Invariant eigendistributions on a semisimple Lie algebra and homology classes on the conormal variety I: an integral formula

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Contents

1. Introduction.

According to a basic result of Harish-Chandra [1965], the invariant eigendistributions on a semisimple complex Lie algebra g_o are locally integrable functions, which for a regular infinitesimal character λ, are given by the formula

$$\theta(x) = \frac{1}{\pi(x)} \sum_{y \epsilon W} m_y e^{\lambda(y \cdot x)} \tag{1}$$

for certain coefficients m_y (notation explained below). There is a remarkable relation between these distributions on g_o and homology classes of top dimension on the conormal variety of the G_o-action on the flag-manifold of g. Namely each such distribution θ may be expressed as a contour integral over such a homology class Γ :

$$\theta(x) = \tfrac{1}{(2\pi)^n n!} \int_{p_\lambda \Gamma} e^{\xi(x)} {\sigma_\lambda}^n (d\xi). \tag{2}$$

(p_λ is a map to the orbit of λ in the compexification g^* of ${g_o}^*$ and σ_λ is the canonical holomorphic two-form on this orbit.) The coefficients m_y have a curious interpretation: every homology class Γ is a linear combination of the fundamental cycles of the conormals of the G_o-orbits on the flag manifold and for these conormals m_y is —up to a sign— the Euler number at a point on a Schubert variety. This is the content of the *Integral Formula* referred to in the title (Theorem 2.1).

What interest there may be in this integral formula should not be sought in another explicit representation of the invariant eigendistributions: one can hardly ask for anything more explicit than Harish-Chandra's formula (1). It is rather the relation to the homology of the conormal variety itself which seems of interest to me: it reveals a structure inherent in the simple exponential expressions (1) which is hardly apparent at first sight. For example, the formula (2) leads to an explicit isomorphism between the representation of W on the homology classes Γ constructed by Springer and Kazhdan-Lusztig [1980] and the representation on the θ permuting the m_y; it explains the relation between the asymptotic behaviour of the θ at 0, nilpotent orbits, and harmonic polynomials discovered by Barbasch and Vogan [1983], the relation between characters and characteristic varieties, and other things, which I shall discuss in the second part of this paper. (See also [Rossmann 1985]).

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Apart from Harish-Chandra's basic regularity theorem quoted above, the proof of the integral formula (2) uses a method I learned from Berline and Vergne [1983], explained below.

Theorems, lemmas, and formulas are numbered independently in each section; a citation (3.2) refers to formula (2) of section 3.

2. Statement of the integral formula.

Even though we shall exclusively be concerned with complex groups, it is notationally and conceptually simpler to start with the real case. For ease of reference we introduce notation in a list:

$g_o =$ a semisimple real Lie algebra.
$g =$ the complexification of g_o.
$G_o =$ the adjoint group of g_o.
$G =$ the adjoint group of g .
$\mathcal{B} =$ the flag manifold of G, realized as the variety of Borel subalgebras b of g.
$\mathcal{B}^* = \{(b, \nu) : b \in \mathcal{B}, \nu \in b^\perp \subset g^*\}$.
$\mathcal{S} = \{(b, \nu) : b \in \mathcal{B}, \nu \in b^\perp \cap i{g_o}^*\} \subset \mathcal{B}^*$.

We note that g/b can be identified with the tangent space to $\mathcal{B}$ at b, $b^\perp = (g/b)^* \subset g^*$ with the cotangent space. The real pairing of g/b and $(g/b)^*$ is taken as $(v, \nu) = \mathit{Re}\ \nu(v)$. $g_o/g_o \cap b$ is then identified with the tangent space to $G_o \cdot b$ at b, $b^\perp \cap i{g_o}^*$ with the conormal of $g_o/g_o \cap b \subset g/b$ in $(g/b)^* = b^\perp$. Correspondingly, $\mathcal{B}^*$ is identified with the cotangent bundle of $\mathcal{B}$ as real manifold, $\mathcal{S}$ with the *conormal variety* of the G_o-action on $\mathcal{B}$, i.e. the union of the conormal bundles of the G_o-orbits in $\mathcal{B}$. This variety $\mathcal{S}$ will play a fundamental role. It is a real-algebraic subvariety of $\mathcal{B}^*$ (not a complex subvariety). The part of $\mathcal{S}$ over any G_o-orbit in $\mathcal{B}$, i.e. the conormal of the orbit, is a smooth vector bundle of fibre-dimension equal to the codimension of the orbit, but $\mathcal{S}$ has singularities along the intersections of the closures of these vector bundles. These closures we call the *components* of $\mathcal{S}$. We record dimensions. Put $\dim_{\mathbf{C}}\mathcal{B} = n$; then

$$\dim_{\mathbf{C}}\mathcal{B}^* = \dim_{\mathbf{R}}\mathcal{S} = 2n.$$

When g_o itself admits a complex structure there is a bijection between the conormal variety $\mathcal{S}$ of the G_o-action on $\mathcal{B}$ and the conormal variety $\mathcal{Z}$ of the K-action (as will be explained presently). Under this bijection, the components of $\mathcal{S}$ correspond to the components of the complex algebraic variety $\mathcal{Z}$. This variety was studied by Steinberg [1976] and reappeared in the work of Kazhdan and Lusztig [1979, 1980]. We continue with notation:

$\theta =$ a Cartan involution of g_o, extended $\mathbf{C}$-linearly to g.
$k_o = \theta$-fixed subalgebra in g_o, k in g.
$\sigma =$ conjugation in g with respect to g_o.
$\tau = \sigma\theta$.
$u = \tau$-fixed subalgebra in g.
$h_o = \theta$-stable Cartan subalgebra in g_o, h in g.
K_o, K, U, H_o, H the corresponding groups.
$s_1 =$ a fixed Borel subalgebra of g containing h (base point for $\mathcal{B}$).
$\Omega_\lambda = G \cdot \lambda$, the G-orbit in g^* of a regular element λ of $h^* = [h, g]^\perp \subset g^*$.
$\sigma_\lambda =$ the canonical holomorphic 2-form on Ω_λ.

Thus U is a compact form in G, K_o a maximal compact subgroup of G_o. The form σ_λ is defined by $\sigma_\lambda(u \cdot \xi, v \cdot \xi) = \xi([u, v])$, the dot denoting the coadjoint action of g on g^*.

There is a map

$$p_\lambda : \mathcal{B}^* \to \Omega_\lambda, u \cdot (s_1, \nu) \to u \cdot (\lambda + \nu)$$

for $u \in U, \nu \in {s_1}^\perp$. This map is well-defined, and for regular λ (as stipulated) it is bijective. It turns the affine bundle $\Omega_\lambda \to \mathcal{B}^*, g \cdot \lambda \to g \cdot s_1$, into the vector bundle $\mathcal{B}^* \to \mathcal{B}$ by means of the cross section $U \cdot \lambda$ of $\Omega_\lambda \to \mathcal{B}$. It is real-analytic, but not holomorphic. It does not respect the action of G, nor even of G_o, only the action of U.

We shall be interested in integrals of the form

$$\int_{p_\lambda \Gamma} e^{\xi(x)} {\sigma_\lambda}^n (d\xi)$$

where Γ is a $2n$-cycle on $\mathcal{S} \subset \mathcal{B}^*$ with arbitrary support. Since $2n = \dim_{\mathbf{R}} \mathcal{S}$, this simply means that Γ is a formal linear combination with integer coefficients of oriented components of $\mathcal{S}$, without boundary in the sense of homology. These integals are to be understood as distributions on the real Lie algebra g_o, x being the variable in g_o. In this sense they converge, i.e.

$$\int_{p_\lambda \Gamma} \left\{ \int_{g_o} f(x) e^{\xi(x)} dx \right\} {\sigma_\lambda}^n (d\xi)$$

converges for all $f \in C_c^\infty(g_o)$.

To see the convergence of the integral, write $\xi = \text{Re } \xi + i \text{Im } \xi$ according to $g = {g_o}^* + i{g_o}^*$. Then $\text{Re}\, \xi$ is bounded along the image $p_\lambda(\mathcal{S})$ of $\mathcal{S}$ in Ω_λ : if $\xi \in p_\lambda(\mathcal{S})$, say $\xi = u \cdot (\lambda + \nu)$ with $u \cdot (s_1, \nu) \in \mathcal{S}$, then $u \cdot \nu \in {g_o}^*$ by the definition of $\mathcal{S}$, so $\text{Re } \xi = \text{Re } u \cdot \lambda$. This gives $|\text{Re } \xi| < C$ with C depending only on λ; $| \cdot |$ is a Euclidean norm on g^* .

On the other hand, for $f \in C_c^\infty(g_o)$ the Fourier transform

$$\varphi(\xi) = \int_{g_o} f(x) e^{\xi(x)} dx$$

is an entire analytic function on g^*, which for each $N = 0, 1, 2, \dots$ satisfies an estimate of the form

$$|\varphi(\xi)| < \frac{A e^{B|Re \xi|}}{1 + |\xi|^N}.$$

The convergence of the above integrals as distributions on g_o amounts to the convergence of the integrals

$$\int_\Gamma \varphi {\sigma_\lambda}^n$$

for all such f. This is clear from the above observations: the estimate for $\varphi(\xi)$ implies that for each $N = 0, 1, 2, \dots$ there is a constant C so that

$$|\varphi(p_\lambda(b, \nu))| < \frac{C}{1 + |\nu|^N} \quad \text{if } (b, \nu) \in \mathcal{S}.$$

Hence the differential form ${p_\lambda}^*(\varphi {\sigma_\lambda}^n)$ on $\mathcal{B}^*$ is rapidly decreasing along the fibres of $\mathcal{S} \to \mathcal{B}$.

Note that the integrand $\varphi\sigma_\lambda{}^n$ is a closed form on Ω_λ, being a holomorphic form of top degree $2n = \dim_{\mathbf{C}}\Omega_\lambda$. For this reason the integral depends only on the homology class of the $2n$-cycle $p_\lambda\Gamma$ on Ω_λ (or of Γ on $\mathcal{B}^*$), provided one defines a homology on these noncompact manifolds which respects the growth properties of the $\varphi(\xi)$. Such $2n$-cycles and their homology classes
will be called contours on Ω_λ or on $\mathcal{B}^*$. This is the point of view taken in [Rossmann 1984]. Of course if the cycles Γ are required to lie on the subvariety $\mathcal{S}$ of $\mathcal{B}^*$ as here, no special homology is required; the convergence of the integrals is built into the definition of $\mathcal{S}$ as we just noted.

From now on we assume that the Lie algebra g_o itself admits a complex structure. To make our definitions explicit we write out what they amount to in this case, although we shall continue to use them in the form given above.

$g_o =$ a complex Lie algebra.
$g = g_o \times g_o$ with g_o embedded as $\{(x, \bar{x})\}$, $x \to \bar{x}$ a conjugation in g_o with respect to a compact form k_o.
$g^* = g_o{}^* \times g_o{}^*$ with $((\xi, \xi'), (x, x')) = \xi(x) + \xi'(x')$.
$G = G_o \times G_o$ with $G_o = \{(a, \bar{a})\}$.
$K_o =$ the compact form of G_o with Lie algebra k_o.
$K =$ diagonal $\{(a, a)\}$ in $G_o \times G_o$.
$U = K_o \times K_o$.
$\mathcal{B} = \mathcal{B}_o \times \mathcal{B}_o$, $\mathcal{B}_o$ the flag manifold of g_o.
$\mathcal{B}^* = \mathcal{B}_o{}^* \times \mathcal{B}_o{}^* = \{(b, b'; \nu, \nu') : b, b' \in \mathcal{B}_o{}^*, \nu \in b^\perp, \nu' \in b'^\perp\}$.
$\mathcal{S} = \{(b, \bar{b}; \nu, \nu') \in \mathcal{B}^* : \nu' = -\bar{\nu}\}$, the conormal variety of the G_o- action .
$\mathcal{Z} = \{(b, b, ; \nu, \nu') \in \mathcal{B}^* : \nu' = -\nu\}$, the conormal variety of the K-action.
$h = h_o \times h_o$. We assume $\bar{h}_o = h_o$.
$W_o =$ the Weyl group of g_o, h_o.
$W =$ the Weyl group of g, $h = W_o \times W_o$ with $W_o = \{(w, w)\}$.
$s_1 = b_o \times \bar{b}_o, b_o$ a Borel subalgebra of g_o, $\bar{b}_o = w_o b_o$, the opposite Borel subalgebra.
$s_w = w^{-1} \cdot s_1 \quad (w \in W)$, base-points for the G_o-action.
$S_w = G_o \cdot s_w \quad (w \in W/W_o)$ the G_o-orbit of s_w
$\mathcal{S}_w =$ the conormal of S_w.
$z_1 = b_o \times b_o$,
$z_w = w^{-1} z_1$ $(w \in W)$, base points for the K-action.
$Z_w = K \cdot z_w (w \in W/W_o)$.
$\mathcal{Z}_w =$ the conormal of Z_w.

One may of course identify $W/W_o = W_o$, but we shall generally not do so. The automorphism $\iota : (x, y) \to (x, \bar{y})$ of g (as real Lie algebra) maps $g_o = \{(x, \bar{x})\}$ to $k = \{(x, x)\}$. The (U-equivariant) induced map $\iota : \mathcal{B} \to \mathcal{B}, (b, b') \to (b, \bar{b}')$, sends the G_o-orbit S_w to the K-orbit $Z_w (w \in W)$. Its cotangent map (under the real pairing) is $\iota : \mathcal{B}^* \to \mathcal{B}^*, (b, b'; \nu, \nu') \to (b, \bar{b}'; \nu, \bar{\nu}')$ and sends the conormal variety $\mathcal{S} = \{(b, b'; \nu, -\bar{\nu}) : \nu \in b^\perp \cap\ \bar{b}'^\perp\}$ of the G_o-action to the conormal variety $\mathcal{Z} = \{(b, b'; \nu, -\nu) : \nu \in b^\perp \cap b'^\perp\}$ of the$K$-action, sending $\mathcal{S}_w$ to $\mathcal{Z}_w$. $\mathcal{Z}$ is the complex subvariety of $\mathcal{B}^*$ referred to above. We shall give $\mathcal{S}$ the complex structure so that the map $\iota : \mathcal{S} \to \mathcal{Z}$ becomes biholomorphic. This amounts to twisting the complex structure on $\mathcal{B}$ by ι and considering $\mathcal{S}$ as a complex subvariety of the holomorphic cotangent bundle of the twisted $\mathcal{B}$. In the same way the G_o-orbits S_w are considered complex submanifolds of the twisted $\mathcal{B}$ and their closures are closures complex submanifolds.

We return to the integral

$$\int_{p_\lambda\Gamma} e^{\xi(x)} \sigma_\lambda^n(d\xi).$$

Here $n = \dim_{\mathbf{C}}\mathcal{B}$ as before, but now $n = 2n_o$ with $n_o = \dim_o\mathcal{B}_o$. Except for a factor $(-1)^n/(2\pi i)^n n!$ this integral may be written as

$$\tfrac{1}{(2\pi i)^n}\int_{p_\lambda \Gamma} e^{\xi_\lambda - \sigma_\lambda}$$

where x_λ is the function $x_\lambda(\xi) = \xi(x)$ on Ω_λ and the exponential is taken in the exterior algebra. (The integral over a k-chain of an inhomogeneous differetial form is the integral of its component of degree k.) The $2n$-cycle Γ on $\mathcal{S}$ is a linear combination of the fundamental cycles of the components of the complex variety S , i.e. of the closures of the conormals $\mathcal{S}_w$; their fundamental cycles will also be denoted $\mathcal{S}_w$. (In contrast to the real case, the components of a complex variety are always without boundary in the sense of homology.) Thus

$$\Gamma = \sum_w m_w \mathcal{S}_w$$

for certain coefficients m_w, the sum running over $w \in W/W_o$. (We shall take the homology coefficients m_w to be complex.) It therefore suffices to consider $\Gamma = \mathcal{S}_w$. For these we shall prove the following *Integral Formula*

2.1 Theorem. *Fix $\lambda \in h^*$, regular. For any $\Gamma \in H_{2n}(\mathcal{S})$. the distribution $\theta = \theta_\Gamma(\lambda)$ on g_o defined by*

$$\theta_\Gamma(\lambda) = \tfrac{1}{(2\pi i)^n}\int_{p_\lambda \Gamma} e^{x_\lambda - \sigma_\lambda}$$

is G_o-invariant and satisfies $D\theta = \chi_\lambda(D)\theta$ for all G_o-invariant constant coefficient operators D on g_o; every G_o-invariant distribution θ satisfying this equation is of the form $\theta = \theta_\Gamma(\lambda)$ for a unique $\Gamma \in H_{2n}(\mathcal{S})$. For $\Gamma = \mathcal{S}_w$, the fundamental cycle of a component of S , the distribution $\theta_\Gamma(\lambda)$ is given by the formula

$$\tfrac{1}{(2\pi i)^n}\int_{p_\lambda \mathcal{S}_w} e^{x_\lambda - \sigma_\lambda} = \frac{1}{\pi(x)}\sum_{y \epsilon W}(-1)^{n_o + l(w) + l(y)}\mathrm{Eu}_y(S_w)e^{y^{-1}\lambda(x)}\,.$$

Explanation: According to a basic theorem of Harish-Chandra [1965], an invariant eigendistribution of the G_o-invariant constant coefficient operators on g_o is a locally integrable function. The formula gives the values of this function on a regular element x of the Cartan subalgebra h_o of g_o. (Without appealing to Harish-Chandra's theorem our argument will only show that the formula holds on the regular set in g_o). $c_\lambda(D)$ is the value of D on λ when D is considered a polynomial on g^*. On the right, $\pi = \prod_{\alpha \epsilon \Delta_+}\alpha$, $\Delta_+$the roots of h in s_1. On the left, $2\pi i$ is of course a complex number. $l(w)$ is the length of $w \in W/W_o$ ($\approx W_o$), so $l(w) = \dim_{\mathbf{C}} S_w + n_o$. $\mathrm{Eu}_y(S_w)$ is the Euler number of the point s_y on the closure of S_w. We shall use the following of the many equivalent definitions of the (local) Euler number (or Euler obstruction) at a point on a complex variety (referring to [MacPherson 1974], [Gonzales-Sprinberg 1981] [Sabbah 1985] for other definitions). The notion of Euler number is local, so we may assume we are dealing with a uniformly d-dimensional algebraic subvariety V of $\mathbf{C}^n$, the point in question being the origin. Let $\mathcal{V} \subset \mathbf{C}^n \times (\mathbf{C}^n)^*$ be the conormal variety (= closure of the conormal bundle of the regular set of) V in $\mathbf{C}^n$. Then

$$\mathrm{Eu}_0(V) = (-1)^{n-d}\lim_{\epsilon \to 0}\frac{1}{\epsilon^{2n}}\int_{\mathcal{V}\cap B_\epsilon}[\tfrac{1}{2\pi i}(d\bar{q}\cdot dq - d\bar{p}\cdot dp)]^n \tag{1}$$

where $q = (q_1, \dots, q_n)$ in $\mathbf{C}^n$, $p = (p_1, \dots, p_n)$ in $(\mathbf{C}^n)^*$, $d\bar{q}\cdot dq = \sum_j d\bar{q}_j dq_j$, and $B_\epsilon = \{|q|^2 + |p|^2 \leq \epsilon^2\}$. We shall verify in an appendix that the above definition of $\mathrm{Eu}_0(V)$ agrees with the definitions found in the literature. In particular, these Euler numbers are integers, $\mathrm{Eu}_p(V) = 1$ if $\overline{V}$ is smooth at p, and $\mathrm{Eu}_p(V) = 0$ if $p \notin \overline{\mathrm{V}}$.

3. Proof of the integral formula.

Fix $\lambda \in h^*$, regular. For $\Gamma \in H_{2n}(\mathcal{S})$ the distribution $\theta_\Gamma(\lambda)$ on g_o is G_o-invariant: G_o being connected, the $2n$-cycle $p_\lambda\Gamma$ on Ω_λ is homotopic to $a \cdot p_\lambda\Gamma$ for any $a \in G_o$ by a homotopy $a(t) \cdot p_\lambda\Gamma, 0 \le t \le 1$; the integral defining $\theta_\Gamma(\lambda)$ is independent of t, since the homotopy respects the condition "$|\mathrm{Re}(\xi)| \le$ const." required for the convergence of the integral defining $\theta_\Gamma(\lambda)$ and Γ is a cycle (not just a chain). That $\theta_\Gamma(\lambda)$ satisfies the differential equation

$$D\theta = \chi_\lambda(D)\theta$$

follows from the fact that the G_o-invariant polynomials on g^* are constant on Ω_λ.

On the other hand, on the regular elements of h_o any solution θ of this differential equation is given by a formula

$$\theta = \frac{1}{\pi} \sum_{y \in W} m_y e^{y^{-1}\lambda}$$

with $m_y = m_{yz}$ for $z \in W_o$. (See [Varadarajan 1977] section I.4.6, equation (17).) So it suffices to show that the $\theta_\Gamma(\lambda)$ corresponding to the $\mathcal{S}_w$ are given by such a formula on h_o and form a basis for such functions on the regular set in h_o. This will follow from the integral formula, because the matrix $[\mathrm{Eu}_y S_w]$ is unipotent-triangular with respect to the Bruhat order on $W/W_o(\approx W_o)$.

The proof of the integral formula uses a method which goes back to Bott [1967]. Berline and Vergne [1983] showed that the method can be used for the evaluation of the type of integral under consideration when the contour $p_\lambda\Gamma$ is replaced by an elliptic orbit of the real group G_o in $g_o{}^*$; our proof ows much to their paper. We summarize the method in Lemma 1 below with the modifications needed for the case at hand, when integration goes over possibly singular varieties or generally over homology-chains rather than over smooth manifolds.

Let M be a real C^∞ manifold, X a C^∞ vector field on M. Introduce the equivariant exterior derivative operator on (generally inhomogeneous) C^∞ differential forms ω by the formula

$$d_X\omega = d\omega + i(X)\omega$$

where $i(X)$ is inner multiplication by X. It satisfies ${d_X}^2 = di(X) + i(X)d = L_X$, the Lie derivative, and $d_X(\alpha\beta) = d_X(\alpha)\beta + (-1)^a\alpha(d_X\beta)$ if α is homogeneous of degree a. (All products of differential forms are exterior products.) There is an equivariant Stokes's theorem: if Γ is a piecewise C^∞, finite m-chain on M which is tangential to X, then

$$\int_\Gamma d_X\omega = \int_{\partial\Gamma} \omega$$

for every C^∞ form ω. This is because along Γ the second summand of $d_X\omega = d\omega + i(X)\omega$ vanishes in degree $m = \dim\Gamma$ in view of the tangential condition. (By definition, the integral over an m-chain of a inhomogenous form is the integral of its component of degree m.) We shall also impose a mild regularity condition on the m-chain Γ : we assume that for every C^∞ form ψ

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^k} \int_{\Gamma \cap B_\epsilon(p)} \psi = 0 \qquad \text{for } k < m = \dim\Gamma.$$

Here $B_\epsilon(p)$ denotes the ϵ-ball in a coordinate system about the point p of M. The intersection with the chain Γ may be defined using a subdivision of Γ and amounts to the set-intersection if Γ is taken to be a chain carried by an oriented variety (which is the only case we will need). The condition is always satisfied, for example, when Γ is a subanalytic chain on a real analytic manifold, as one can see from [Federer 1969] sections 4.3.18–19 and [Hardt 1975]. It is in particular satisfied for fundamental cycles of complex analytic varieties, as is indeed clear from the usual proof of local integrability over such cycles, see [Griffiths-Harris 1978, page 32, for example]. We shall assume the condition satisfied by all chains Γ without special mention.

3.1 Lemma. *Assume:*
(1) *X has only isolated zeros.*
(2) *Γ is an m-chain on M tangential to X without zeros of X on its boundary.*
(3) *ω is a C^∞ form satisfying $d_X\omega = 0$.*
(4) *φ is an X-invariant, C^∞ one-form on a neighbourhood of Γ which satisfies*

$$\varphi(X) = \sum_j {x_j}^2 + o(\sum_j {x_j}^2)$$

on Γ in some coordinate $\{x_j\}$ system on M around each zero p of X. Then:

$$\int_\Gamma \omega = (-1)^n \sum_p \omega_0(p) \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2\mathrm{n}}} \int_{\Gamma \cap \mathrm{B}_\epsilon(\mathrm{p})} (\mathrm{d}\varphi)^{\mathrm{n}} + \int_{\partial\Gamma} \frac{\varphi(\mathrm{d}\varphi)^{\mathrm{n}-1}}{\varphi(\mathrm{X})^{\mathrm{n}}} \omega$$

if $m = 2n$ is even and $= 0$ otherwise.

Explanation. $\omega_0(p)$ is the value at p of the degree-zero component of ω. $B_\epsilon(p) = \{\sum {x_j}^2 \le \epsilon^2\}$ in the coordinate system referred to. Observe that for small ϵ the boundary $\partial[\Gamma \cap B_\epsilon(p)]$ is a cycle on the sphere $S_\epsilon(p) = \{\sum {x_j}^2 = \epsilon^2\}$. A form φ as in (4) always exists when X leaves invariant a Riemann metric g and has only nondegenerate zeros in the sense that in a coordinate system around p :

$$X = \sum a_{ij} x_i \frac{\partial}{\partial x^j} + \text{higher terms in } x, \qquad \text{with } \det(a_{ij}) \neq 0,$$

as one can then take $\varphi(Y) = g(X, Y)$. Alternatively, a suitable φ may be constructed locally around each zero of X and patched together with an X-invariant partition of unity, which always exists when X leaves a Riemann metric invariant. Note that the condition on the behaviour of $\varphi(X)$ around a zero p of X concerns only its restriction to Γ.

Proof of Lemma 3.1. Let $\theta = \varphi/\varphi(X)$. The assumptions imply that $d_X\theta = 1 + d\theta$. This form is invertible on $\{X \neq 0\}$:

$$(1 + d\theta)^{-1} = \sum_j (-1)^j (d\theta)^j.$$

For any C^∞ form ω on $\{X \neq 0\}$ satisfying $d_X\omega = 0$ one has

$$\omega = d_X(\theta(d_X\theta)^{-1}\omega).$$

Therefore

$$\int_{\Gamma}\omega = \lim_{\epsilon\to 0}\sum_p \int_{\Gamma-\Gamma\cap B_\epsilon(p)} d_X(\theta d_X\theta)^{-1}\omega)$$
$$= \lim_{\epsilon\to 0}\sum_p - \int_{\partial[\Gamma\cap B_\epsilon(p)]}\theta(d_X\theta)^{-1}\omega + \int_{\partial\Gamma}\theta(d_X\theta)^{-1}\omega$$

We claim that

$$\lim_{\epsilon\to 0}\int_{\partial[\Gamma\cap B_\epsilon(p)]}\theta(d\theta)^{j-1}\omega = 0 \qquad \text{for } 2j<m$$

or equivalently

$$\lim_{\epsilon\to 0}\int_{\partial[\Gamma\cap B_\epsilon(p)]}\frac{\varphi(d\varphi)^{j-1}}{\varphi(X)^j}\omega = 0 \qquad \text{for } 2j<m.$$

For by assumption we may choose coordinates around p so that

$$\varphi(X) = \sum_j {x_j}^2 + o(\sum_j {x_j}^2) \qquad \text{on } \Gamma.$$

Since $B_\epsilon(p)$ is the ϵ-ball in this coordinate system we may therefore replace $\varphi(X)$ by ϵ^2 on $S_\epsilon(p) = \{\sum {x_j}^2 = \epsilon^2\}$ when taking the limit. We may also assume $\omega = dx_1dx_2\cdots dx_k$ in coordinates around p, and in particular $d\omega = 0$. This allows us to rewrite the integral using Stokes's theorem as

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^{2j}}\int_{\Gamma\cap B_\epsilon(p)}(d\varphi)^j\omega;$$

and this is indeed $=0$ for $2j<m$ by the regularity assumption on Γ. On the other hand, when $2j=m$ we get the formula of the lemma.

Return to the proof of the integral formula for

$$\tfrac{1}{(2\pi i)^n}\int_{p_\lambda \mathcal{S}_w} e^{x_\lambda-\sigma_\lambda}.$$

Use the same letter x for the vector field induced by x on Ω_λ through the coadjoint action, $d_x = d + i(x)$ for the equivariant derivative. The form $x_\lambda - \sigma_\lambda$ on Ω_λ satisfies $d_x(x_\lambda-\sigma_\lambda)=0$, hence so does $e^{x_\lambda-\sigma_\lambda}$. When $x\in k_o$ the same is true of the form ${p_\lambda}^* e^{x_\lambda-\sigma_\lambda}$ on $\mathcal{B}^*$ when x is considered a vector field on $\mathcal{B}^*$, because $p_\lambda:\mathcal{B}^*\to\Omega_\lambda$ respects the action of K_o. However, Lemma 1 does not apply directly, because the integral is taken in the distribution sense as

$$\int_{p_\lambda\Gamma}\left\{\int_{g_o} f(x)e^{\xi(x)}dx\right\}{\sigma_\lambda}^n(d\xi)$$

(suppressing the normalizing factor) with $f\in C_c^\infty(g_o)$. We first show that this distribution can be restricted to k_o, meaning (here):

3.2 Lemma.

$$\int_{p_\lambda \mathcal{S}_w}\left\{\int_{k_o} f(x)e^{\xi(x)}dx\right\}{\sigma_\lambda}^n(d\xi)$$

converges for $f \in C_c^\infty(k_o)$ and defines a distribution on k_o.

Proof of Lemma 3.2. Take $f \in C_c^\infty(k_o)$ and put

$$\varphi(\xi) = \int_{k_o} e^{\xi(x)} f(x)dx,$$

an entire analytic function on g^*. We have for each $N = 0, 1, 2\dots$ an estimate

$$|\varphi(\xi)| \leq \frac{Ae^{B|Re\xi_k|}}{1 + |\xi_k|^N} \tag{1}$$

where $\xi_k \in {k_o}^*$ is the restriction of ξ to k_o, $|\cdot|$ is a norm on g^*, and $\xi = \mathrm{Re}\,\xi + i\mathrm{Im}\xi$ according to $g^* = {g_o}^* + i{g_o}^*$, as earlier.

Recall that $|\mathrm{Re}\,\xi| \leq$ const. on $p_\lambda\mathcal{S}$, hence also $|\mathrm{Re}\,\xi_k| \leq$ const. on $p_\lambda\mathcal{S}$. It therefore suffices to show that on $p_\lambda\mathcal{S}$ one has

$$|\xi|^2 \leq a|\xi_k|^2 + b \tag{2}$$

(with $a > 0$, necessarily); for this will guarantee that on $\mathcal{S}$ the function $\varphi \circ p_\lambda$ on $\mathcal{B}^*$ is rapidly decreasing along the fibres of $\mathcal{S} \to \mathcal{B}$.

To see (2), identify $g = g^*$ by the Killing form $(\cdot, \cdot)$ and write $\xi = \xi_k + \xi_{k^\perp}$ according to $g = k + k^\perp$, and $\xi = \xi_R + i\xi_I$ according to $g = g_o + ig_o$. We specify the norm on g by setting $|\xi|^2 = -(\xi, \tau\xi)$, τ the conjugation in g with respect to the compact form u introduced earlier. Observe that the involution θ of g is given by $\theta\xi = \xi_k - \xi_{k^\perp}$. We have

$$(\xi, \xi) = (\xi_k, \xi_k) + (\xi_{k^\perp}, \xi_{k^\perp}) = -(\xi, \theta\xi) + 2(\xi_k, \xi_k).$$

or

$$(\xi, \theta\xi) = -(\xi, \xi) + 2(\xi_k, \xi_k). \tag{3}$$

On the other hand

$$(\xi, \theta\xi) = (\xi_R, \theta\xi_R) - (\xi_I, \theta\xi_I) + 2i(\xi_R, \theta\xi_I),$$

or

$$(\xi, \theta\xi) = |\xi_R|^2 - |\xi_I|^2 + 2i(\xi_R, \theta\xi_I). \tag{4}$$

From (3),

$$\mathrm{Re}(\xi, \theta\xi) = -\mathrm{Re}(\xi, \xi) + 2\mathrm{Re}(\xi_k, \xi_k)$$

while from (4)

$$\mathrm{Re}(\xi, \theta\xi) = -|\xi_R|^2 + |\xi_I|^2.$$

So

$$|\xi_I|^2 = |\xi_R|^2 - \mathrm{Re}(\xi, \xi) + 2\mathrm{Re}(\xi_k, \xi_k)$$

On $p_\lambda\mathcal{S}$, $|\xi_R|^2 \le$ const. (as noted earlier) and $(\xi,\xi) = (\lambda,\lambda)$. Thus on $p_\lambda\mathcal{S}$

$$|\xi|^2 \le c(|\xi_R|^2 + |\xi_I|^2) \le a|\xi_k|^2 + b.$$

This proves (2) and thereby Lemma 2.

Because of Lemma 2 it suffices to show that the integral formula holds on k_o, i.e. that

$$\tfrac{1}{(2\pi i)^n}\int_{p_\lambda\mathcal{S}_w} e^{x_\lambda - \sigma_\lambda}$$

interpreted as a distribution on k_o in accordance with Lemma 2 coincides on the regular set with the K_o-invariant function whose values on the regular elements of $h_o \cap k_o$ is given by the right side of the integral formula. (This is seen as in [Duflo-Heckman-Vergne], A.6.(3)–(4).) Choose a K_o-invariant, positive-definite real-bilinear inner product g$(\cdot,\cdot)$ on g^*. For $f \in C_c^\infty(k_o)$,

$$\begin{aligned}\int_{p_\lambda\mathcal{S}_w}\Big\{\int_{k_o} f(x)e^{x_\lambda-\sigma_\lambda}dx\Big\} &= \lim_{r\to\infty}\int_{p_\lambda\mathcal{S}_w\cap\{g(\xi,\xi)\le r\}}\Big\{\int_{k_o} f(x)e^{x_\lambda-\sigma_\lambda}dx\Big\}\\ &= \lim_{r\to\infty}\int_{k_o} f(x)\Big\{\int_{p_\lambda\mathcal{S}_w\cap\{g(\xi,\xi)\le r\}} e^{x_\lambda-\sigma_\lambda}\Big\}dx\end{aligned}$$

Assume the support of f consists of regular elements. The inner integral is now of the form to which Lemma 1 applies: When $x \in k_o$ is regular, then the vector field $\xi \to x\cdot\xi$ induced by the coadjoint action has finitely many non-degenerate zeros on Ω_λ, namely the $w\lambda, w \in W$, if $x \in h_o$, and leaves invariant the metric $g(\xi,\eta) = -(\xi,\tau\eta)$. The $2n$-cycle $p_\lambda\mathcal{S}_w$ on Ω_λ is tangential to this vector field, since $\mathcal{S}_w$ is K_o-stable (being a component of $\mathcal{S}$) and p_λ is K_o-equivariant. According to the remark after Lemma 1 the form $\varphi = \varphi_x$ on Ω_λ may be taken to be

$$\varphi_{x,\xi}(\eta) = g(x\cdot\xi,\eta)$$

when $\eta \in g^*$ is a tangent vector to Ω_λ at $\xi \in \Omega_\lambda$. Actually it will ultimately be convenient to make a different choice for φ_x individually around each zero of the vector field, but we do assume φ_x to be given by this formula at least for $g(\xi,\xi)$ sufficiently large. This is possible as the different local choices of φ_x may be put together with a K_o-invariant partition of unity, as was already noted after Lemma 1. That lemma now gives for sufficiently large r :

$$\begin{aligned}\tfrac{1}{(2\pi i)^n}\int_{p_\lambda\mathcal{S}_w\cap\{g(\xi,\xi)\le r\}} e^{x_\lambda-\sigma_\lambda} = \frac{(-1)^n}{(2\pi i)^n}\sum_{w\in W} e^{\lambda(yx)}\lim_{\epsilon\to 0}\frac{1}{\epsilon^{2n}}\int_{\mathcal{S}_w\cap B_\epsilon(s_y)}(\varphi_x)^n\\ + \frac{1}{(2\pi i)^n}\int_{\Sigma_w(r)}\psi_x e^{x_\lambda-\sigma_\lambda}.\end{aligned} \tag{5}$$

In the first integral on the right the integration has been transferred from Ω_λ to $\mathcal{B}^*$ by the map p_λ so that the form φ_x is here a form on $\mathcal{B}^*$ with the requisite properties. In the second integral

$$\Sigma_w(r) = \partial[p_\lambda\mathcal{S}_w \cap \{g(\xi,\xi)\le r\}] = p_\lambda\mathcal{S}_w \cap \{g(\xi,\xi) = r\}$$

and the form ψ_x is

$$\psi_x = \frac{\varphi_x(d\varphi_x)^{n-1}}{\varphi_x(x\cdot\xi)^n} = \frac{g(x\cdot\xi,-)g(x\cdot -,-)^{n-1}}{g(x\cdot\xi,x\cdot\xi)^n}\,. \tag{6}$$

Here $g(x\cdot\xi,-)$ is the one-form $\varphi_{x,\xi}$ on Ω_λ introduced above, $g(x\cdot -,-)$ is its exterior derivative $d\varphi_x$.

We show that the second term on the right side of (5) $= 0$ as distribution on the regular set in k_o :

3.3 Lemma. *For $f \in C_c^\infty(k_o)$ with support on the regular set,*

$$\lim_{r \to \infty} \int_{\Sigma_w(r)} \left\{ \int_{k_o} f(x)\psi_x e^{x_\lambda - \sigma_\lambda} dx \right\} = 0.$$

Proof of Lemma 3.3. One sees from (6) that as function of $x \in g$ and $\xi \in g^*$, $\psi_{x,\xi}$ and all of its partials with respect to x (in a linear coordinate system) are bounded on $|x \cdot \xi| > \epsilon$ (any $\epsilon > 0$), by homogeneity in x and ξ. We may assume this inequality holds for x in the support of f and $|\xi| > R$ (R large). This is because the finitely many points ξ on Ω_λ satisfying $x \cdot \xi = 0$ remain in a bounded subset of Ω_λ as x varies over the support of f. Thus

$$\left| \int_{k_o} f(x)\psi_{x,\xi} e^{\xi(x)} dx \right| \le A e^{B|Re\xi|}$$

An integration by parts gives

$$\left| \xi_j \int_{k_o} f(x)\psi_{x,\xi} e^{\xi(x)} dx \right| = \left| \int_{k_o} \frac{\partial}{\partial x_j} [f(x)\psi_{x,\xi}] e^{\xi(x)} dx \right| \le A' e^{B'|Re\xi|}$$

Continuing this way we get for each $N = 0, 1, 2, \ldots$ an estimate

$$\left| \int_{k_o} f(x)\psi_{x,\xi} e^{\xi(x)} dx \right| \le \frac{A e^{B|Re\xi|}}{|\xi|^N}$$

Since $|\text{Re } \xi|$ is bounded on $p_\lambda \mathcal{S}$ the lemma follows.

From formula (5) and Lemma 3 we get:

$$\tfrac{1}{(2\pi i)^n} \int_{p_\lambda \mathcal{S}_w} e^{x_\lambda - \sigma_\lambda} = \tfrac{1}{(2\pi i)^n} \sum_{y \in W} e^{\lambda(yx)} \lim_{\epsilon \to 0} \tfrac{1}{\epsilon^{2n}} \int_{\mathcal{S}_w \cap B_\epsilon(s_y)} (d\varphi_x)^n. \tag{7}$$

We now have to evaluate the limits in this formula. It is here that the singularities of the variety $\mathcal{S}_w$ come in.

We start by constucting the form φ_x entering into formula (7). For that purpose we shall make use of the complex structure on $\mathcal{S}$. We do so by transferring the integral (7) from $\mathcal{S}$ to $\mathcal{Z}$ by the map $\iota : \mathcal{B}^* \to \mathcal{B}^*$ and replacing p_λ by $q_\lambda = p_\lambda \circ \iota$. The x-equivariance conditions required for (7) remain satisfied as $x \in k_o$ and ι is K_o-equivariant (in fact U-equivariant). As a consequence, we shall now deal with $\mathcal{Z}$ instead of $\mathcal{S}$.

φ_x should then be a C^∞ one-form on a neighbourhood of $\mathcal{Z}$ in $\mathcal{B}^*$, invariant under the vector field on $\mathcal{B}^*$ induced by the infinitesimal action of a regular element x in $h_o \cap k_o$ and satisfying

$$\varphi_x(x) = \sum_j {x_j}^2 + o(\sum_j {x_j}^2) \qquad \text{on } \mathcal{Z}$$

in a suitable coordinate system $\{x_j\}$ on $\mathcal{B}^*$ around each point $z_y = (z_y, 0)$. As mentioned, this may be done locally around each point z_y separately. We first introduce coordinates in a neighbourhood of z_y on $\mathcal{B}$ as follows. Write $g = {n_y}^- + z_y$, h-stable decomposition. Choose a basis $v_\alpha, \alpha \in -y^{-1}\Delta_+$, of root vectors for ${n_y}^-$ (Δ_+ the roots of h in z_1). Write a general $v \in {n_y}^-$ as $v = \sum_\alpha q_\alpha v_\alpha$ with $q_\alpha \in \mathbf{C}$. (Here and elsewhere the sum runs over $\alpha \in -y^{-1}\Delta_+$.)

The map ${n_y}^- \to \mathcal{B}, v \to \exp(v)z_y$ is biholomorphic to a neighbourhood of z_y in $\mathcal{B}$, so that we can introduce coordinates q_α around z_y by writing $b = \exp\left(\sum_\alpha q_\alpha v_\alpha\right) \cdot z_y$. The coordinates q_α on $\mathcal{B}$ extend to canonical coordinates q_α, p_α on its holomorphic cotangent bundle $\mathcal{B}^*$: $p_\alpha = \partial/\partial q_\alpha$ as function on $\mathcal{B}^*$. In these coordinates q_α, p_α the action of $\exp x, x \in h$ is given by

$$\exp x : (q_\alpha, p_\alpha) \to (e^\alpha q_\alpha, e^{-\alpha} p_\alpha),$$

where $\alpha = \alpha(x)$.

The canonical one-form on the cotangent bundle $\mathcal{B}^*$ is

$$\sum_\alpha p_\alpha dq_\alpha.$$

From the definition of $\mathcal{Z}$ as the union of the conormals of the K orbits on $\mathcal{B}$ it is clear that this one-form vanishes on vectors tangential to $\mathcal{Z}$:

$$\sum_\alpha p_\alpha(t){q_\alpha}'(t) = 0$$

for any differentiable curve $(q_\alpha(t),\, p_\alpha(t))$ which lies on $\mathcal{Z}$,i.e.

$$\sum_\alpha p_\alpha dq_\alpha = 0 \qquad \text{on } \mathcal{Z}.$$

Define a map $f : \mathcal{B}^* \to \mathcal{B}$ in these coordinates on a neighbourhood of the point $(z_y, 0)$ by the formula

$$f(q,p) = q + \bar{p} \quad \text{ i.e. } \quad f_\alpha(q,p) = q_\alpha + \bar{p}_\alpha$$

the bar denoting complex conjugation. Observe that the map f is $H_o \cap K_o$ -equivariant. Since $\mathcal{Z}$ is a union of conormals, the following lemma should be geometrically plausible.

3.4 Lemma. *On $\mathcal{Z}$,*

$$|q + \bar{p}|^2 = |q|^2 + |p|^2 + o(|q|^2 + |p|^2)$$

Proof of Lemma 3.4. Write

$$|q + \bar{p}|^2 = |q|^2 + |p|^2 + q \cdot p + \bar{q} \cdot \bar{p}.$$

It suffices to show that

$$\frac{|q \cdot p|}{|q|^2 + |p|^2} \to 0 \qquad \text{as } (q,p) \to (0,0) \text{ on } \mathcal{Z}.$$

So assume $(q_k, p_k) \to (0,0)$ on $\mathcal{Z}$, $(q_k, p_k) \neq (0,0)$. To show :

$$\limsup_k \frac{|q_k \cdot p_k|}{|q_k|^2 + |p_k|^2} = 0.$$

Put

$$\lambda_k = \frac{1}{\sqrt{|q_k|^2 + |p_k|^2}}.$$

Pass to subsequence of (q_k, p_k) for which ${\lambda_k}^2 |q_k \cdot p_k|$ converges to the lim sup and then to a further subsequence for which $(\lambda_k q_k, \lambda_k p_k)$ converges as well, say to (v, w), so that the lim sup $= v \cdot w$. The coordinate vector (v, w) lies on the tangent cone to the closure of $\mathcal{Z}$ at $z_y = (0, 0)$ as an analytic subvariety of the (q, p)-coordinate space. According to a result of Whitney [1965], the vectors on the tangent cone may also be realized as tangent vectors of differentiable arcs on the closure of $\mathcal{Z}$. So we can write

$$(v, w) = (q'(0), p'(0))$$

where $(q(t), p(t))$ is a differentiable arc on $\mathcal{Z}$ with $(q(0), p(0)) = (0, 0)$. Since $p(t) \cdot q'(t) \equiv 0$ on $\mathcal{Z}$ we get by differentiation at $t = 0$ that $p'(0) \cdot q'(0) = 0$, i.e. $v \cdot w = 0$. This proves the lemma.

For the form φ_x in formula (7) we take around z_y :

$$\varphi_x = \sum_\alpha \tfrac{1}{\alpha} \bar{f}_\alpha df_\alpha$$

where $\alpha = \alpha(x)$ for the fixed regular element $x \in h_o \cap k_o$ and $f_\alpha = q_\alpha + \bar{p}_\alpha$ as before. φ is C^∞, invariant under $H_o \cap K_o$, and $\varphi_x(x) = \sum_\alpha \bar{f}_\alpha f_\alpha = |q + \bar{p}|^2 = |q|^2 + |p|^2 + o(|q|^2 + |p|^2)$ on $\mathcal{Z}$. So φ_x satisfies all requirements. The ϵ-ball $B_\epsilon(z_y)$ in formula (7) is $B_\epsilon(z_y) = \{|q|^2 + |p|^2 \le \epsilon^2\}$, and the limit to be calculated is:

$$\lim_{\epsilon \to 0} \tfrac{1}{\epsilon^{2n}} \int_{\mathcal{Z}_w \cap B_\epsilon(z_y)} (d\varphi_x)^n. \tag{8}$$

The relation $\varphi_x = \sum_\alpha \frac{1}{\alpha} \bar{f}_\alpha df_\alpha$ gives $d\varphi_x = \sum_\alpha \frac{1}{\alpha} d\bar{f}_\alpha df_\alpha$ and

$$\begin{aligned}
(d\varphi_x)^n &= (\sum_\alpha \tfrac{1}{\alpha} d\bar{f}_\alpha df_\alpha)^n \\
&= n! \prod_\alpha (\tfrac{1}{\alpha} d\bar{f}_\alpha df_\alpha) \\
&= \frac{1}{\pi} n! \prod_\alpha (d\bar{f}_\alpha df_\alpha) \\
&= \frac{1}{\pi} (\sum_\alpha d\bar{f}_\alpha df_\alpha)^n \\
&= \frac{(-1)^{l(y)}}{\pi} (d\varphi_o)^n
\end{aligned}$$

where

$$\pi = \prod_{\alpha \epsilon \mathit{\Delta}_+} \alpha, \tag{9}$$

and

$$\varphi_o = \sum_\alpha \bar{f}_\alpha df_\alpha = \sum_\alpha (\bar{q}_\alpha + p_\alpha)(dq_\alpha + d\bar{p}_\alpha).$$

Since $\sum_\alpha p_\alpha dq_\alpha = 0$ on $\mathcal{Z}$ one finds that

$$d\varphi_o = \sum_\alpha d\bar{q}_\alpha dq_\alpha - d\bar{p}_\alpha dp_\alpha \qquad \text{on } \mathcal{Z}.$$

Therefore

$$\begin{aligned}\lim_{\epsilon \to 0} \frac{1}{\epsilon^{2n}} \int_{\mathcal{Z}_w \cap B_\epsilon(z_y)} (d\varphi_x)^n & \\ &= \frac{(2\pi i)^n (-1)^{l(y)}}{\pi} \lim_{\epsilon \to 0} \int_{\mathcal{Z}_w \cap B_\epsilon} \Big(\frac{1}{2\pi i} \sum_\alpha d\bar{q}_\alpha dq_\alpha - d\bar{p}_\alpha dp_\alpha\Big)^n \\ &= \frac{(2\pi i)^n (-1)^{l(w)+l(y)}}{\pi} \mathrm{Eu}_{z_y} \mathcal{Z}_w.\end{aligned}$$

In this formula one can again replace $\mathcal{Z}$ by $\mathcal{S}$ and z_y by s_y.

Substituting then into (7) and replacing the product of the roots in ${z_1}^\perp$ as in (9) by the product of the roots in ${s_1}^\perp$ (which introduces a factor $(-1)^{n_o}$) completes the proof of the integral formula.

4. Euler numbers on G/P and some examples.

In this section we collect some simple observations concerning the Euler numbers which enter into the integral formula.

We now denote the fixed Borel subgroup of G_o by B and realize B as $G_o/B \times G_o/B$. The orbits of the diagonal $K \subset G_o \times G_o$ on $G_o/B \times G_o/B$ are in obvious closure-preserving one-to-one correspondence with the orbits of B on G_o/B (Schubert cells): $K \cdot (wB,\, 1B) \leftrightarrow B \cdot wB$, $w \in W_o$. The Euler number at $(yB,\, 1B)$ on the closure of $K \cdot (wB,\, 1B)$ equals the Euler number at yB on the closure of $B \cdot wB$ (Schubert variety). We agree that "Euler number on …" means "Euler number on the closure of …", as before.

Notation:

$P \supset B =$ a parabolic subgroup of G_o,
$\Delta^+ = \{$roots of h_o in b $\}$.
${\Delta_P}^+ = \{\alpha \in \Delta^+ : -\alpha$ a root in $p\}$ =positive roots for the reductive part of P.
$W_P =$ the Weyl group of the reductive part of P.
$[W_o/W_P] = \{w \in W_o : w \cdot {\Delta_P}^+ \subset -\Delta^+\}$, a system of coset representatives for W_o/W_P.
N^- opposite to the unipotent radical N of B.
$N^-(P) = N^- \cap w_P N^- w_P$, where
$w_P \in W_P, w_P \cdot {\Delta_P}^+ \subset -{\Delta_P}^+$.

Then $N^- = N^-(P)(N^- \cap P)$. The following lemma is well known and easy to prove.

4.1 Lemma. *For $w, y \in [W_o/W_P], ByP < BwP$ iff $ByB < BwB$. If so,*

$$\overline{(BwP)} \cap (yN^-P) = \Big(N \cap yN^-(P)y^{-1}\Big) \cdot \Big(\overline{BwB}) \cap (N^-yw \cap ywN^-)\Big) \cdot P.$$

(direct decompositon).

Explanation. $ByP < BwP$ means $ByP \subset \overline{BwP}$ The lemma may be viewed as saying that in the affine neighbourhood $yN^- \cdot P$ of the point yP in G_o/P the image of the second factor provides a cross section to the Schubert cell $B \cdot yP$ (represented by the first factor) in the Schubert variety $\overline{B \cdot wP}$. Such transversals are familiar from the work of Kazhdan-Lusztig [1980].

4.2 Corollary. For $y, w \in [W_o/W_P]$ with $y < w$

$$\mathrm{Eu}_{yP}(\overline{B \cdot wP}) = \mathrm{Eu}_{yB}(\overline{B \cdot wB}) = \mathrm{Eu}_y[(\overline{BwB}) \cap (N^- yw \cap wyN^-)]$$

Proof of Corollary 4.2. It is clear from the lemma that the Euler number of the Schubert variety $\overline{B \cdot wP} \subset \overline{G_o/P}$ at yP is the same as the Euler number of the Bruhat variety $\overline{BwP}$ in G_o at y. Compare:

$$\begin{aligned} &(1) \quad (\overline{BwP}) \cap (yN^-P) = (N \cap yN^-(P)y^{-1}) \cdot [(\overline{BwB}) \cap (N^-y \cap yN^-)] \cdot P \\ &(2) \quad (\overline{BwB}) \cap (yN^-P) = (N \cap yN^-(P)y^{-1}) \cdot [(\overline{BwB}) \cap (N^-y \cap yN^-)] \cdot P. \end{aligned}$$

The first factors on the right, being affine spaces, may be omitted in calculating Euler numbers, hence the corollary.

4.3 Example. We calculate some Euler numbers for G_o of type A in rank < 3. It will be convenient to work with $G_o = GL(n, \mathbf{C})$ rather than with $PGL(n, \mathbf{C})$.

For $n = 2, 3$ the Euler numbers $\mathrm{Eu}_y\ BwB$ are $= 1$ or $= 0$ depending on whether $y \le w$ or not, as the Schubert varieties are all smooth in this case.

For $n = 4$ the Euler numbers are again $= 1$ or $= 0$ as above with the these exceptions:
$w = (3412)$ and $y = 1$, or (2134);
$w = (4231)$ and $y = 1$, (2134), (1243), or (2143)
in which case the Euler numbers $= 2$. The elements w, y of $W_o = S_4$ are here written as permutations of (1234). As sample we take $n = 4, w = (4231), y = (2143), y, w \in W_o = S_4$. That $\mathrm{Eu}_y\ BwB = 2$ may be seen as follows.

In $GL(n,\mathbf{C})$ with $B = \{$upper triangular $\}$the Bruhat variety $\overline{BwB}$ consists of all invertible $n \times n$ matrices $[a_{ij}]$ satisfying for all $1 \le k \le m \le n$:

$$\mathrm{rank}\,[a_{ij} : w_m(k) + 1 \le i \le n, 1 \le j \le m] \le m - k$$

Here $w_m(1) \le w_m(2) \le \ldots \le w_m(m)$ is the increasing rearrangement of the first m terms of the permutation $w = (w(1), w(2), \ldots, w(n))$ in $W_o = S_n$. On the other hand, the affine space $yN^- \cap N^-y$ may be described thus: it consists of all $n \times n$ matrices with entries $= 1$ as in the permutation matrix $y \in W_o = S_m$ (i.e. in the places $ij, i = y(j)$) and 0's above and to the right of these 1's. The remaining entries are arbitrary. If one applies this recipe to the w and y above one finds that in this case $(yN^- \cap N^-y) \cap \overline{BwB}$ consists of matrices of the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ * & * & 0 & 1 \\ * & * & 0 & 0 \end{bmatrix}$$

for which the $2{\times}2$ submatrix of $*$'s has det $= 0$. Thus $V = \{z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}\}$. Its conormal variety is

$$\mathcal{V} = \text{closure of } \{(z, w) \in \mathbf{C}^4 \times \mathbf{C}^4 | 0 \ne z \in V, w \in \mathbf{C}z'\}$$

where

$$z' = \begin{bmatrix} z_{22} & -z_{21} \\ -z_{12} & z_{11} \end{bmatrix}$$

The bi-projective tangent cone of $\mathcal{V}$ at (0,0) is

$$Z = \{([z],[z']) \in \mathbf{PC}^3 \times \mathbf{PC}^3 | z \in C\}$$

where $C \subset \mathbf{PC}^3$ is the projective tangent cone of $\mathcal{V}$ at 0. One finds

$$\mathrm{Eu}_0(V) = (-1)^{d+1} \int_Z (1+\omega)^{-1}(1-\omega')^{-1}$$

Since both ω and ω' pull back to ω under $\mathbf{PC}^3 \to P\mathbf{C}^3 \times P\mathbf{C}^3, [z] \to ([z],[z'])$, we get

$$\mathrm{Eu}_0(V) = \left(\sum_{j=0}^{2} (-1)^j \right) \int_C \omega^2 = \int_C \omega^2 = \deg(C) = 2.$$

5. Appendix: Euler numbers.

Recall the definition of Euler number given at the end of section 1. We may assume we are dealing with a uniformly d-dimensional algebraic subvariety V of $\mathbf{C}^n$, the point in question being the origin. Let $\mathcal{V} \subset \mathbf{C}^n \times (\mathbf{C}^n)^*$ be the conormal variety (= the closure of the conormal bundle of the regular set of) V in $\mathbf{C}^n$. Then

$$\mathrm{Eu}_0(V) = (-1)^{n-d} \lim_{\epsilon \to 0} \tfrac{1}{\epsilon^{2n}} \int_{\mathcal{V} \cap B_\epsilon} [\tfrac{1}{2\pi i}(d\bar{q} \cdot dq - d\bar{p} \cdot dp)]^n \tag{1}$$

where $q = (q_1, \cdots, q_n)$ in $\mathbf{C}^n$, $p = (p_1, \cdots, p_n)$ in $(\mathbf{C}^n)^*$, $d\bar{q} \cdot dq = \sum_j d\bar{q}_j dq_j$ and $B_\epsilon = \{|q|^2 + |p|^2 \leq \epsilon^2\}$.

The above limit in (1) can be written as an integral over the bi-projective tangent cone Z of V at (0,0) as follows. The tangent cone $\mathcal{V}_0$ of $\mathcal{V}$ at (0,0) is a subvariety of $\mathbf{C}^n \times (\mathbf{C}^n)^*$ stable under scalar multiplications in either factor. Z is the corresponding subvariety of $\mathbf{C}^{n-1} \times (\mathbf{PC}^{n-1})^*$ and

$$\mathrm{Eu}_0(V) = (-1)^{d+1} \int_Z (1+\omega)^{-1}(1-\omega')^{-1} \tag{2}$$

where

$$\omega = \tfrac{1}{2\pi i} \bar{\partial}\partial log|q|^2 \qquad \text{and} \qquad \omega' = \tfrac{1}{2\pi i} \bar{\partial}\partial log|p|^2$$

are the Kähler 2-forms of the Fubini-Study metrics on $\mathbf{PC}^n$ and $(\mathbf{PC}^{n-1})^*$.

From [Griffith-Harris 1978, p. 391, Thie 1967] one sees first of all that

$$\begin{aligned} \lim_{\epsilon \to 0} \tfrac{1}{\epsilon^{2n}} \int_{\mathcal{V} \cap B_\epsilon} [\tfrac{1}{2\pi i}(d\bar{q} \cdot dq - d\bar{p} \cdot dp)]^n &= \int_{\mathcal{V}_0 \cap B_1} [\tfrac{1}{2\pi i}(d\bar{q} \cdot dq - d\bar{p} \cdot dp)]^n \\ &= \tfrac{1}{(2\pi i)^n} \sum_{j+k=n} (-1)^k \tfrac{n!}{j!k!} \int_{\mathcal{V}_0 \cap B_1} (d\bar{q} \cdot dq)^j (d\bar{p} \cdot dp)^k \end{aligned}$$

It should be noted that the tangent cone $\mathcal{V}_0$ must here be counted with multiplicity, i.e. considered as algebraic cycle associated to the tangent cone scheme (See [Whitney 1965] or [Mumford 1976] for more details on tangent cones, [Fulton 1984] for algebraic cycles, [Thie 1967] for the reduction of limits of integrals to the tangent cone.)

To evaluate the second integral in (3) we change coordinates as follows. Choose a holomorphic section $u : \mathbf{CP}^{n-1} \to \mathbf{C}^n - \{0\}$ defined on an open subset of $\mathbf{CP}^{n-1}$. Put $q = su$ with $s \in \mathbf{C}^\times$. Then

$$d\bar{q} \cdot dq = |s|^2 d\bar{u} \cdot du + |u|^2 d\bar{s}ds - \bar{s}dsd\bar{u} \cdot u + sd\bar{s}\bar{u} \cdot du.$$

Here $\alpha \cdot \beta = \sum \alpha_j \beta_j$ for vector valued forms $\alpha = (\alpha_1, \cdots), \beta = (\beta_1 . \cdots)$.

In the integral (3) we only need keep those terms in the expansion of $(d\bar{q} \cdot dq)^j$ which involve ds and $d\bar{s}$ only in the factor $d\bar{s}ds$. Indicating other terms by dots we have:

$$\begin{aligned}
(d\bar{q} \cdot dq)^j &= j|s|^{2(j-1)} d\bar{s}ds|u|^2 \left\{ (d\bar{u} \cdot du)^{j-1} - (j-1)(d\bar{u}du)^{j-2} \frac{(d\bar{u} \cdot u)(\bar{u} \cdot du)}{|u|^2} \right\} + \cdots \\
&= j|s|^{2(j-1)} d\bar{s}ds|u|^2 \left\{ d\bar{u} \cdot du - \frac{(d\bar{u} \cdot u)(\bar{u} \cdot du)}{|u|^2} \right\}^{j-1} + \cdots \\
&= j|s|^{2(j-1)} d\bar{s}ds|u|^2 \{|u|^2 \bar{\partial}\partial log|u|^2\}^{j-1} + \cdots \\
&= j|su|^{2j} \frac{d\bar{s}ds}{|s|^2} \{\bar{\partial}\partial log|u|^2\}^{j-1} + \cdots
\end{aligned}$$

Write similarly $p = tv$ and substitute into (3) to find

$$(3) = \sum_{j+k=n} (-1)^k \tfrac{n!}{(j-1)!(k-1)!} \int_{\mathcal{V}_0 \cap \left\{|su|^2 + |tv|^2 < 1\right\}} |su|^{2j} |tv|^{2k} \frac{d\bar{t}dt}{|t|^2} \frac{d\bar{s}ds}{|s|^2} \omega^{j-1} {\omega'}^{k-1}$$

Integrate first over s and t, using the beta-integral

$$B(a, b) = \int_0^1 (1 - r)^{a-1} r^{b-1} dr = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)};$$

the above expression becomes

$$(3) = \sum_{j+k=n} (-1)^k \int_Z \omega^{j-1} {\omega'}^{k-1} = (-1)^{n+1} \int_Z (1 + \omega)^{-1} (1 - \omega')^{-1}$$

This gives the desired expression for the Euler number as an integral over the bi-projective tangent cone Z.

In terms of the Chern classes of the tautological line bundles $\mathcal{O}(1)$ on $\mathbf{PC}^{n-1}$ and $\mathcal{O}'(-1)$ on $(\mathbf{CP}^{n-1})'$ we have

$$\omega = -c_1(\mathcal{O}(-1)) = c_1(\mathcal{O}(-1)^*) = c_1(\mathcal{O}(1)), \omega' = -c_1(\mathcal{O}'(-1)).$$

Write $\mathrm{c} = 1 + c_1$ for the total Chern class. Then (2) becomes

$$\mathrm{Eu}_0(V) = (-1)^{(d+1)} \int_Z c(\mathcal{O}(1))^{-1} c(\mathcal{O}'(-1))^{-1}.$$

This agrees with the formula for $\mathrm{Eu}_0(V)$ given by Sabbah [1985, Lemme (1.2.2)], except that the bi-projective tangent cone Z of $\mathcal{V}$ at (0,0) is there replaced by the fundamental cycle of $\zeta^{-1}(0)$ where ζ is defined as follows.

Write $P\mathcal{V} \subset \mathbf{C}^n \times (\mathbf{CP}^{n-1})^*$ for the projective conormal variety of $V \subset \mathbf{C}^n$ with projection $\tau : P\mathcal{V} \to V$. Then ζ is the blow-up of $P\mathcal{V}$ along $\tau^{-1}(0)$ composed with τ. This means that $\zeta^{-1}(0)$ is the bi-projective variety associated to the normal cone N of the fibre $\mathcal{V}(0)$ over $0 \in V$ in $\mathcal{V}$. On the other hand, Z is associated in the same way to the tangent cone $\mathcal{V}_0$ of $\mathcal{V}$ at (0,0). So to show that our definition agrees with Sabbah's it suffices to show that $N = \mathcal{V}_0$.

Let $I \subset \mathbf{C}[q,p]$ be the ideal of $\mathcal{V} \subset \mathbf{C}^n \times (\mathbf{C}^n)^*$. The ideal of $\mathcal{V}(0)$ in the coordinate ring $\mathbf{C}[q,p]/I$ of V is then $J = (q)/I$ where (q) is the ideal gererated by $q_1, \ldots, q_n$ in $\mathbf{C}[q,p]$. By definition [Fulton 1984, Appendix B.5], the (affine) normal cone N of $\mathcal{V}(0)$ in $\mathcal{V}$ is Spec of the graded ring

$$\begin{aligned} R &= \oplus_{k>0} J^k/J^{k+1} \\ &= \oplus_{k>0} (q)^k/I \cap (q)^k + (q)^{k+1} \\ &= \oplus_{k>0} (q)^k/I_k \end{aligned}$$

where I_k consists of polynomials $f_k(q,p)$, homogeneous of degree k in q, which occur as q-leading terms of polynomials $f(q,p)$ in I :

$$f(q,p) \equiv f_k(q,p) \qquad \mathrm{mod}\, (q)^{k+1}. \tag{4}$$

Hence

$$R = \mathbf{C}[q,p]/I_*$$

where $I_* = \sum I_k$ in $\mathbf{C}[q,p]$ and the grading in R is according to degree in q.

On the other hand, the tangent cone $\mathcal{V}_0$ to $\mathcal{V}$ at (0,0) is Spec of the graded ring

$$R' = \mathbf{C}[q,p]/I^*.$$

where I^* is generated by polynomials $f^k(q,p)$, homogeneous of degree k in (q,p), which occur as (q,p)-leading terms of polynomials $f(q,p)$ in I :

$$f(q,p) = f^k(q,p) \quad \mathrm{mod}\ (q,p)^{k+1}. \tag{5}$$

The grading on R' is by degree in (q,p). Since the ideal I of $\mathcal{V}$ is homogeneous in p, it suffices to take polynomials $f(q,p)$ which are homogeneous in p in (4) and (5), hence $I_* = I^*$, and therefore $N = \mathcal{V}_0$ as schemes.

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