B}^*(V)$. When V is a closed subvariety of $\mathcal{N}$, then $H_{2n}(\mathcal{Z}(V))$ is naturally a W-submodule of $H_{2n}(\mathcal{Z})$, by (2.9). When V is only constuctible then $H_{2n}(\mathcal{Z}(V))$ is naturally a W-subquotient:

$$H_{2n}(\mathcal{Z}(V)) \approx H_{2n}(\mathcal{Z}(\bar{V}))/H_{2n}(\mathcal{Z}(\partial V))$$

where $\bar{V}$ is the closure of V and ∂V is the topological boundary of V.

The decomposition of $\mathcal{N}_o$ into G_o-orbits $\mathcal{O}$ (equivalently: the decomposition of $\mathcal{N} \cap k^\perp$ into K-orbits) leads to a filtration of $H_{2n}(\mathcal{Z})$ according to the closure relations among the orbits :

$$\mathcal{O}' \subset \bar{\mathcal{O}} \text{ implies } H_{2n}(\mathcal{Z}(\bar{\mathcal{O}})) \subset H_{2n}(\mathcal{Z}(\bar{\mathcal{O}}')).$$

The subquotients of this filtration are

$$H_{2n}(\mathcal{Z}(\mathcal{O})) = H_{2n}(\mathcal{Z}(\bar{\mathcal{O}}))/\sum_{\mathcal{O}' < \mathcal{O}} H_{2n}(\mathcal{Z}(\bar{\mathcal{O}}')).$$

Where $\mathcal{O}' < \mathcal{O}$ means $\bar{\mathcal{O}}' \underset{\neq}{\subset} \bar{\mathcal{O}}$. The associated graded group is

$$gr H_{2n}(\mathcal{Z}) \approx \oplus H_{2n}(\mathcal{Z}(\mathcal{O})), \tag{1}$$

sum over all G_o-orbits on $\mathcal{N}_o$. (1) is a W-decomposition; but one shoud keep in mind that the $H_{2n}(\mathcal{Z}(\mathcal{O}))$ are naturally W-subquotients of $H_{2n}(\mathrm{Z}\ )$, not W-submodules. This in spite of the fact that $H_{2n}(\mathcal{Z}(\mathcal{O}))$

may naturally be realized as a subgroup of $H_{2n}(\mathcal{Z})$: it has as a basis the fundamental cycles of the components of $\mathcal{Z}(\mathcal{O})$ and is therefore isomorphic to the subgroup of $H_{2n}(\mathcal{Z})$ spanned by the fundamental cycles of the closures in $\mathcal{Z}$ of these components. But the latter subgroup of $H_{2n}(\mathcal{Z})$ is generally not W-stable. It should also be noted that according to Steinberg [1976] each $\mathcal{Z}(\mathcal{O})$ has uniform dimension $=\dim\mathcal{Z} = n$, so its components are certain $\mathcal{Z}_w$'s. More precisely, the components of $\mathcal{Z}(\mathcal{O})$ are dense parts of those $\bar{\mathcal{Z}}_w$ for which $\mathcal{O}$ intersects the fibre $\mathcal{Z}_w \cap k^\perp$ of $\mathcal{Z} \to \mathcal{B}$ over $\mathcal{Z}_w$ densely. We denote them $\mathcal{Z}_w(\mathcal{O}) := \mathcal{Z}(\mathcal{O}) \cap \bar{\mathcal{Z}}_w$.

From the fibration $\mathcal{Z}(\mathcal{O}) \to \mathcal{O}$ one gets that

$$H_{2n}(\mathcal{Z}(\mathcal{O})) \approx H_{2e(\nu)}(\mathcal{B}^\nu)^{A_o(\nu)} \tag{2}$$

where $\nu \in \mathcal{O}$, $e(\nu) = \dim_{\mathbf{C}}\mathcal{B}^\nu = 2e_o(\nu)$, $\quad e_o(\nu) = \dim_{\mathbf{C}}\mathcal{B}_{o\nu}$, and $A_o(\nu)$ is the component group of the stabilizer of ν in G_o, a quotient of the fundamental group of $\mathcal{O}$. $H_{2e(\nu)}(\mathcal{B}^\nu)^{A_o(\nu)}$ denotes the $A_o(\nu)$-invariants in $H_{2e(\nu)}(\mathcal{B}^\nu)$.

The isomorphism (1) is explicitly seen as follows. As just mentioned, $H_{2n}(\mathcal{Z}(\mathcal{O}))$ has a basis consisting of the fundamental cycles of those $\mathcal{Z}_w$ of $\mathcal{Z}$ which make up $\mathcal{Z}(\mathcal{O})$. On such a $\mathcal{Z}_w$ the fibration $\mathcal{Z}(\mathcal{O}) \to \mathcal{O}$ restricts to a fibration $\mathcal{Z}_w(\mathcal{O}) \to \mathcal{O}$ whose fibre over ν is exactly an $A_o(\nu)$-orbit of components of $\mathcal{B}^\nu = {\mathcal{B}_o}^\nu \times {\mathcal{B}_o}^\nu$ (according to Steinberg [1976]); and these $A_o(\nu)$-orbits of components of $\mathcal{B}^\nu$ form a basis of $H_{4e(\nu)}(\mathcal{B}^\nu)^{A_o(\nu)}$.

The action of the component group $A(\nu) = A_o(\nu) \times A_o(\nu)$ of the $G-$stabilizer of ν on $H_{2e(\nu)}(\mathcal{B}^\nu) = H_{2e_o(\nu)}(\mathcal{B}_{o\nu}) \otimes H_{2e_o(\nu)}(\mathcal{B}_{o\nu})$ commutes with the action of $W = W_o \times W_o$, as we know, and the invariants of the diagonal $A_o(\nu)$ in $A_o(\nu) \times A_o(\nu)$ decompose as

$$\sum_\phi \chi_{\nu,\bar{\phi}} \otimes \chi_{\nu,\phi}$$

where ϕ runs over the irreducible characters of $A_o(\nu)$ which occur in $H_{2e_o(\nu)}(\mathcal{B}_{o\nu})$ and $\chi_{\nu,\phi}$ is the character of W_o on the subspace of $H_{2e_o(\nu)}(\mathcal{B}_{o\nu})$ which transforms according to ϕ. (*A priori* one might have to extend scalars to $\mathbf{C}$ for this decomposition, but it follows from the known structure of the $A_o(\nu)$ that it suffices to work over $\mathbf{Q}$.) If one knew that

the representation of $W = W_o \times W_o$ on $H_{2n}(\mathcal{Z}, \mathbf{C})$ is the biregular representation on $\mathbf{C}[W/W_o] = \mathbf{C}[W_o]$, (3)

which is the main result of the Kazhdan-Lusztig [1980] paper, this argument (which they attribute to Springer) would prove

4.1 Springer's Theorem. *The $\chi_{\nu,\phi}$, are exactly the irreducible characters of W_o.*

In the present context the missing link (3) will be supplied by the Integral Formula, as we shall now show.

5. Coherent families of eigendistributions.

To bring in the Integral Formula we need to pass from the conormal variety $\mathcal{Z}$ of the K-action to the conormal variety $\mathcal{S}$ of the G_o-action by means of the involution $\iota : (x, y) \to (x, \bar{y})$ of $g = g_o \times g_o$ which interchanges $g_o = \{(x, \bar{x})\}$ and $k = \{(x, x)\}$. Use this automorphism ι to transfer the represention of W on $H_*(\mathcal{Z})$ constructed above from $\mathcal{Z}$ to $\mathcal{S}$. It is still induced by the transformations $a_\lambda(w) = p_{w\lambda}(w)^{-1} \circ p_\lambda$ with $p_\lambda : \mathcal{B}^* \to \Omega_\lambda, u \cdot (b_1, \nu) \to u \cdot (\lambda + \nu)$, except that for $\mathcal{S}$ we need to take $b_1 = s_1 = b_o \times \bar{b}_o$ as base-point in the definition of p_λ (because $\iota a_{b_1,\lambda}(w)\iota = a_{\iota b_1,\iota\lambda}(w) \sim a_{\iota b_1,\lambda}(w)$).

Recall the Integral Formula: for $\Gamma \in H_{2n}(\mathcal{S}, \mathbf{C})$, and regular $\lambda \in h^*$,

$$\tfrac{1}{(2\pi i)^n} \int_{p_\lambda \Gamma} e^{x_\lambda - \sigma_\lambda} = \frac{1}{\pi(x)} \sum_{y \in W} m_y e^{y^{-1}\lambda}. \tag{1}$$

We also know that for $\Gamma = \mathcal{S}_w$, m_y is up to a sign the local Euler number of $\mathcal{S}_w$ at s_y :

$$m_y = (-1)^{n_o + l(w) - l(y)} \mathrm{Eu}_y(S_w) = (-1)^{n_o + l(w) - l(y)} \mathrm{Eu}_y(Z_w) \qquad (\text{for } \Gamma = \mathcal{S}_w).$$

(1) is interpreted as an invariant eigendistribution on g_o, which we denoted $\theta_\Gamma(\lambda)$:

$$\theta_\Gamma(\lambda) = \tfrac{1}{(2\pi i)^n} \int_{p_\lambda \Gamma} e^{x_\lambda - \sigma_\lambda}$$

The right side of (1) may be written as

$$\theta_\Gamma(\lambda) = \frac{1}{\pi} \sum_{y \in W/W_o} m_y \varphi_y(\lambda)$$

where

$$\varphi_y(\lambda) = \frac{1}{\pi} \sum_{w \in W_o} e^{(yw)^{-1}\lambda}. \tag{2}$$

Any family $\theta(\lambda)$ of G_o-invariant eigendistributions depending on $\lambda \in h^*$ which for regular λ is given by a formula

$$\theta(\lambda) = \frac{1}{\pi} \sum_{y \in W} m_y e^{y^{-1}\lambda}. \tag{3}$$

with $m_{yz} = m_y$ for $z \in W_o$, or equivalently

$$\theta(\lambda) = \frac{1}{\pi} \sum_{y \in W/W_o} m_y \varphi_y(\lambda) \tag{4}$$

will be referred to as a *coherent family* of invariant eigendistributions on ${g_o}^*$. Is is evident that a coherent family θ is determined by its value $\theta(\lambda)$ at a single regular λ. Furthermore, any $\theta(\lambda)$ given by (3) or (4) for a single regular λ extends uniquely to a coherent family: to see this it suffices to know that the $\varphi_y(\lambda)$ extend, and that will become clear shortly. We shall denote the space of coherent families of invariant eigendistributions on g_o by $CH(g_o, \mathbf{C})$ or $CH(g_o, \mathbf{Z})$ depending on whether we use complex or integral coefficients m_y in (3) or (4). The $\mathbf{C}$ or $\mathbf{Z}$ will be omitted when understood or unimportant.

The Weyl group $W = W_o \times W_o$ operates on the θ in the obvious way:

$$(w \cdot \theta)(\lambda) = \theta(w^{-1}\lambda)$$

and the resulting representation of W on $CH(g_o, \mathbf{C})$ is evidently the biregular representation on $\mathbf{Z}[W/W_o] \approx \mathbf{Z}[W_o]$, with the φ_w, $w \in W/W_o$, corresponding to the basis elements wW_o of $\mathbf{Z}[W/W_o]$. On the other hand, we have the representation of W on $H_{2n}(\mathcal{S}, \mathbf{Z})$ constructed above.

5.1 Theorem. *The map*

$$H_{2n}(\mathcal{S},\mathbf{Z}) \to CH(g_o,\mathbf{Z}), \quad \Gamma \to \theta_\Gamma,$$

given by the Integral Formula is a $W-$*isomorphism.*

More precisely, the homology classes $w\cdot\mathcal{S}_1$, $w\in W/W_o$, *form a basis for* $H_{2n}(\mathcal{S},\mathbf{Z})$. *They correspond to the basis* $(-1)^{n_o}\varphi_w, \quad w\in W/W_o$, *of* $CH(g_o,\mathbf{Z})$ *under the bijection* $\Gamma\to\theta_\Gamma$.

Proof. That the map $H_{2n}(\mathcal{S}) \to CH(g_o)$ is a bijection we know from the Integral Formula : its matrix $(-1)^{n_o+l(w)-l(y)}\mathrm{Eu}_y(\mathcal{S}_w)$ with respect to the bases $\mathcal{S}_w$ of $H_{2n}(\mathcal{S})$ and φ_w of $CH(g_o)$, is integral and unipotent-triangular with respect to the Bruhat order on W/W_o.

It suffices to prove the second assertion which says explicitly that for regular $\lambda\in h^*$,

$$\tfrac{1}{(2\pi i)^n}\int_{p_\lambda w\mathcal{S}_1} e^{x_\lambda-\sigma_\lambda} = \frac{(-1)^{n_o}}{\pi(x)}\sum_{y\in W_o} e^{(wy)^{-1}\lambda(x)}. \tag{5}$$

For $w=1$ this is a special case of the Integral Formula : the K-orbit $\mathcal{Z}_1 = K\cdot z_1$ is smooth, so $\mathrm{Eu}_y(Z_1)=1$ or 0, according as $y\in W_o$ or not. To see that (5) holds for all $w\in W$ we only need to identify the $p_\lambda w\mathcal{S}_1$. The cycle $p_\lambda\mathcal{S}_1$ is the image under the map

$$p_\lambda : \mathcal{B}^* \to \Omega_\lambda, u\cdot(b_1,\nu)\to u\cdot(\lambda+\nu)$$

of the conormal bundle of the closed G_o-orbit $G_o\cdot s_1$, where $s_1 = b_o\times\bar{b}_o$. That conormal bundle is

$$G_o\cdot\{(s_1,\nu)|\nu\in {s_1}^\perp\cap i{g_o}^\perp\}$$

Since $G_o=K_oB_o$ and $b_o\subset s_1$ this conormal is also

$$= K_o\cdot\{(b_1,\nu)|\nu\in i{b_o}^\perp\}.$$

where ${b_o}^\perp\subset {g_o}^*$ is the orthogonal of b_o in ${g_o}^*$ (not in g^*). Its image under p_λ becomes $K_o\cdot\{\lambda+i{b_o}^\perp\}$.

From the definition of the action of W on $H_{2n}(\mathcal{S})$ one finds that the cycle $p_\lambda w\cdot\mathcal{S}_1$ on Ω_λ which figures in (5) is properly homotopic to $K_o\cdot\{w^{-1}\lambda+i{b_o}^\perp\}$:

$$p_\lambda w\cdot\mathcal{S}_1 \sim K_o\cdot\{w^{-1}\lambda+i{b_o}^\perp\} \text{ on } \Omega_\lambda.$$

taken with the appropriate orientation. So (5) says

$$\tfrac{1}{(2\pi i)^n}\int_{K_o\cdot\{w^{-1}\lambda+i{b_o}^\perp\}} e^{x_\lambda-\sigma_\lambda} = \frac{(-1)^{n_o}}{\pi(x)}\sum_{y\in W_o} e^{(wy)^{-1}\lambda(x)}. \tag{6}$$

We know that this formula holds for $w=1$ (and regular λ). Replacing λ by $w\lambda$ one sees that it holds for all $w\in W$. Furthermore, writing the integral (6) as a double integral, first over ${b_o}^\perp$, then over K_o, one sees that (6) exists for all $\lambda\in h^*$ (regular or not). This means that

$$\varphi_w(\lambda) \textit{ is entire analytic in } \lambda\in h^*$$

as promised above. This finishes the proof of the theorem. We record explicitly the special case when $\Gamma=\mathcal{S}_w$:

5.2 Corollary.

$$\mathcal{S}_w = \sum_{y \in W/W_o} (-1)^{l(w)-l(y)} \mathrm{Eu}_y(S_w) y \cdot \mathcal{S}_1.$$

or equivalently

$$\mathcal{Z}_w = \sum_{y \in W/W_o} (-1)^{l(w)-l(y)} \mathrm{Eu}_y(Z_w) y \cdot \mathcal{Z}_1.$$

6. Characters and characteristic cycles.

As a first and immediate application of Theorem 5.1 we derive a formula for the global character of a (g, K)-module as a contour integral over the characteristic cycle of the corresponding $(\mathcal{D}, K)$-module.

We start with some general remarks about global characters. A (g, K)-module M with infinitesimal character $\lambda \in h$ is the Harish-Chandra module of an admissible representation of G_o, whose global character is an invariant eigendistribution $\Theta = ch(M)$ on G_o. It follows from results of Harish-Chandra [1965] that one has an identity of distributions in a neighbourhood of 0 in g_o :

$$\Theta(\exp x) = j(x)^{-1}\theta(x) \tag{1}$$

where θ is an invariant eigendistribution on g_o with infinitesimal character λ and

$$j(x) = \det{}^{1/2}(\frac{e^{adx/2} - e^{-adx/2}}{adx}) = \frac{\Delta(x)}{\pi(x)} \quad \text{on } h_o.$$

Here $\pi = \prod_{\alpha \in \Delta_+} \alpha$ as before, and

$$\Delta = \prod_{\alpha \in \Delta_+} (e^{\alpha/2} - e^{-\alpha/2})$$

is the Weyl denominator. The products are over the roots α of h_o in ${b_o}^{\perp}$. If $\lambda \in h^*$ is regular, as we shall now assume, the Integral Formula allows us to write $\theta = \theta_\Gamma(\lambda)$ for a unique $\Gamma \in H_{2n}(\mathcal{S})$. We shall call this homology class Γ, or any $2n$-cycle representing it, the *character contour* of the (g, K)-module M, and denote it $\mathcal{C}(M)$. It should be noted that $\mathcal{C}(M)$ depends on the choice of the element λ in the W-orbit in h^* determined by the infinitesimal character of M and on the Borel subalgebra s_1 used to define the map p_λ.

Recall the Beilinson-Bernstein [1981] correspondence between (g, K)-modules and $(\mathcal{D}, K)$-modules. We consider only the case of regular integral infinitesimal character; one may as well assume (as we now do) that the infinitesimal character is represented by $-\rho$ where $\rho = (\rho_o, \rho_o)$ is half the sum of the roots of h in $z_1 = b_o \times b_o$.

The Beilinson-Bernstein correspondence based on the data $(z_1, -\rho)$ associates to each (g, K)-module M with infinitesimal character -ρ the $(\mathcal{D}, K)$-module $\mathcal{M} = \mathcal{D} \otimes M$. ($\mathcal{D}$ is the sheaf of differential operators on $\mathcal{B}$ as in [Beilinson-Bernstein]; the tensor product is over $U(g)$ as explained there.) This correspondence is an equivalence of categories.

For each K-orbit $\mathcal{Z}_y = K \cdot y^{-1} z_1$ there is an induced $\mathcal{D}$-module which we shall denote $\mathcal{I}_y$. Write I_y for the corresponding (g, K)-module. It is known that I_y is an induced (principal series) (g, K)-module. The corresponding representation of G_o has global character

$$\frac{1}{\Delta}\sum_{w\in W} e^{-(yw)^{-1}\rho}.$$

$(Thisisaspecialcaseoftherelationbetweentwoofthethreeclassificationsof($

g,K)-$modules, of Hecht - Miličić - Schmid - Wolf$[1986].)

Under the correspondence (1) between invariant eigendistributions on G_o and on g_o the global character corresponds to the distribution

$$\frac{1}{\pi}\sum_{w\in W} e^{-(yw)^{-1}\rho}. \tag{2}$$

In analogy with the data $(z_1, -\rho)$ for the Beilinson-Bernstein correspondence it seems natural to base the character contours on the data $(s_1, -\sigma)$ which correspond to $(z_1, -\rho)$ under the automorphism ι interchanging k and $g_o : s_1 = b_o \times \bar{b}_o$ and $\sigma = (\rho_o, -\rho_o)$. We therefore write (2) as

$$\frac{1}{\pi}\sum_{w\in W} e^{-(w_\iota yw)^{-1}\sigma} = \varphi_{w_1 y}(-\sigma) \tag{3}$$

where $w_\iota \in W$ is the Weyl group element with $w_\iota\rho = \sigma$.

It follows from Theorem 5.1 that the character contour of the (g, K)-module I_y (based on the data $(s_1, -\sigma)$ is

$$\mathcal{C}(I_y) = w_\iota y \cdot \mathcal{S}_1 \tag{4}$$

On the other hand, to a $(\mathcal{D}, K)$-module $\mathcal{M}$ one can associate a *characteristic cycle*; this is an algebraic cycle of complex dimension n on $\mathcal{B}^*$ which is known to lie on the conormal variety $\mathcal{Z}$ of the K-action on $\mathcal{B}$ for these modules $\mathcal{M}$. It determines therefore a homology class in $H_{2n}(\mathcal{Z})$, denoted $Ch(\mathcal{M})$. By a result of Tanisaki [1985] (or by the corresponding result of Kasihwara-Tanisaki [1984] for $(\mathcal{D}, B)$-modules, which amounts to essentially the same thing when g_o itself is complex, as here) the characteristic cycle of the induced $\mathcal{D}$-module $\mathcal{I}_y$ is

$$Ch(\mathcal{I}_y) = y \cdot \mathcal{Z}_1. \tag{5}$$

Since the $\mathcal{I}_y$ and I_y form bases for the respective K-groups one finds by comparing (4) and (5) :

6.1 Theorem. *The character contour* $\mathcal{C}(M)$ *of a* (g, K)*-module* M *and the characteristic variety* $Ch(\mathcal{M})$ *of the corresponding* $\mathcal{D}$*-module* $\mathcal{M}$ *are related by*:

$$Ch(\mathcal{M}) = w_\iota \cdot \iota\mathcal{C}(M). \tag{6}$$

In this equation ι denotes the map from the conormal variety of K-action of $\mathcal{B}$ to the conormal variety of the G_o-action induced by the involutionι of g which interchanges k and g_o and induces the W-isomorphism of the homology groups of the conormal varieties. It is further understood that the data $(z_1, -\rho)$ entering into the Beilinson-Bernstein correspondence and the data $(s_1, -\sigma)$ entering into the definition of character contours are related by this automorphism ι. Explicitly, (6) says that the global character ch(M) of M is given by the formula

$$ch(M)(\exp x) = j(x)^{-1} \int_{p_{-\sigma}\iota w_\iota{}^{-1}Ch(\mathcal{M})} e^{x_\sigma - \sigma_\sigma}.$$

7. Asymptotics at zero

Recall the W-filtration of $\mathcal{Z}$ by the inverse images $\mathcal{Z}(\mathcal{O})$ of nilpotent K-orbits $\mathcal{O}$ on $\mathcal{N} \cap k^\perp$, which gave rise to the decomposition (4.1) of $grH_*(\mathcal{Z})$. Passing from $\mathcal{Z}$ to $\mathcal{S}$, the analogous filtration of $\mathcal{S}$ by the inverse images $\mathcal{S}(\mathcal{O})$ of G_o-orbits on $\mathcal{N} \cap ig_o{}^*$ gives a filtration of $H_{2n}(\mathcal{S})$ by the subgroups $H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$ according to the closure relation among the $\mathcal{O}$'s leading to the decomposition

$$grH_{2n}(\mathcal{S}) \approx \sum_{\mathcal{O}} H_{2n}(\mathcal{S}(\mathcal{O})). \tag{1}$$

As in (4.2),

$$H_{2n}(\mathcal{S}(\mathcal{O})) \approx H_{2e(\nu)}(\mathcal{B}^\nu)^{A_o(\nu)} \tag{2}$$

and the inclusion $\mathcal{B}^\nu \overset{\subseteq}{\to} \mathcal{B}$ induces a W-injection

$$H_{2e(\nu)}(\mathcal{B}_\nu)^{A(\nu)} \overset{\subseteq}{\to} H_{2e(\nu)}(\mathcal{B}) \tag{3}$$

(Note that there is $A_o(\nu)$ in (2), $A(\nu) = A_o(\nu) \times A_o(\nu)$ in (3).) We choose the same base point $b_1 = s_1$ to define the W-action on $H_{2e(\nu)}(\mathcal{B}^\nu)^{A(\nu)}$ and on $H_{2e(\nu)}(\mathcal{B})$.)

We recall Borel's description of the cohomology ring of $\mathcal{B}$ [Borel 1953]. For $\lambda \in h^*$ denote τ_λ the U-invariant 2-from on $\mathcal{B}$ which at the base point s_1 is given by

$$\tau_\lambda(x \cdot s_1, y \cdot s_1) = \lambda([x, y]) \qquad \text{for } x, y \in u \tag{4}$$

Set

$$\omega_\lambda = -\tfrac{1}{2\pi i}\tau_\lambda$$

The map $\lambda \to \omega_\lambda$ extends to a map $f \to \omega_f$ from the ring $\mathbf{C}[h]$ of polynomial functions on h to the algebra of differential forms on $\mathcal{B}$. It annihilates the W-invariants with zero constant term in $\mathbf{C}[h]$, denoted I^+, and induces an isomorphism

$$\mathbf{C}[h]/I^+ \overset{\approx}{\to} H^*(\mathcal{B}), \quad [f] \to [\omega_f], \tag{5}$$

where $[f]$ is the class of f, $[\omega_f]$ the class of ω_f (in de Rahm cohomology). The transpose of (5) is an isomorphism

$$H_*(\mathcal{B}, \mathbf{C}) \overset{\approx}{\to} \mathcal{H}(h^*), \quad \gamma \to c_\gamma \tag{6}$$

where $\mathcal{H}(h^*)$ consists of the W-*harmonic polynomials* on h^*, i.e. the polynomials annihilated by the W-invariant constant coefficient operators without constant term. Explicitly, the isomorphism (6) is defined by

$$\int_\gamma \omega_f = \langle c_\gamma, f\rangle. \tag{7}$$

The right side is the natural pairing $\langle x^i, \lambda^j\rangle/j! = \langle x, \lambda\rangle^j \delta_{ij}$. c_γ may also be defined directly by the equation

$$c_\gamma(\lambda) = \int_\gamma e^{\omega_\lambda} \tag{8}$$

as one sees by taking $f = e^\lambda$ in (7) using $\langle c, e^\lambda\rangle = c(\lambda)$. Composing (3) and (6) gives a map

$$H_{2e(\nu)}(\mathcal{B}^\nu)^{A(\nu)} \to \mathcal{H}_{e(\nu)}(h^*) \tag{9}$$

where $\mathcal{H}_{e(\nu)}(h^*)$ denotes the homogeneous polynomials of degree $e(\nu)$ in $\mathcal{H}(h^*)$. Using further (2) we get a map

$$H_{2n}(\mathcal{S}(\bar{\mathcal{O}})) \to \mathcal{H}_{e(\nu)}(h^*), \quad \Gamma \to c_\Gamma, \tag{10}$$

This last map (10) is of course not an isomorphism: it factors through the projection of $H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$ onto $H_{2n}(\mathcal{S}(\mathcal{O}))$ as well as through the projection of $H_{2e(\nu)}(\mathcal{B}^\nu)^{A_o(\nu)}$ onto $H_{2e(\nu)}(\mathcal{B}^\nu)^{A(\nu)}$.

7.1 Theorem. *For any nilpotent G_o-orbit $\mathcal{O}$ in $i{g_o}^*$ and any $2n$-cycle $\Gamma \in H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$ over $\bar{\mathcal{O}}$,*

$$\tfrac{1}{(2\pi i)^n} \int_{p_\lambda \Gamma} e^{x_\lambda - \sigma_\lambda} = c_\Gamma(\lambda) \tfrac{1}{(2\pi i)^d} \int_{\mathcal{O}} e^{x_{\mathcal{O}} - \sigma_{\mathcal{O}}} + o(|\lambda|^e) \tag{11}$$

Explanation and remarks. $d = d(\mathcal{O}) = \dim_{\mathbf{C}} \mathcal{O}$, $\quad e = e(\mathcal{O}) = \dim_{\mathbf{C}} \mathcal{B}^\nu (\nu \in \mathcal{O})$; $x_{\mathcal{O}}$ is x considered as a function on $\mathcal{O}$; $\sigma_{\mathcal{O}}$ is the canonical 2-form on $\mathcal{O}$:

$$\sigma_{\mathcal{O}}(x \cdot \nu, y \cdot \nu) = \langle \nu, [x, y]\rangle.$$

The equation (11) is understood as an identity of distributions on g_o, as usual. It will be abbreviated to

$$\theta_\Gamma(\lambda) = c_\Gamma(\lambda)\theta_{\mathcal{O}} + o(|\lambda|^e). \tag{12}$$

with

$$\theta_\Gamma(\lambda) = \tfrac{1}{(2\pi i)^n} \int_{p_\lambda \Gamma} e^{x_\lambda - \sigma_\lambda},$$

$$\theta_{\mathcal{O}} = \tfrac{1}{(2\pi i)^d} \int_{\mathcal{O}} e^{x_{\mathcal{O}} - \sigma_{\mathcal{O}}}.$$

The equation (11) can also be written as an asymptotic relation as $x \to 0$ in g_o in the sense of Barbasch and Vogan [1980]:

$$\theta_\Gamma(\lambda) \sim c_\Gamma(\lambda)\theta_{\mathcal{O}} \text{ as } x \to 0 \text{ in } g_o. \tag{13}$$

This means that $c_\Gamma(\lambda)\theta_{\mathcal{O}}$ is the leading term in the asymptotic expansion of $\theta_\Gamma(\lambda)$ at $x = 0$ described in Theorem 1.3 of [Barbasch-Vogan 1980].

Proof of the theorem. Fix the G_o-orbit $\mathcal{O}$ on $\mathcal{N} \cap i{g_o}^*$ and $\Gamma \in H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$. For $f \in C_c^\infty(g)$, set

$$\varphi(\xi) = \int_g e^{\xi(x)} f(x) dx.$$

The theorem says that for all such f

$$\tfrac{(-1)^n}{(2\pi i)^n n!} \int_{p_\lambda \Gamma} \varphi {\sigma_\lambda}^n = c_\Gamma(\lambda) \tfrac{(-1)^d}{(2\pi i)^d d!} \int_{\mathcal{O}} \varphi {\sigma_{\mathcal{O}}}^d + o(|\lambda|^e).$$

To determine the asymptotic behaviour of the integral on the left we need some general remarks on symplectic structure. Let $\mathcal{Q}$ be an orbit of the complex group G on the nilpotent cone $\mathcal{N}$ in g^*. As before, put

$$\mathcal{B}^*(\mathcal{Q}) = \{(b, \nu) | \nu \in \mathcal{Q}\}$$

Then there are injections

$$\mathcal{B}^*(\mathcal{Q}) \overset{\subseteq}{\to} \mathcal{B} \times \mathcal{Q} \tag{15}$$

$$\mathcal{B}^*(\mathcal{Q}) \overset{\subseteq}{\to} \Omega_\lambda \tag{16}$$

the first map being the natural inclusion, the second one the restriction of the bijection

$$p_\lambda : \mathcal{B}^* \to \Omega_\lambda, u \cdot (s_1, \nu) \to u \cdot (\lambda + \nu)$$

($u \in U, \nu \in {s_1}^\perp$). In view (15) and (16) we can restrict to $\mathcal{B}^*(\mathcal{Q})$ the 2-form $\tau_\lambda + \sigma_Q$ on $\mathcal{B} \times \mathcal{Q}$ and the 2-form σ_λ on Ω_λ.

7.2 Lemma. *The 2-forms $\tau_\lambda + \sigma_{\mathcal{Q}}$ and σ_λ agree on $\mathcal{B}^*(\mathcal{Q})$.*

Proof of the lemma. Because of U-invariance it suffices to show that the forms agree at a point (s_1, ν) with $\nu \subset {s_1}^\perp \cap \mathcal{Q}$. Let

$$u(t) \cdot (s_1, v(t) \cdot \nu) \tag{17}$$

be a smooth curve on $\mathcal{B}^*(\mathcal{Q})$ with $u(t) \in U,\ v(t) \in G,\ v(t) \cdot \nu \in {s_1}^\perp,\ u(0) = 1,\ v(0) = 1$. Its tangent vector at $t = 0$ is

$$(u' \cdot s_1, (u' + v') \cdot \nu) \tag{18}$$

$u', v' \in g$ being the tangent vectors of $u(t)$ and $v(t)$ at $t = 0$. The 2-form $\tau_\lambda + \sigma_Q$ assigns to two such tangent vectors at (s_1, ν) the value

$$\langle\lambda, [u', u'']\rangle + \langle\nu, [u', u'']\rangle + \langle\nu, [u', v'']\rangle + \langle\nu, [v', u'']\rangle + \langle\nu, [v', v'']\rangle. \tag{19}$$

The curve on Ω_λ corresponding to (17) under the injection (16) is

$$u(t)\cdot\lambda + u(t)v(t)\cdot\nu$$

and its tangent vector at $\lambda + \nu$, corresponding to the vector (18) at (s_1, ν), is

$$u'\cdot(\lambda+\nu) + v'\cdot\nu.$$

The 2-form σ_λ assigns to two such tangent vectors at $\lambda + \nu$ the value

$$\langle\lambda, [u', u'']\rangle + \langle\nu, [u', u'']\rangle + \langle\nu, [u', v'']\rangle + \langle\nu, [v', u'']\rangle + \langle\lambda+\nu, [v', v'']\rangle. \tag{20}$$

The last term of (20) $= 0$, as $\lambda + {s_1}^\perp \subset \Omega_\lambda$ is an isotropic (in fact Lagrangian) submanifold of Ω_λ, which is clear. The same is true of the last term of (19), but this is less clear: it is known that, for any $b, b^\perp \cap \mathcal{Q}$ is a co-isotropic subvariety of $\mathcal{Q}$ [Joseph 1984, Lemmas 7.5 and 9.6(i)] of dimension $= 1/2 = \dim\mathcal{Q}$ (as follows from [Spaltenstein 1977] and [Steinberg 1976, section 4]), hence in fact Lagrangian. Thus (19) agrees with (20), proving the lemma.

Remark. The fact that $b^\perp \cap \mathcal{Q}$ is Lagrangian on $\mathcal{Q}$ is mentioned in [Ginsburg 1986, Proposition 4.3]. The proof indicated there uses general facts from symplectic geometry. [NB(2003). These facts (moment maps preserving Poisson brackets) imply also that the form $\sigma_{\mathcal{Q}}$ is the canonical 2-form on $\mathcal{B}^*$ over the open orbit $\mathcal{Q}$, where $\mathcal{B}^* \to \{\mathcal{N}$ is regular.]

We return to the proof of the theorem. Apply the above lemma to the G-orbit Q containing the G_o-orbit $\mathcal{O}$. The fibration

$$\mathcal{B}^\nu \overset{\subseteq}{\to} \mathcal{B}^*(\mathcal{Q}) \to \mathcal{Q} \tag{21}$$

restricts to

$$\mathcal{B}^\nu \overset{\subseteq}{\to} \mathcal{S}(\mathcal{O}) \to \mathcal{O}. \tag{22}$$

For Γ it suffices to take a component of $\mathcal{S}(\mathcal{O})$. As explained for $\mathcal{Z}(\mathcal{O})$ in connection with (4.2), a component of $\mathcal{S}(\mathcal{O})$ is a dense part of an $\mathcal{S}_w$, denoted $\mathcal{S}_w(\mathcal{O})$. The fibration (22) restricts to a fibration

$$\mathcal{B}^\nu{}_w \overset{\subseteq}{\to} \mathcal{S}_w(\mathcal{O}) \to \mathcal{O}. \tag{23}$$

According to Steinberg [1976] the fibre $\mathcal{B}^\nu{}_w = \mathcal{S}_w(\mathcal{O}) \cap \mathcal{B}^\nu$ over $\nu \in \mathcal{O}$ in (23) is a single orbit of components of $\mathcal{B}^\nu$ under the component group $A_o(\nu)$ of the stabilizer of ν in G_o. These components all have the same dimension $e = e(\mathcal{O}) = \dim_{\mathbf{C}}\mathcal{B}^{\nu}$, by a result of Spaltenstein [1977]. The fundamental class of such a component in $H_{2e}(\mathcal{B})$ is independent of ν in $\mathcal{O}$, as $\mathcal{O}$ is connected. Note, incidentally, that

$$d + e = n$$

because of the fibration (14). Now calculate:

$$\begin{aligned}\tfrac{(-1)^n}{(2\pi i)^n n!}\int_{p_\lambda \mathcal{S}_w(\mathcal{O})} \varphi {\sigma_\lambda}^n &= \tfrac{(-1)^n}{(2\pi i)^n n!}\int_{\mathcal{S}_w(\mathcal{O})} (\varphi \circ p_\lambda)(\tau_\lambda + \sigma_{\mathcal{O}})^n \\ &= \tfrac{(-1)^n}{(2\pi i)^n d! e!}\int_{\mathcal{O}} \left\{ \int_{{\mathcal{B}^\nu}_w} \varphi(p_\lambda(b,\nu)){\tau_\lambda}^e(db) \right\} {\sigma_{\mathcal{O}}}^d(d\nu) + o(|\lambda|^e) \qquad (24)\end{aligned}$$

where we used the fibration (22) and Lemma 7.2 to write the integral over $\mathcal{S}_w(\mathcal{O})$ as an integral over the fibre ${\mathcal{B}^\nu}_w$ followed by an integral over the base $\mathcal{O}$.

In the integral (24) write $(b,\nu) = u\cdot(s_1,\nu_1)$ with $u \in U$ and $\nu_1 \in {s_1}^\perp$. Then the integrand becomes

$$\begin{aligned}\varphi(p_\lambda(b,\nu)) &= \varphi(u\cdot(\lambda+\nu_1)) \\ &= \varphi(u\cdot\nu_1) + o(|\lambda|) \\ &= \varphi(\nu) + o(|\lambda|)\end{aligned}$$

Thus

$$\begin{aligned}(24) &= \tfrac{(-1)^n}{(2\pi i)^n d! e!}\int_{\mathcal{O}} \varphi(\nu) \left\{ \int_{{\mathcal{B}^\nu}_w} {\tau_\lambda}^e(db) \right\} {\sigma_{\mathcal{O}}}^d(d\nu) + o(|\lambda|^e) \\ &= \tfrac{(-1)^d}{(2\pi i)^d d!}\int_{\mathcal{O}} \varphi(\nu)\langle c, e^\lambda\rangle {\sigma_{\mathcal{O}}}^d(d\nu) + o(|\lambda|^e) \\ &= \tfrac{(-1)^d}{(2\pi i)^d d!} c(\lambda)\int_{\mathcal{O}} \varphi(\nu){\sigma_{\mathcal{O}}}^d(d\nu) + +o(|\lambda|^e) \qquad (25)\end{aligned}$$

where $c = c_{\nu w} \in \mathcal{H}_e(h^*)$ is the harmonic polynomial on h^* representing the fundamental class of ${\mathcal{B}^\nu}_w$ in $H_*(\mathcal{B})$ under Borel's isomorphism (6). As in (7), $\langle c, f\rangle$ is the natural pairing of polynomials on h^* and formal power series on h , so that $\langle c, e^\lambda\rangle = c(\lambda)$. This proves the theorem.

8. Harmonic polynomials

We study in some more detail the harmonic polynomials $c_\Gamma \in \mathcal{H}_e(h^*)$ associated to a $2n$-cycle $\Gamma \in H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$ over $\bar{\mathcal{O}}$ by the relation

$$\theta_\Gamma(\lambda) = c_\Gamma(\lambda)\theta_{\mathcal{O}} + o(|\lambda|^e). \qquad (1)$$

These polynomials, or variants thereof, have a history. When θ_Γ represents a coherent family of virtual characters of Harish-Chandra modules, then c_Γ is the *character polynomial* studied by King [1979] using results of Joseph [1980], except that in King's definition the polynomials are not canonically normalized: there they are defined up to a constant factor which depends on an arbitrary regular element in the

Cartan h_o. King shows that these polynomials are essentially the same as the polynomials introduced by Jantzen [1977], Joseph [1980], and Vogan [1978]. The approach taken here is logically independent of (but strongly inspired by) these developments; it opens up some (from the previous point of view) surprising perspectives, among them the relation of the c_Γ to the homology of the flag variety, and a formula for the c_Γ in terms of Euler numbers.

Write

$$\Gamma = \sum_{y \in W/W_o} m_y y \cdot \mathcal{S}_1. \tag{2}$$

Then

$$\theta_\Gamma(\lambda) = \frac{1}{\pi} \sum_{y \in W} m_y e^{y^{-1}\lambda}. \tag{3}$$

Expanding the exponential gives

$$\theta_\Gamma(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{\pi} \sum_{y \in W} m_y y^{-1} \lambda^k. \tag{4}$$

Comparing (1) and (4) one finds that

$$\sum_{y \in W} m_y y^{-1} \lambda^k = 0 \quad \text{for } k < e \tag{5}$$

and

$$\sum_{y \in W} m_y y^{-1} \lambda^e = c_\Gamma(\lambda) p_{\mathcal{O}} \tag{6}$$

where

$$p_{\mathcal{O}} = e! \pi \theta_{\mathcal{O}}. \tag{7}$$

From equation (7) it appears that $p_{\mathcal{O}}$ is a G_o-invariant distribution on g_o (the W_o-invariant polynomial π on h_o extends to a G_o-invariant polynomial on g_o), but from (6) one sees that on h_o

$$p_{\mathcal{O}} \text{ is a } W_o\text{-invariant polynomial on } h, \text{ homogeneous of degree } e = e(\mathcal{O}). \tag{8}$$

(In this interpretation we used implicitly Harish-Chandra's regularity theorem, which guarantees that $\theta_{\mathcal{O}}$ is a locally integrable function on g_o.)

The relation (6) is an amazingly powerful tool for analyzing the polynomials c_Γ. First of all we get from (6) a fromula for the c_Γ for arbitrary $\Gamma \in H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$:

$$\text{If } \Gamma = \sum_{y \in W/W_o} m_y y \cdot \mathcal{S}_1 \text{ , then } c_\Gamma = \frac{1}{p_{\mathcal{O}}(x)} \sum_{y \in W} m_y y \cdot x^e \tag{9}$$

For any $x \in h$ with $p_{\mathcal{O}}(x) \neq 0$. In particular, for $\Gamma = \mathcal{S}_w$ (any $w \in W/W_o$) we take for $\mathcal{O} = \mathcal{O}(w)$ the unique nilpotent G_o-orbit in $i{g_o}^*$ which intersects ${s_1}^{\perp} \cap i{g_o}^*$ densely (which amounts to $\mathcal{O}$ intersecting ${b_o}^{\perp} \cap w^{-1}{b_o}^{\perp}$ densely if we think of $\mathcal{O}$ as a G_o-orbit on ${g_o}^*$ and represent $w \in W/W_o$ as $(1, w)$); then $\mathcal{S}_w \in H_{2n}(\mathcal{S}(\bar{\mathcal{O}}(w)))$ and according to Corollary 5.2

$$\mathcal{S}_w = \sum_{y \in W/W_o} (-1)^{l(w)-l(y)} \mathrm{Eu}_y(S_w) y \cdot \mathcal{S}_1. \tag{10)}$$

$\Gamma = \mathcal{S}_w \in H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$ corresponds to the fundamental cycle $[{\mathcal{B}^\nu}_w] \in H_{2e}(\mathcal{B})$ of the component $\mathcal{B}^\nu_w$ of $\mathcal{B}^\nu$ under the map

$$H_{2n}(\mathcal{S}(\bar{\mathcal{O}})) \to H_{2n}(\mathcal{S}(\mathcal{O})) \approx H_{2e}(\mathcal{B}^\nu)^{A_o(\nu)} \to H_{2e}(\mathcal{B}).$$

From the definition of c_Γ the corresponding $c_\Gamma = c_w$ is explicitly given by the formula

$$c_w(\lambda) = \int_{{\mathcal{B}^\nu}_w} e^{\omega_\lambda}. \tag{11}$$

On the other hand, the formula (9) says that

$$c_w = \frac{1}{p_{\mathcal{O}}(x)} \sum_{y \in W} (-1)^{l(w)-l(y)} \mathrm{Eu}_y(S_w) y \cdot x^e \tag{12}$$

for any $x \in h$ with $p_{\mathcal{O}}(x) \neq 0$. Comparing (11) and (12) leads to the curious integral formula

$$\int_{{\mathcal{B}^\nu}_w} e^{\omega_\lambda} = \frac{1}{p_{\mathcal{O}}(x)} \sum_{y \in W} (-1)^{l(w)-l(y)} \mathrm{Eu}_y(S_w) \langle \lambda, y \cdot x \rangle^e \tag{12}$$

for any $x \in h$ with $p_{\mathcal{O}}(x) \neq 0$

The polynomials c_Γ have some positiviy and integrality properties which should be mentioned. When $\lambda \in h^*$ is the s_1-heighest weight of a finite-dimensional representation V_λ of G, the integral on the left of (13) can be thought of as follows.

Embed the flag-manifold $\mathcal{B}$ of G into the projective space PV_λ of V_λ by

$$i_\lambda : \mathcal{B} \to PV_\lambda, b \to [b\text{-highest weight vector}]. \tag{14}$$

Then

$$\omega_\lambda = {i_\lambda}^* \omega_{PV_\lambda}$$

where ω_{PV_λ} is the Kähler 2-form of the Fubini-Study metric on PV_λ. Thus for a complex subvariety V of $\mathcal{B}$,

$$\int_V e^{\omega_\lambda} = \mathrm{vol}_\lambda(V) = \frac{1}{(\mathrm{dim} V)!} \mathrm{deg}_\lambda(V) \tag{15}$$

where $\mathrm{vol}_\lambda(\mathcal{V})$ is the volume of V with respect to the metric on $\mathcal{B}$ coming from the embedding (14) and $\deg_\lambda(V)$ is the degree of the projective variety $i_\lambda(V)$ in PV_λ. (See [Griffith-Harris 1978], p. 171, for example.) The formula (11) says that for λ regular, s_1-positive, integral

$$c_w(\lambda) = \mathrm{vol}_\lambda(\mathcal{B}^\nu{}_w) = \frac{1}{e!}\deg_\lambda(\mathcal{B}^\nu{}_w). \tag{16}$$

Thus

$$e!c_w(\lambda) \text{ is positive integral for regular, } s_1\text{-positive, integral } \lambda. \tag{17}$$

We want a more explicit formula for the W_o-invariant polynomial $p_{\mathcal{O}}$. For that purpose we make a particular choice for $\Gamma \in H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$ as follows.

We know that the W-quotient $H_{2n}(\mathcal{S}(\mathcal{O}))$ of $H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$ is $\approx$ $H_{2e}(\mathcal{B}^\nu)^{A_o(\nu)}$ and contains the $A(\nu)$-invariants $H_e(\mathcal{B}^\nu)^{A(\nu)}$ as an irreducible subrepresentation (isomorphic with the space $\mathcal{H}_e(h^*)$ of W-harmonic polynomials on h^*). Let

$$\chi_{\mathcal{O}} = \text{ character of } W_o \text{ on } H_{2e}(\mathcal{B}_{o\nu})^{A_o(\nu)}.$$

(Note that we passed from the irreducible representation $H_{2e_o}(\mathcal{B}^\nu)^{A(\nu)}$ of $W = W_o \times W_o$ to the corresponding irreducible representation $H_{2e}(\mathcal{B}_{o\nu})^{A_o(\nu)}$ of W_o.) It follows from these remarks that we can choose $\Gamma = \Gamma_{\mathcal{O}}$ in $H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$ so that

$$m_y = \frac{e_o}{|W|}\chi_{\mathcal{O}}(y) \tag{18}$$

where

$$e_o = \deg\chi_{\mathcal{O}} = \dim\mathcal{B}_{o\nu}$$

and we identify $W/W_o = W_o$ via $(y_1, y_2) \to y_1{y_2}^{-1}$ to think of $\chi_{\mathcal{O}}$ as a function on W/W_o (or as a W_o-bi-invariant function on W). Explicitly:

$$\chi_{\mathcal{O}}(y) = \chi_{\mathcal{O}}(y_1{y_2}^{-1})$$

when we write $y \in W = W_o \times W_o$ as $y = (y_1, y_2)$. With this choice of m_y the element

$$\sum_{y\in W} m_y y^{-1} = \frac{e_o}{|W|}\sum_{y\in W}\chi_{\mathcal{O}}(y)y^{-1} \tag{19}$$

of the group ring of W operates as the projection on the W_o-invariants of type $\chi_{\mathcal{O}} \otimes \chi_{\mathcal{O}}$, which have dimension $= 1$ in the irreducible W-module of this type.

Write out the equation (6) with m_y given by (18):

$$c_{\mathcal{O}}(\lambda)p_{\mathcal{O}}(x) = \frac{e_o}{|W|}\sum_{y\in W}\chi_{\mathcal{O}}(y)\langle\lambda, y\cdot x\rangle^e. \tag{20}$$

If one identifies h and h^* by a W-invariant, $\mathbf{C}$-bilinear, symmetric, inner product, then the right side of (20) is symmetric in λ and x, since $\chi_{\mathcal{O}}(y) = \chi_{\mathcal{O}}(y^{-1})$ as function of W. As a consequence,

$$c_{\mathcal{O}} = \text{const.} p_{\mathcal{O}} \tag{21}$$

for some constant depending only on $\mathcal{O}$. In particular, $p_{\mathcal{O}}$ belongs to $\mathcal{H}_e(h^*)$, and consequently:

$$p_{\mathcal{O}} = \tfrac{e_o}{c_{\mathcal{O}}(\lambda)|W|} \sum_{y \in W} \chi_{\mathcal{O}}(y) y^{-1} \cdot \lambda^e \tag{22}$$

is the (up to a constant factor) unique W_o-invariant, harmonic polynomial on h_o which transforms under W by $\chi_{\mathcal{O}} \otimes \chi_{\mathcal{O}}$. In (22) we can take any λ for which $c_{\mathcal{O}}(\lambda) \neq 0$. For the same values (18) of m_w the relations (5) and (6) imply:

The W-representation of type $\chi_{\mathcal{O}} \otimes \chi_{\mathcal{O}}$ occurs with multiplicity one in degree $e = e(\mathcal{O})$ in $\mathbf{C}[h^*]$ and not at all in lower degree.

This follows from (5), (6), and (22), since the λ^k span the homogeneous polynomials of degree k on h^* and each irreducible subspace of type $\mathbf{C}[h^*]$ contains exactly one W_o-invariant (up to scalar multiples). On the other hand, $\mathbf{C}[h^*] \approx \mathbf{C}[{h_o}^*] \otimes \mathbf{C}[{h_o}^*]$ as $W = W_o \times W_o$ module, so we can pass from W to W_o to get the following result of Borho-Macpherson [1981]:

8.1 Lemma. *The W_o-representation of type $\chi_{\mathcal{O}}$ occurs with multiplicity one in degree $e_o = e_o(\mathcal{O})$ in $\mathbf{C}[{h_o}^*]$ and not at all in lower degree.*

To make use of the formula (22) for $p_{\mathcal{O}}$ one needs to know when $c_{\mathcal{O}}(\lambda) \neq 0$. This happens exactly when the linear functional

$$\mathbf{C}_e[h^*] \to \mathbf{C}, c \to c(\lambda)$$

has a nonzero value on the W_o-invariant therein. That functional is the restriction of the natural pairing of polynomials on h^* with the formal power series e^{λ} on h. Hence :

$$c_{\mathcal{O}}(\lambda) \neq 0 \text{ iff } \lambda^e \text{ has a non-zero component along the } W_o\text{-invariant in } \mathbf{C}_e[h^*].$$

A simple sufficient condition is:

$$c_{\mathcal{O}}(\lambda) \neq 0 \text{ whenever } \lambda \text{ is regular in } h^*.$$

To see this , recall that for regular λ the map

$$\mathcal{H}(h^*) \to \mathbf{C}[W], c \to \sum_w c(w \cdot \lambda) w$$

is a W-isomorphism. Under this isomorphism the linear functional $c \to c(\lambda)$ corresponds to evaluation at 1 in the regular representation of W, hence has the non-zero component deg χ along each irreducible character χ of W_o (considered as a W_o-bi-invariant function on W). We summarize what has transpired in a theorem.

8.2 Theorem. *Let $\Gamma \in H_{2n}(\mathcal{S}(\bar{\mathcal{O}}))$ be a $2n-$cycle Γ over $\bar{\mathcal{O}}$ on the conormal variety $\mathcal{S}$ of the G_o-orbits on $\mathcal{B}$, $c_\Gamma \in \mathcal{H}_e(h^*)$ the associated harmonic polynomial on h^* :*

$$\theta_\Gamma(\lambda) = c_\Gamma(\lambda)\theta_{\mathcal{O}} + o(|\lambda|^e).$$

If $\Gamma = \sum_{y \in W/W_o} m_y y \cdot \mathcal{S}_1$, then $c_\Gamma = \frac{1}{p_{\mathcal{O}}(x)} \sum_{y \in W} m_y y \cdot x^e$. Here $p_{\mathcal{O}}$ is the (up to a constant factor) unique W_o-invariant, harmonic polynomial on h which transforms under W by $\chi_{\mathcal{O}} \otimes \chi_{\mathcal{O}}$; $\chi_{\mathcal{O}}$ is the (irreducible) character of W_o on $H_{2e}(\mathcal{B}_{o\nu})^{A_o(\nu)}$; $x \in h$ is arbitary subject to $p_{\mathcal{O}}(x) \neq 0$, which is the case whenever x is regular. The W_o-representation of type $\chi_{\mathcal{O}}$ occurs with multiplicity one in degree $e_o = e_o(\mathcal{O})$ in $\mathbf{C}[{h_o}^*]$ and not at all in lower degree. In particular, if $\Gamma = \mathcal{S}_w$, the fundamental cycle to the conormal of the G_o-orbit S_w, then $c_\Gamma = c_w$ is given by

$$c_w = \frac{1}{p_{\mathcal{O}}(x)} \sum_{y \in W} (-1)^{l(w)-l(y)} Eu_y(S_w) y \cdot x^e.$$

9. Univalence

If one uses explicitly the identification $h = h_o \times h_o$ and ${h_o}^* = h_o$ one can get a formula for $c_\Gamma(\lambda)$ which does not involve the arbitrary regular $x \in h$ and an analogous formula for $p_{\mathcal{O}}$. To see this we can place ourselves momentarily in the following general situation envisaged by Lusztig and Spaltenstein [1979].

Until further notice, let h_o denote *any* finite dimensional complex vector space, W_o *any* finite group of linear transformations of h_o. $\mathbf{C}[h_o]$ denotes the complex polynomial functions on h_o, $\mathbf{C}[W_o]$ the group algebra of W_o. We shall assume that ${h_o}^* \approx h_o$ as W_o-module so that we can identify the two whenever convenient. (Nevertheless we generally distinguish ${h_o}^*$ fom h_o in notation. The assumption ${h_o}^* \approx h_o$ could be avoided at the expense of some complications irrelevant here.) Call an irreducible representation σ of W_o *univalent* if it occurs with multiplicity one in the homogeneous polynomials of degree e_o on h_o and does not occur in lower degree (some e_o defined by this condition). Fix such a univalent representation σ of W_o and write χ_σ for its character. Denote by $\mathbf{C}_\sigma[W_o]$ the subspace of $\mathbf{C}[W_o]$ transforming by $\sigma \otimes \sigma$ under the biregular representation of $W_o \times W_o$. For any $a = \sum a(w)w$ in $\mathbf{C}_\sigma[W_o]$ define functions φ_a and f_a on ${h_o}^* \times h_o$ by

$$\varphi_a(\lambda, x) = \sum_{y \in W_o} a(y) e^{\langle \lambda, y \cdot x \rangle} \tag{1}$$

$$f_a(\lambda, x) = \frac{1}{e_o!} \sum_{y \in W_o} a(y) \langle \lambda, y \cdot x \rangle^{e_o} \tag{2}$$

$f_a(\lambda, x)$ is evidently a polynomial on ${h_o}^* \times h_o$, homogeneous of degree e_o in either variable separately. Write $\mathbf{C}^{e_o, e_o}[{h_o}^* \times h_o]$ for the space of all such polynomials, and indicate by a subscript "σ" the part thereof transforming by $\sigma \otimes \sigma$ under $W_o \times W_o$.

9.1 Lemma. *The map*

$$\mathbf{C}_\sigma[W_o] \to \mathbf{C}^\sigma_{e_o, e_o}[{h_o}^* \times h_o], a \to f_a,$$

is a $W_o \times W_o-$isomorphism. One has

$$\varphi_a(\lambda, x) = f_a(\lambda, x) + o(|\lambda|^{e_o}, |x|^{e_o})$$

with $o(|\lambda|^{e_o}, |x|^{e_o})$ *indicating a power series in* λ *and* x *whose terms are of degree* $> e_o$ *both in* λ *and in* x.

Proof. This is obvious from the univalence property of σ.

Write $h = h_o \times h_o$ and $W = W_o \times W_o$ with W_o embedded as the diagonal in W. An element $a \in \mathbf{C}_\sigma[W_o]$ may be considered as an element of $\mathbf{C}[W]$ via

$$a(w_1, w_2) = a(w_1 {w_2}^{-1})$$

As element of $\mathbf{C}[W]$, a is right W_o-invariant and transforms by $\sigma \otimes \sigma$ by W on the left. We denote the subspace of these a in $\mathbf{C}[W]$ by $\mathbf{C}_\sigma[W/W_o]$. For such an a define functions Φ and F on $h^* \times h$ by

$$\Phi_a(\lambda, x) = \sum_{y \in W} a(y) e^{\langle \lambda, y \cdot x \rangle} \tag{3}$$

$$F_a(\lambda, x) = \frac{1}{e!} \sum_{y \in W} a(y) \langle \lambda, y \cdot x \rangle^e \tag{4}$$

where we put $e = 2e_o$. $\Phi_a(\lambda, x)$ and $F_a(\lambda, x)$ transform according to $\sigma \otimes \sigma$ in λ and are W_o-invariant in x. $F(\lambda, x)$ is also polynomial, homogeneous of degee e in λ and x separately.

9.2 Lemma.

$$\Phi_a(\lambda, x) = F_a(\lambda, x) + o(|\lambda|^e, |x|^e).$$

Furthermore,

$$F_a(\lambda, x) = f_a(\lambda_1, \lambda_2) f_\sigma(x_1, x_2)$$

where $\lambda = (\lambda_1, \lambda_2), \quad x = (x_1, x_2)$ *and* ${h_o}^*$ *is identified with* h_o; f_a *is defined as in* (2) :

$$f_a(\lambda_1, \lambda_2) = \frac{1}{e_o!} \sum_{y \in W_o} a(y) \langle \lambda_1, y \cdot \lambda_2 \rangle^{e_o}$$

and f_σ *is defined by*

$$f_\sigma(x_1, x_2) = \frac{1}{e_o!} \sum_{y \in W_o} \chi_\sigma(y) \langle x_1, y \cdot x_2 \rangle^{e_o} \tag{5}$$

Proof. A typical term in the expansion of $\Phi_a(\lambda, x)$ looks like

$$\frac{1}{k_1! k_2!} \sum_{y \in W} a(y) \langle \lambda_1, y_1 \cdot x_1 \rangle^{k_1} \langle \lambda_2, y_2 \cdot x_2 \rangle^{k_2}$$

where $y = (y_1, y_2)$. By univalence, this $= 0$ if either k_1 or $k_2 < e_o$. For the same reason

$$F_a(\lambda, x) = \frac{1}{e_o! e_o!} \sum_{y \in W} a(y) \langle \lambda_1, y_1 \cdot x_1 \rangle^{e_o} \langle \lambda_2, y_2 \cdot x_2 \rangle^{e_o} \tag{6}$$

The assumption that $a \in \mathbf{C}[W]$ transforms by $\sigma \otimes \sigma$ on the left and is invariant by W_o on the right allows one to write for $w \in W$

$$a(y) = \frac{\deg\sigma}{|W|} \sum_{w \in W} \chi_\sigma(y^{-1}w)a(w)$$

where $\chi_\sigma(y) = \chi_\sigma(y_1 {y_2}^{-1})$ if $y = (y_1, y_2)$ in $W = W_o \times W_o$. In terms of the pairing $\langle a, b\rangle = \sum a(w)b(w)$ on $\mathbf{C}[W]$ this may be written as

$$a(y) = \frac{\deg\sigma}{|W|} \langle y \cdot \chi_\sigma, a\rangle.$$

Also, in terms of the pairing $\langle \lambda^i, x^j\rangle / i! = \langle \lambda, x\rangle^i \delta_{ij}$ on polynomials on h_o and ${h_o}^*$, one checks that

$$\langle \lambda_1, y_1 \cdot x_1\rangle^{e_o} \langle \lambda_2, y_2 \cdot x_2\rangle^{e_o} = \frac{1}{e_o! e_o!} \langle (\lambda_1 \otimes \lambda_2)^{e_o}, y \cdot (x_1 \otimes x_2)^{e_o}\rangle.$$

Thus (6) can be written as

$$F_a(\lambda, x) = \frac{1}{e_o! e_o!} \frac{\deg\sigma}{|W|} \sum_{y \in W} \langle y \cdot \chi_\sigma, a\rangle \langle (\lambda_1 \otimes \lambda_2)^{e_o}, y \cdot (x_1 \otimes x_2)^{e_o}\rangle.$$

In this equation $(\lambda_1 \otimes \lambda_2)^{e_o}$ and $(x_1 \otimes x_2)^{e_o}$ may be replaced by their components $(\lambda_1 \otimes \lambda_2)^{e_o}_\sigma$ and $(x_1 \otimes x_2)^{e_o}_\sigma$ which transform by $\sigma \otimes \sigma$ under W, by Schur's relations for irreducible matrix coefficients, which give further that

$$F_a(\lambda, x) = \frac{1}{e_o! e_o!} \langle (\lambda_1 \otimes \lambda_2)_\sigma{}^{e_o}, a\rangle \langle \chi_\sigma, (x_1 \otimes x_2)_\sigma{}^{e_o}\rangle \tag{7}$$

where the pointed brackets denote a W-invariant, nondegenerate, bilinear form on $\mathbf{C}_\sigma[W] \times \mathbf{C}^{\sigma}_{e_o, e_o}[h]$. Lemma 9.1 says that under the identification $h = h_o \times h_o$ such a form is given by

$$\langle (x_1 \otimes x_2)_\sigma{}^{e_o}, a\rangle = \sum_{y \in W_o} a(y) \langle x_1, y \cdot x_2\rangle^{e_o}.$$

Thus (7) becomes

$$F_a(\lambda, x) = f_a(\lambda_1, \lambda_2) f_\sigma(x_1, x_2)$$

with f_a and f_σ as specified. This is just the assertion of the lemma.

Returning now to the situation of section 8, we have :

9.3 Addendum to theorem 8.2. *The polynomials c_Γ and $p_\mathcal{O}$ are given by*

$$c_\Gamma(\lambda) = k_\mathcal{O} \sum_{y \in W_o} m_y \langle \lambda_1, y \cdot \lambda_2\rangle^{e_o} \tag{8}$$

$$p_\mathcal{O}(x) = \frac{1}{e! k_\mathcal{O}} \sum_{y \in W_o} \chi_\sigma(y) \langle x_1, y \cdot x_2\rangle^{e_o} \tag{9}$$

for some constant $k_{\mathcal{O}}$ depending only on $\mathcal{O}$.

(Note that in these equations $m_y = m_{(y,1)}$ in accordance with the identification $W/W_o = W_o$.)

Proof. This follows by comparing the relation (8.6):

$$\sum_{y\in W} m_y \langle \lambda, y\cdot x\rangle^e = c_\Gamma(\lambda) p_{\mathcal{O}}(x)$$

with the relation

$$\frac{1}{e!}\sum_{y\in W} m_y \langle \lambda, y\cdot x\rangle^e = f_a(\lambda_1,\lambda_2) f_\sigma(x_1,x_2)$$

from Lemma 9.2 when we take $\chi_\sigma = \chi_{\mathcal{O}}$ and $a =$ the component of m which transforms by $\chi_{\mathcal{O}} \otimes \chi_{\mathcal{O}}$ on the left. In the formula

$$f_a(\lambda_1,\lambda_2) = \sum_{y\in W_o} a(y)\langle \lambda_1, y\cdot\lambda_2\rangle^{e_o}$$

we may then again replace $a(y)$ by m_y, because the component of m of type $\chi_{\mathcal{O}}$ is the only one which contributes in degree e_o in this sum, as follows from

$$\Gamma \in H_{2n}(\mathcal{S}(\bar{\mathcal{O}})) \approx H_{2n}(\mathcal{S}(\mathcal{O})) + \sum_{\mathcal{O}'<\mathcal{O}} H_{2n}(\mathcal{S}(\mathcal{O}')).$$

10. Nilpotent orbital integrals

For each G_o-orbit $\mathcal{O}$ in $ig_o{}^*$ let $\mu_{\mathcal{O}}$ be the distribution on $g_o{}^*$ defined by

$$\langle \mu_{\mathcal{O}}, f\rangle = \tfrac{1}{(2\pi i)^d}\int_{\mathcal{O}} f e^{-\sigma_{\mathcal{O}}} \tag{1}$$

with $d = \dim_{\mathbf{C}}\mathcal{O}$. $\mu_{\mathcal{O}}$ is a tempered distribution on $ig_o{}^*$. (Of course, we use $ig_o{}^*$ rather than $g_o{}^*$ or g_o only to conform with the definition of the distributions θ_Γ, which will simplify the notation.) We denote by $\widehat{\mu}_{\mathcal{O}}$ the Fourier transform of $\mu_{\mathcal{O}}$, a tempered distribution on g_o, given by a locally integrable function. In this sense :

$$\widehat{\mu}_{\mathcal{O}}(\xi) = \tfrac{1}{(2\pi i)^d}\int_{\mathcal{O}_\lambda} e^{x_{\mathcal{O}}-\sigma_{\mathcal{O}}}.$$

For regular $\lambda \in ih^*$ we set $\mathcal{O}_\lambda = G_o\cdot\lambda$ and $\mu_\lambda = \mu_{\mathcal{O}_\lambda}$. Since $G_o\cdot\lambda = K\cdot(\lambda + ib_o{}^\perp) = p_\lambda \mathcal{S}_1$ for such λ we find that its Fourier transform is

$$\widehat{\mu}_\lambda = \frac{1}{(2\pi i)^n}\int_{\mathcal{O}_\lambda} e^{\lambda-\sigma_\lambda} = \frac{1}{\pi}\sum_{y\in W_o} e^{y^{-1}\lambda} \tag{2}$$

as tempered distributions on g_o.

Now take for $\mathcal{O}$ a nilpotent G_o-orbit in $i{g_o}^*$. From (8.7) we know that its Fourier transform $\widehat{\mu}_{\mathcal{O}} = \theta_{\mathcal{O}}$ is

$$\widehat{\mu}_{\mathcal{O}} = \frac{1}{e!}\frac{p_{\mathcal{O}}}{\pi} \tag{3}$$

where $p_{\mathcal{O}}$ is the distribution on g_o which on h_o is given by the up to scalars unique W_o-invariant polynomial of degree $e = e(\mathcal{O})$ on h transforming according to the irreducible character $\chi_{\mathcal{O}} \otimes \chi_{\mathcal{O}}$ under $W = W_o \times W_o$. Using (8.20) we have for any regular $\lambda \in h^*$ a formula

$$\widehat{\mu}_{\mathcal{O}} = \text{const.}\frac{1}{\pi}\sum_{y\in W} c_{\mathcal{O}}(y) y^{-1}\cdot\lambda^e \tag{4}$$

for a constant depending on $\mathcal{O}$ and λ, namely

$$\text{const.} = \frac{e_o}{e! c_{\mathcal{O}}(\lambda)|W|}$$

in the notation of section 8. Alternatively, using (9.9):

$$\widehat{\mu}_{\mathcal{O}}(x) = \text{const.}\sum_{y\in W_o} \chi_\sigma(y)\langle x_1, y\cdot x_2\rangle^{e_o} \tag{5}$$

where this time the constant depends only on O , namely

$$\text{const.} = \frac{1}{k_{\mathcal{O}} e!}.$$

(4) is a rather explicit formula for the invariant eigendistributions $\widehat{\mu}_{\mathcal{O}}$ on g_o, which have infinitesimal character 0. Compare this with Harish-Chandra's formula for eigendistributions with regular infinitesimal character λ :

$$\theta = \frac{1}{\pi}\sum_{y\in W} m_y e^{y^{-1}\lambda}. \tag{6}$$

(Of course, (6) is part of the genesis of (4).)

Formula (3) quickly leads to a formula for $\mu_{\mathcal{O}}$ itself as follows. Let $\partial_{\mathcal{O}}$ be the constant coefficient operator on h^* corresponding to the polynomial $p_{\mathcal{O}}$ on h. It satisfies

$$\partial_{\mathcal{O},\lambda} e^{\langle\lambda,x\rangle}|_{\lambda=0} = p_{\mathcal{O}}(x).$$

where the subscript λ on $\partial_{\mathcal{O},\lambda}$ means "differentiation with respect to λ".
Differentiating equation (2) with respect to λ and evaluating at $\lambda = 0$ one gets

$$\lim_{\lambda\to 0} \partial_{\mathcal{O},\lambda}\widehat{\mu}_\lambda(x) = \lim_{\lambda\to 0} \partial_{\mathcal{O},\lambda}\left\{\frac{1}{\pi}\sum_{y\in W_o} e^{\langle\lambda, y\cdot x\rangle}\right\} = \frac{|W_o|}{\pi} p_{\mathcal{O}}(x) = e!|W_o|\widehat{\mu}_{\mathcal{O}}(x).$$

Inverting the Fourier transforms we arrive at

10.1 Theorem. *Let $\mathcal{O}$ be a nilpotent G_o-orbit in $i{g_o}^*$. Then*

$$\mu_{\mathcal{O}} = e!|W_o| \lim_{\lambda \to 0} \partial_{\mathcal{O},\lambda}\mu_\lambda \tag{7}$$

and

$$\widehat{\mu}_{\mathcal{O}}(x) = \text{const.}(\lambda, \mathcal{O})\frac{1}{\pi}\sum_{y\in W}\chi_{\mathcal{O}}(y)\langle\lambda, y\cdot x\rangle^e \tag{8}$$

$$= \text{const.}(\mathcal{O})\frac{1}{\pi}\sum_{y\in W_o}\chi_\sigma(y)\langle x_1, y\cdot x_2\rangle^{e_o} \tag{9}$$

The formula (7) has some history. For $\mathcal{O} = \{0\}$, $\chi_{\mathcal{O}} =$ sign representation of W_o, $p_{\mathcal{O}} = \text{const.}\pi$, and (7) becomes Harish-Chandra's Limit Formula

$$\mu_{\mathcal{O}} = \text{const.} \lim_{\lambda \to 0} \partial_{\pi,\lambda}\mu_\lambda(x).$$

For special orbits Formula (7) was proved by Barbasch-Vogan [1982, 1983] by case by case computations and conjectured to hold in general. This conjecture was then proved by Hotta-Kashiwara [1984] (using the theory of holonomic systems).

11. A conjecture of Joseph

The asymptotic relation

$$\theta_\Gamma(\lambda) = c_\Gamma(\lambda)\theta_{\mathcal{O}} + o(|\lambda|^e) \tag{1}$$

may be used to prove a conjecture of Joseph [1984, Conjecture 9.8]. The conjecture may be explained as follows. Let

$$\mathcal{T} = \{(b,\nu)|b\in\mathcal{B}_o, \nu\in b^\perp\cap {b_o}^\perp\}. \tag{2}$$

This is the conormal variety of the action of the fixed Borel subgroup B_o of G_o on the flag manifold $\mathcal{B}_o$ of G_o. (The complexification of G_o does not enter into the picture here.) $\mathcal{T}$ is also the inverse image ${\mathcal{B}_o}^*({b_o}^\perp)$ of ${b_o}^\perp$ under the Springer map ${\mathcal{B}_o}^* \to \mathcal{N}_o$. Its dimension is $n_o = \dim_{\mathbf{C}}{\mathcal{B}_o}^*$ and our general construction gives a representation of W_o on $H_*(\mathcal{T})$. In top degree, $H_{2n_o}(\mathcal{T})$ has as basis the fundamental cycles of the components of $\mathcal{T}$. These components are the closures of the conormal bundles $\mathcal{T}_w$ of the B_o-orbits $B_o\cdot w^{-1}b_o$:

$$\mathcal{T}_w = \{B_o\cdot(w^{-1}b_o,\nu)|\nu\in {b_o}^\perp\cap w^{-1}b_o). \tag{3}$$

For each $w\in W_o$ there is a unique G_o-orbit $\mathcal{O} = \mathcal{O}(w)$ on $\mathcal{N}_o$ which intersects ${b_o}^\perp\cap w^{-1}b_o$ densely; and for each $\mathcal{O}$ the components of $\mathcal{O}\cap {b_o}^\perp$ are the closures in $\mathcal{O}\cap {b_o}^\perp$ of the dense parts of the $B_o\cdot({b_o}^\perp\cap w^{-1}b_o)$ cut out by $\mathcal{O}$ when $\mathcal{O}(w) = \mathcal{O}$. For a given $\mathcal{O}$ these components $\mathcal{V}(w)$ of $\mathcal{O}\cap {b_o}^\perp$ all have the same dimension $= \frac{1}{2}\dim\mathcal{O}$. (For these results see Steinberg [1976] and Joseph [1984].) To each such subvariety $\mathcal{V}(w)$ of $\mathcal{N}_o$ Joseph associates a polynomial $p_{\mathcal{V}(w)}$ on h_o^*, homogeneous of degree $n_o - \frac{1}{2}\dim\mathcal{O}$. This degree $= e_o = \dim{\mathcal{B}_o}_\nu$ for $\nu\in\mathcal{O}$. Joseph's conjecture says that if one writes

$$\mathcal{Z}_w = \sum_{y\in W_o} A(w,(y,1))(y,1)\cdot\mathcal{Z}_1 \tag{4}$$

in the homology $H_{2n}(\mathcal{Z})$ of the conormal variety Z , then

$$p_{\mathcal{V}(w)} = \text{const.} \sum_{y \in W_o} A(w,(y,1)) y \cdot {\rho_o}^{e_o}, \tag{5}$$

and the exponent $e_o = \dim \mathcal{B}_{o\nu}$ in this relation is the least for which the right side is non-zero. ρ_o denotes half the sum of the roots of h_o in ${b_o}^{\perp}$, which must here be thought of as an element of h_o by meansof a W_o-invariant bilinear form (Joseph identifies h_o and ${h_o}^*$); in (5) we take $(y,1)$ as element of $W = W_o \times W_o$ and use the W-action on the homology of $\mathcal{Z}$ defined earlier (which by section 12 is also the action constructed by Kazhdan-Lusztig [1980], to which Joseph refers. Joseph writes this action of $W_o \times W_o$ as a W_o–W_o bimodule.)

To prove the conjecture we consider the projection $\mathcal{B}^* = {\mathcal{B}_o}^* \times {\mathcal{B}_o}^* \to \mathcal{B}_o$ through the second factor. It restricts to a fibration

$$\mathcal{T} \overset{\subseteq}{\to} \mathcal{Z} \to \mathcal{B}_o \tag{6}$$

with $\mathcal{T}$ as in (2) as fibre over b_o. The components $\mathcal{Z}_w$ of $\mathcal{Z}$ meet the fibre $\mathcal{T}$ in the components $\mathcal{T}_w$ of $\mathcal{T}$. For any $\nu \in {b_o}^{\perp}$ there is a diagram

$$\begin{array}{ccccc} {\mathcal{B}_o}^{\nu} & \overset{\subseteq}{\to} & \mathcal{T} & \to & {b_o}^{\perp} \\ \cap & & \cap & & \cap \\ {\mathcal{B}_o}^{\nu} & \overset{\subseteq}{\to} & \mathcal{Z} & \to & \mathcal{N} \end{array} \tag{7}$$

For a given G_o-orbit $\mathcal{O}$ on $\mathcal{N}_o$ and $\nu \in \mathcal{O} \cap {b_o}^{\perp}$, $\mathcal{Z}_w$ meets the fibre $\mathcal{B}^{\nu}$ of $\mathcal{Z} \to \mathcal{N}$ in a single $A_o(\nu)$-orbit of components; $\mathcal{T}_w$ meets the fibre $\mathcal{B}_{o\nu}$ of $\mathcal{T} \to {b_o}^{\perp}$ in a union of components which lie in a single $A_o(\nu)$-orbit and determine an element of $H_{2e_o}(b_{o\nu})^{A_o(\nu)}$. (For these facts see Spaltenstein [1977].) According to a result of Hotta [1984], the map which sends $p_{\mathcal{V}(w)}$ to this element of $H_{2e_o}(\mathcal{B}_{o\nu})^{A_o(\nu)}$ extends to a W_o-isomorphism of the space spanned by the $p_{\mathcal{V}(w)}$, $\mathcal{O}(w) = \mathcal{O}$, onto $H_{2e_o}(\mathcal{B}_{o\nu})^{A_o(\nu)}$.

On the other hand, the inclusion $\mathcal{B}_{o\nu} \overset{\subseteq}{\to} \mathcal{B}_o$ gives a W_o-injection $H_{2e_o}(\mathcal{B}_{o\nu})^{A_o(\nu)} \to H_{2e_o}(\mathcal{B}_o) \approx \mathcal{H}(h_o)$, as we know from sections 3 and 7. Hence the element of $H_{2e_o}(\mathcal{B}_{o\nu})^{A_o(\nu)}$ mentioned above gives rise to another (harmonic) polynomial on ${h_o}^*$, homogeneous of degree e_o. Because of Lemma 8.1, this polynomial must be $p_{\mathcal{V}(w)}$ up to a (non-zero) factor depending only on $\mathcal{O}$.

To make use of the relation (1) note that the fibration (6) leads to a $W_o \times \{1\}$-isomorphism $H_{2n}(\mathcal{Z}) \to H_{2n_o}(\mathcal{T})$ which sends the fundamental cycle of $\mathcal{Z}_w$ to the fundamental cycle of $\mathcal{T}_w$. From Theorem 5.1 we know that the relation

$$\mathcal{Z}_w = \sum_{y \in W_o} A(w,(y,1))(y,1) \cdot \mathcal{Z}_1 \tag{4}$$

implies that

$$\theta_{\mathcal{S}_w} = \tfrac{(-1)^n}{\pi} \sum_{y \in W_o} A(w,(y,1)) e^{(y^{-1},1)\lambda}. \tag{8}$$

Of course, we also know that

$$A(w,y) = (-1)^{l(w)-l(y)} Eu_y(Z_w), \tag{9}$$

but this will not even be needed. Rather, we use the formula (9.8) to write

$$c_w(\lambda) = k_{\mathcal{O}} \sum_{y \in W_o} A(w,(y,1))\langle \lambda_1, y \cdot \lambda_2 \rangle^{e_o}. \tag{10}$$

On the other hand, the polynomial c_w represents the fundamental cycle of the component $\mathcal{B}^\nu_w$ of $\mathcal{B}^\nu$ in $H_{2e}(\mathcal{B}) \approx \mathcal{H}(h)$. Since $\mathcal{B}^\nu = \mathcal{B}_{o\nu} \times \mathcal{B}_{o\nu}$ that component must factor $\mathcal{B}^\nu_w = \mathcal{C}_w \times \mathcal{C}'_w$ as a product of a pair of components of $\mathcal{B}_{o\nu}$ depending on w. These two components correspond to two harmonic polynomials p_w , p'_w on $h_o{}^*$ and the factorization $\mathcal{B}^{\nu *}{}_w = \mathcal{C}_w \times \mathcal{C}'_w$ means that

$$c_w(\lambda) = p_w(\lambda_1)p'_w(\lambda_2) \tag{11}$$

if $\lambda = (\lambda_1,\, \lambda_2)$ in $h^* = h_o{}^* \times h_o{}^*$. From the discussion around diagram (7) it is clear that

$$p_w = \text{const.} p_{\mathcal{V}(w)}.$$

To prove Joseph's conjecture we may therefore replace $p_{\mathcal{V}(w)}$ by p_w in (5). From (10) and (11) one finds

$$p_w(\lambda_1) = \frac{k_{\mathcal{O}}}{p'_w(\lambda_2)} \sum_{y \in W_o} A(w,(y,1))\langle \lambda_1, y \cdot \lambda_2 \rangle^{e_o}$$

provided $p'_w(\lambda_2) \neq 0$, which know to be the case for regular λ_2. Choosing $\lambda_2 = \rho_o$ we get the formula conjectured by Joseph:

$$p_{\mathcal{V}(w)} = \text{const.} \sum_{y \in W_o} A(w,(y,1)) y \cdot \rho_o{}^{e_o}.$$

For the record we point out once more that we have actually proved the more precise formula

$$p_{\mathcal{V}(w)} = \text{const.} \sum_{y \in W/W_o} (-1)^{l(w)-l(y)} Eu_y(Z_w) y \cdot \rho_o{}^{e_o}. \tag{12}$$

for a constant depending only on $\mathcal{O}$.

12. Appendix. Comparison with the Kazhdan-Lusztig construction

For every simple reflection s of g_o, h_o ("simple" with respect to the fixed Borel b_o of g_o) Kazhdan and Lusztig [1980] define a proper homotopy equivalence α_s of the conormal variety $\mathcal{Z}$ of the K-action on $\mathcal{B} = \mathcal{B}_o \times \mathcal{B}_o$:

$$\mathcal{Z} = \{(b, b', \nu, \nu') | b, b' \in \mathcal{B}_o, \nu' = -\nu \in b^\perp \cap b'^\perp\}.$$

Their procedure is equivalent to the following. Choose a neighbourhood $\mathcal{U}$ of $\mathcal{Z}$ in $\mathcal{B}_o \times \mathcal{B}_o \times \mathcal{N}_o \times \mathcal{N}_o$ so that the inclusion $i : \mathcal{Z} \to \mathcal{U}$ has a proper homotopy inverse $p : \mathcal{U} \to \mathcal{Z}$. Choose a complex valued continuous function μ on $b_o{}^\perp$ so that $(k \exp{(t\nu)}\, s \cdot b_o, k' \cdot b_o, k \cdot \nu, k' \cdot \nu') \in \mathcal{U}$ for $(k \cdot b_o, k \cdot b_o, k \cdot \nu, k' \cdot \nu') \in \mathcal{Z}$ and $|t| > |\mu(\nu)|$. Here $k, k' \in K_o$, the fixed maximal compact subgroup of G and $\exp{(\nu)}$ is defined for $\nu \in b^\perp$ by thinking of ν as an element of the nilradical n of b. (One may take μ real, positive, as do Kazhdan and Lusztig.) Define

$$a_s : \mathcal{Z} \to \mathcal{U}, (k \cdot b_o, k' \cdot b_o, k \cdot \nu, k' \cdot \nu') \to (k \exp{(\mu(\nu)\nu)} s \cdot b_o, k' \cdot b_o, k \cdot \nu, k' \cdot \nu'). \tag{1}$$

Then α_s is defined to be the proper homotopy class of

$$p \circ a_s \circ i : \mathcal{Z} \to \mathcal{Z}. \tag{2}$$

In our construction the proper homotopy equivalence $a(s,1) : \mathcal{Z} \to \mathcal{Z}$ which gives the homotopy action of $(s,1) \in W = W_o \times W_o$ on $\mathcal{Z}$ is defined by the same procedure, except that $\mathcal{U}$ is taken as a suitable neighbourhood of $\mathcal{Z}$ in $\mathcal{B}^* = \{(b, b', \nu, \nu') | \nu \in b^{\perp}, \nu' \in b'^{\perp}\}$ and the map (1) is replaced by the map

$$a_{(\lambda,\lambda')}(s,1) : \mathcal{Z} \to \mathcal{U}, (k \cdot b_o, k' \cdot b_o, k \cdot \nu, k' \cdot \nu') \to (h \cdot b_o, h' \cdot b_o, h \cdot \eta, h' \cdot \eta') \tag{3}$$

where $(\lambda, \lambda') \in h^* = {h_o}^* \times {h_o}^*$ is chosen sufficiently close to (0,0), but regular, and $k, k', h, h' \in K_o$, $\quad \nu, \nu', \eta, \eta' \in {b_o}^{\perp}$ are related by

$$(k \cdot (\lambda + \nu), k' \cdot (\lambda' + \nu')) = (h \cdot (s\lambda + \eta), h' \cdot (\lambda' + \eta')).$$

(Of course here $h' = k'$ and $\eta' = \nu'$ and $k \cdot \nu = -k' \cdot \nu'$ on $\mathcal{Z}$.) To prove that (1) and (3) give rise to the same homotopy equivalence of $\mathcal{Z}$ by the construction (2) it suffices to show that if we take for $\mathcal{U}$ in (3) the $\mathcal{U}$ in (1), then (1) and (2) are properly homotopic as maps $\mathcal{Z} \to \mathcal{U}$.

We may assume that $\mathcal{U}$ consists of all points (b, b', ν, ν') of $\mathcal{B}_o \times \mathcal{B}_o \times \mathcal{N}_o \times \mathcal{N}_o$ satisfying

$$(\nu, b^{\perp}) < \epsilon, \text{dist}\,(\nu', b'^{\perp}) < \epsilon, \text{dist}\,(\nu, \nu') < \epsilon \tag{4}$$

where "dist" refers to a U-invariant Euclidean distance in g^*. The homotopy between these two maps will take the form

$$\mathcal{Z} \to \mathcal{U}, (k \cdot b_o, k' \cdot b_o, k \cdot \nu, k' \cdot \nu') \to (k \cdot b_t, k' \cdot b_o, k \cdot \nu_t, k' \cdot \nu'); \tag{5}$$

b_t and ν_t will depend only on $(t, \nu) \in [0,1] \times {b_o}^{\perp}$; $\quad t \in [0,1]$ is the homotopy parameter.

In order that (5) be a homotopy from (1) to (3) we need that

$$(b_t, \nu_t) = \begin{cases} (\exp(\mu(\nu)\nu) s \cdot b_o, \nu) \text{ for } t = 0, \\ (h \cdot b_o, h \cdot \eta) \text{ for t} = 1 \end{cases} \tag{6}$$

where $h \in K_o$ and $\nu, \eta \in {b_o}^{\perp}$ are related by

$$(\lambda + \nu) = h \cdot (s \cdot \lambda + \eta). \tag{7}$$

It will be convenient to write (5) in the equivalent form

$$\mathcal{Z} \to \mathcal{U}, k \cdot (b_o, b, \nu, -\nu) \to k \cdot (b_t, b, \nu_t, -\nu). \tag{5$'$}$$

We need to insure that the image of (5′) remains in the neighbourhood $\mathcal{U}$ of $\mathcal{Z}$ in $\mathcal{B}_o \times \mathcal{B}_o \times \mathcal{N}_o \times \mathcal{N}_o$. For this it suffices that

$$\text{dist}\,(\nu_t, {b_t}^{\perp}) < \epsilon \text{ and dist}\,(\nu_t, \nu) < \epsilon$$

or equivalently

$$\text{dist}\,(\nu, {b_t}^{\perp}) < \epsilon \text{ and dist}\,(\nu_t, \nu) < \epsilon. \tag{8}$$

Let $p_s = b_o + s \cdot b_o$ be the parabolic subalgebra of g_o associated to the simple root α. Both $\exp{(\mu(\nu)\nu)s \cdot b_o}$ and $h \cdot b_o$ in (6) depend only on ν mod the nilradical ${p_s}^{\perp}$ of p_s. Furthermore, since

$$\exp{(-\frac{1}{\lambda_\alpha}\nu)} \cdot \lambda \equiv \lambda + \nu \quad \text{mod } {p_s}^{\perp}$$

with $\lambda_\alpha = \langle \lambda, \check{\alpha} \rangle$, $\check{\alpha}$ the coroot of the simple root α belonging to s, we get from (7) that

$$h \cdot (s \cdot \lambda + \eta) \equiv \exp{(-\frac{1}{\lambda_\alpha}\nu)} \cdot \lambda \quad \text{mod } {p_s}^{\perp},$$

which may be written as

$$hbs \cdot \lambda = \exp{(-\frac{1}{\lambda_\alpha}\nu)} \cdot \lambda$$

for some $b \in B_o$; hence

$$h \cdot b_o = \exp{(-\frac{1}{\lambda_\alpha}\nu)} s \cdot b_o.$$

We shall verify presently:

$$\begin{aligned} &\text{For any } \epsilon > 0 \text{ there is a constant } R \text{ so that for all } \nu \in {b_o}^{\perp} \\ &\text{dist}\,(\nu, \exp{(t\nu)} s \cdot {b_o}^{\perp}) < \epsilon \text{ for } |t| > R. \end{aligned} \tag{9}$$

If one applies this with $R = 1/|\lambda_\alpha|$ (λ sufficiently close to 0), one sees that we may take

$$\mu(\nu) \equiv -\frac{1}{\lambda_\alpha}\text{const.} \tag{10}$$

in equation (1). With this choice of $\mu(\nu)$ we can take

$$b_t \equiv \exp{(-\frac{1}{\lambda_\alpha}\nu)} s \cdot b_o = k \cdot b_o, \tag{11}$$

independent of t, to satisfy the equations (6) in the b-component.

To insure that (6) also holds in the ν-component, choose ν_t continuous in $(t, \nu) \in [0,1] \times \mathcal{N}_o$ with

$$\nu_t = \begin{cases} \nu & \text{for } t = 0 \\ h \cdot \eta & \text{for } t = 1 \end{cases}$$

and so that

$$\text{dist}\,(\nu_t, \nu) < \epsilon \text{ for all } t \in [0,1]. \tag{12}$$

This is possible (for λ close to 0) because the given endpoints of the path ν_t satisfy

$$|h \cdot \nu - \nu| \leq \text{const.}|\lambda|$$

in view of (7). Then both inequalities in (8) are satisfied, as required.

It only remains to check the assertion (9). This assertion concerns only the subalgebra of g generated by the root vectors for $\pm\alpha$, so that we may assume $g = sl(2, \mathbf{C})$ in order to prove (9). With this assumption, rewrite (9):

$$\frac{1}{|t|}\operatorname{dist}(t\nu, \exp(t\nu)s \cdot {b_o}^{\perp}) < \epsilon \text{ for } |t| > R.$$

Replacing $t\nu$ by ν the condition (9) may be replaced by

$$\operatorname{dist}(\nu, \exp(\nu)s \cdot {b_o}^{\perp}) < |t|\epsilon \text{ for } |t| > R.$$

which says simply that

$$\operatorname{dist}(\nu, \exp(\nu)s \cdot {b_o}^{\perp}) < \text{const.} \tag{13}$$

with "const." independent of $\nu \in {b_o}^{\perp}$.

That (13) holds as long as $|\nu|$ remains bounded is obvious. On the other hand,

$$\operatorname{dist}(\nu, \exp(\nu)s \cdot {b_o}^{\perp}) \to 0 \quad \text{as } |\nu| \to \infty$$

because

$$\exp(\nu)s \cdot b_o \to b_o \quad \text{as } |\nu| \to \infty. \tag{14}$$

This last assertion says that on $\mathbf{CP}^1$ the point $[0:1]$ approaches $[1:0]$ under the right action of $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ as $|c| \to \infty$ (as does every other point). —We mention this triviality only because utlimately the whole construction of the α_s and the $a(s,1)$ comes down to this.

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