# EQUIVARIANT MULTIPLICITIES ON COMPLEX VARIETIES

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**Abstract**

We define the *equivariant multiplicity* of an isolated fixed-point of the action of an algebraic torus on a complex variety and prove an integral formula for it which generalizes the Lelong-number formula for the classical multiplicity of a singular point on a complex variety. As an application we prove a *Localization Formula* in equivariant cohomology for the fundamental cycles of (possibly singular) compact subvarietes stable under the torus action. In the special case of Schuberty varieties this formula leads to a geometric interpretation of certain polynomials arising from the operators $A_w$ used to describe the homology ring of the flag manifold.

**0. Introduction**. The purpose of this paper is to introduce the concept of *equivariant multiplicity* of a non-degenerate fixed-point of an algebraic torus acting on a complex analytic variety, to prove some its basic properties, and to give some applications. As in the classical case, there is an algebraic and an analytic definition of *multiplicity*; their equivalence is perhaps the main point here.

The concept of *equivariant multiplicity* is not abstruse. In its algebraic form it was introduced by Joseph [9] and, in a special case, gave rise to the *Joseph polynomial*s, of importance for the representation theory semisimple Lie groups. In its analytic form the concept is present, though unrecognized, in the *localization formula* of equivariant cohomology, when this formula is extended to varieties (as will be done here). The case of *Schubert varieties* is particularly noteworthy, because there the equivariant multiplicity sheds some light on a construction of Bernstein-Gelfand-Gelfand [3] through a result of Arabia [1]. These matters will be explained in more detail below.

**1. Multiplicities Of R,H-Modules**. Let $H = (\mathbb{C}^x)^r$ be an algebraic torus acting algebraically on $\mathbb{C}^N$. We may assume that action is of the form

$$h\cdot(z_1, \dots, z_N) = (h^{\alpha_1} z_1, \dots, {}^{\alpha_N} z_N)$$

for certain characters $h \to h^{\alpha_k}$ of H. Generally we write characters $e^\lambda$: $H \to \mathbb{C}^x$ as

$$e^\lambda(h) = h^\lambda = e^{\lambda(x)} \quad \text{if} \quad h = \exp x$$

with $x \in h$, the Lie algebra of H. The $\lambda \in h^*$ we call weights, a term also applied to the $e^\lambda$.

Let $R = R_N$ denote the graded ring of polynomials in $z_1, \dots, z_N$. H acts on R through its action on $z = (z_1, \dots, z_N)$:

$$(h\cdot f)(z) = f(h^{-1}\cdot z) \quad .$$

By an R,H-module we mean a graded R-module M which is also an H-module so that

$$h\cdot(fm) = (h\cdot f)(h\cdot m) \quad .$$

In addition we require:

(i) M is finitely generated as an R-module.

(ii) M is locally finite and holomorphic as H-module
The conditions imply that M has finitely many generators as R-module which may be chosen to be weight vectors for H.

R,H-modules admit a character theory. To explain the construction we remark that the most natural definition of $\mathrm{ch}_M$ as a formal Poincaré series

$$\mathrm{ch}_M = \sum_\lambda (\dim M_\lambda)\, e^\lambda \tag{1.1}$$

may not make sense, since the dimension of the weight spaces $M_\lambda = \{ m \in M \mid h\cdot m = h^\lambda m \}$ may well be infinite. We therefore proceed somewhat indirectly.

**(1.1) Lemma.** *To every R,H-module one can associate a fractional virtual character $ch_M$ of H, uniquely characterized by the following properties.*
*(i) If M is finite dimensional, then $ch_M$ is the usual character of H.*
*(ii) Additive: If $0 \to P \to M \to Q \to 0$ is exact, then $ch_M = ch_P + ch_Q$.*
*(iii) Multiplicative: If F is a finite-dimensional H-module, then $ch_{M\otimes F} = ch_M\, ch_F$.*
*Furthermore, $ch_M$ is of the form*

$$ch_M = \frac{f}{D}$$

*where $f = \sum_\lambda c_\lambda e^\lambda$ with $c_\lambda \in \mathbb{Z}$ (finite sum) and $D = \prod_k (1 - e^{-\alpha_k})$.*

**Explanation**. A *fractional virtual character* of H is by definition a quotient of virtual characters
$\sum_\lambda m_\lambda\, e^\lambda$ (finite sum, $m^\lambda \in \mathbb{Z}$) with denominator not identically zero. It may be interpreted as formal object or as (densely defined) function on H. We shall take the latter point of view.
The tensor product $M \otimes F$ (over $\mathbb{C}$) of a finitely generated R-module M and a finite-dimensional H-module F is an R,H-module in an obvious way.

Proof. Assuming a character theory with the stated properties exists, $\mathrm{ch}_M$ may be calculated as follows. As mentioned, one may choose a finite set of generators for M over R consisting of weight vectors for H to obtain a map

$$R \otimes F_1 \to M \to 0 \quad .$$

Applying the same process to its kernel (which is also finitely generated, since R is Noetherian) one constructs a resolution

$$\cdots \to R \otimes F_2 \to R \otimes F_1 \to M \to 0 \quad .$$

By Hilbert's Syzygy Theorem (Zariski-Samuel [13], p.240) , this resolution breaks off at the (N+1)-st step:

$$0 \to R \otimes F_{N+1} \cdots \to R \otimes F_2 \to R \otimes F_1 \to M \to 0.$$

Because of the additive and multiplicative properties on ch,

$$\mathrm{ch}_M = \sum_{k=1}^{N+1} (-1)^k\, \mathrm{ch}_R \cdot \mathrm{ch}_{F_k} \quad .$$

Each $\mathrm{ch}_{F_k}$ is a genuine character. To find the character of R we momentarily identify $R_k = R/(z_{k+1}, \ldots, z_N)$ and consider the exact sequence

$$0 \to z_k R_k \to R_k \to R_{k-1} \to 0 \quad .$$

As R,H-module, $z_k R_k$ is the tensor product of the R-module $R_k$ and the one-dimensional H-module of weight $-\alpha_k$. In view of the properties of ch,

$$\mathrm{ch}_{k-1} = (1 - e^{-\alpha_k})\mathrm{ch}_k$$

where $\mathrm{ch}_k$ is the character of $R_k$. Thus $\mathrm{ch}_0 = \left(\prod_k (1 - e^{-\alpha_k})\right) \mathrm{ch}_N$ , i.e. $\mathrm{ch}_R = \frac{1}{D}$, where $D = \prod_k (1 - e^{-\alpha_k})$. There results the formula

$$\mathrm{ch}_M = \frac{f}{D} \tag{1.2}$$

where $f = \sum (-1)^k\, \mathrm{ch}_{F_k}$ is of the required type.

$\mathrm{ch_M}$ is therefore uniquely determined by the properties (i) $-$ (iii) of the lemma. Changing point of view, one may define $\mathrm{ch_M}$ by (1.2): standard arguments from homological algebra show that ch is well-defined and has the required properties.

The weights of H on an R,H-module M are of the form $\lambda = \lambda_j -$ (a sum of $\alpha_k$'s) where the $\lambda_j$ are the weights of some generators of M as R-module. Assume there are elements $x \in h$ so that $\mathrm{Re}\,\alpha_k(x) > 0$ for all k. The multiplicites $\dim M_\lambda$ are then necessarily finite and for such $x \in h$ the character $\mathrm{ch_M}(\exp x)$ is given by the convergent series

$$\mathrm{ch_M}(\exp x) = \sum_\lambda (\dim M_\lambda)\, e^{\lambda(x)} \qquad .$$

This will be called the convergent case. That case prevails in particular when the action of H on $\mathbb{C}^N$ contains the scalar multiplications, i.e. when h contains an element $x_1$ so that $\alpha_k(x_1) = 1$ for all k. In general this situation may be achieved by replacing H by $H \times \mathbb{C}^x$ where $s \in \mathbb{C}^x$ acts by multiplication on $\mathbb{C}^N$ and by $s^{-k}$ on the k$-$th graded piece of an R,H-module $M = \sum M_k$.

The following lemma is due to Joseph [9] (in the convergent case; the general case is a consequence thereof).

**(1.2) Lemma**. *Let M be an R,H-module, J its annihilator in R, and n the Krull-dimension of R/J. Write*

$$ch_M = \frac{1}{D} \sum_\lambda c_\lambda\, e^\lambda \qquad , \quad (\textit{finite sum, } c_\lambda \in \mathbb{Z}),$$

*and define a homogeneous polynomial $e_M$ of degree $N-n$ on h by*

$$e_M = \frac{1}{(N\text{-}n)!} \sum_\lambda c_\lambda\, \lambda^{N-n} \tag{1.3}$$

*Then*

$$ch_M\,(exp\, x) = \frac{1}{\pi(x)} \Big( e_M(x) + o(|x|^{N-n}) \Big) \tag{1.4}$$

*where $\pi(x) = \prod_k \alpha_k(x)$ .*

**Definition**. **$e_M$ is called the H-equivariant multiplicity of the R,H-module M.**

**Remark**. In the convergent case the polynomial $e_M$ may also be defined by a classical construction of Hilbert and Samuel, as follows. Fix $x \in h$ with $\alpha_k(x) < 0$ for all k (convergent case) and consider

$$\sum_{\lambda(x) \le s} \dim M_\lambda \tag{1.5}$$

as function of s. (The sum is finite because of the condition on x.) Joseph [9] shows that asymptotically as $s \to \infty$ this function is of the form

$$\frac{e_M(x)}{\pi(x)} \frac{s^n}{n!} + o(s^n); \tag{1.6}$$

where $e_M$ is given by (1.3); $\pi$ and n are defined as above. Furthermore, in the convergent case $e_M$ is always non-zero; but in general $e_M$ may be zero.

The classical case of Hilbert and Samuel concerns $H = \mathbb{C}^x$ acting on $\mathbb{C}^N$ by scalar multiplication and $M = R/J$, J a homogeneous ideal. $e_M$ may then be thought of as a number: $e_M(x) = e_M\, x^{N-n}$ ; as number, $e_M$ is the classical multiplicity defined in algebraic geometry: it is the multiplicity of the point 0 on the affine cone in $\mathbb{C}^N$ defined by the homogeneous ideal J or, equivalently, the degree of the corresponding projective variety in $\mathbb{CP}^{N-1}$. (Mumford [10], §6C.) These remarks explain the notation and terminology introduced above.

**Comment**. In contrast to the classical case, the function of s defined by (1.5) is generally not polynomial for large s. (Otherwise $e_M(x)/\pi(x)$ would have to be integral for $\alpha_k(x) \in \mathbb{N}$ (Hartshorne [8], p.49), which is generally not the case.)

**2. Multiplicities On H-Varieties**. Let X be an N-dimensional complex analytic manifold, with a holomorphic action by $H = (\mathbb{C}^{x})^{r}$. Let Z be an H-stable n-dimensional analytic subvariety of X. Let p be a fixed-point of H on Z. One may introduce analytic coordinates $z = (z_1, \ldots, z_N)$ around $p = (0, \ldots, 0)$ on X so that the action of H on X is locally of the form

$$h{\cdot}z = (h^{\alpha_1} z_1, \ldots, h^{\alpha_N} z_N).$$

We shall say that such a coordinate system *linearizes* the H action around p. (It may be constructed using the exponential map of a Kähler metric which is invariant under the compact real form of H.) $\alpha_1, \ldots, \alpha_N$ are the weights of the linear action of H on the tangent space of X at p; if they are all non-zero we say that the fixed-point is *non-degenerate*. In this situation we shall define a notion of equivariant multiplicity of p on Z, related to the classical notion of multiplicity of a point on a complex analytic variety. It will help to first recall the classical notion. We give two (equivalent) definitions of the *classical multiplicity* $e_p$ of p on Z in X .

**Algebraic definition**. *$e_p$ is the multiplicity of the local ring $\mathcal{O}_{Z,p}$ as an $\mathcal{O}_{X,p}$ - module.* (Mumford [10], P.121).

**Analytic definition**. *$e_p$ is given by the formula*

$$e_p = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{2n}} \int_{Z\cap B_\epsilon} \omega^n \qquad (2.1)$$

*where $B_\epsilon = \{ \| z \| < \epsilon\}$ is the $\epsilon$-ball in a coordinate system $z_1, \ldots, z_N$ around $p = (0, \ldots, 0)$ and $\omega$ is the (1, 1) form*

$$\omega = -\frac{1}{2\pi i} \sum_k dz_k\, d\,\overline{z}_k \qquad (2.2)$$

(Griffiths-Harris [6], p. 391).

We now turn to the equivariant case. In the situation described above, let $\mathcal{O}_{X,p}$ denote the local ring of X at p, $\mathcal{M}_{X,p}$ its maximal ideal. The associated graded ring

$$\mathrm{gr}\,\mathcal{O}_{X,p} = \sum_{k=0}^{\infty} \mathcal{M}^{k}_{X,p}/\mathcal{M}^{k+1}_{X,p}$$

may be identified with the graded ring of polynomials in $z_1, \cdots, z_N$ and the graded local ring of Z at p, $\mathrm{gr}\,\mathcal{O}_{Z,p}$, is an $\mathrm{gr}\,\mathcal{O}_{X,p}$,H-module in the sense of section 1. Again we give two definitions of the equivariant multiplicity $e_p$ of p on Z in X.

**Algebraic definition**. *$e_p$ is the multiplicity of $gr\,\mathcal{O}_{Z,p}$ as an $gr\,\mathcal{O}_{X,p}, H$ - module.*

Analytic definition. *$e_p$ is given by the formula*

$$\frac{e_p(x)}{\pi_p(x)} = \frac{1}{\epsilon^{2n}} \int_{Z\cap B_\epsilon} \omega(x)^n \qquad .(2.3)$$

*Here $\pi_p = \prod_k \alpha_k$ as before. $x \in h$ is assumed to satisfy $\alpha_k(x) \neq 0$ for all k. $B_\epsilon = \{ \| z \| < \epsilon\}$ is any sufficiently small $\epsilon$-ball in a coordinate system $z_1, \ldots, z_N$ around $p = (0, \ldots, 0)$ which linearizes the H-action. $\omega(x)$ is the (1,1)-form*

$$\omega(x) = -\frac{1}{2\pi i} \sum_k \frac{1}{\alpha_k(x)}\, dz_k\, d\,\overline{z}_k. \qquad (2.4)$$

*(The products of differential forms are exterior products.) The integral on the right side of (2.3) is independent of $\epsilon$ (as will be shown).*

The equivalence of the definitions will be proved after some remarks.

**Remarks**. (1) In limits of integrals of the type met in the analytic definition of $e_p$, the variety Z may be replaced by its tangent cone C at p, as explained in Griffiths-Harris [6], p.391, and proved in detail in Thie [12]. Because of homogeneity, the integral in formula (2.1) then becomes independent of $\epsilon$, just like the integral in

(2.3). After passing to the tangent cone the formulas (2.1) and (2.3) may therefore be written

$$e_p = \int_{C\cap B_1} \omega^n \quad (2.5)$$

and

$$\frac{e_p(x)}{\pi_p(x)} = \int_{C\cap B_1} \omega(x)^n \; . \quad (2.6)$$

The tangent cone C must then however be counted with the appropriate multiplicity, i.e. the integral over C must be interpreted as an integral over the cycle associated to C as scheme (Fulton [5], p.15; on p.79 Fulton defines $e_p$ by the projective equivalent of (2.5) ).

(2) The equivariant multiplicity reduces to the classical multiplicity in case $H = \mathbb{C}^x$ acts by scalar multiplication in the linearizing coordinates. Starting with any variety and any point thereon, this situation may be achieved by passing to the tangent cone. In this way the classical multiplicity may be considered a special case of the equivariant multiplicity.

(3) $e_p$ depends on the embedding of Z in the manifold X, although the quotient $e_p/\pi_p$ depends only on the action of H on Z. (It is for this reason that " in X" was added to "equivariant multiplicity of p on Z".) Numerator and denominator of $e_p$ could be unambigously normalized by passing to the smallest subspace of the tangent space of X at p which contains the tangent cone of Z. (This is precisely the Zariski-tangent space of Z at p.) The effect of the normalization is to cancel those factors $\alpha_k$ for which $z_k \equiv 0$ on the tangent cone from numerator and denominator of $e_p/\pi_p$ .

**(2.1) Theorem**. *The two definitions of "equivariant multiplicity" are equivalent.*

The proof will consist of a reduction to the classical case. We shall need a lemma.

**(2.2) Lemma**. *Let $x \in h$ with $\alpha_k(x)$ purely imaginary and $\neq 0$ for all k. Denote by $L_x$ (resp. $i(x)$) the Lie derivative (resp. inner multiplication) by the corresponding vector field on X. Let $\theta(x)$ be any $C^\infty$ one-form defined in a neighbourhood of p on X, except at p itself, so that*

$$L_x\theta(x) = 0, \quad \text{and} \quad i(x)\theta(x) \equiv 1 \quad .$$

*Such forms exist, and if B is any sufficiently small neighbourhood of p in X, then*

$$\int_{\partial(Z\cap B)} \theta(x)(d\theta(x))^{n-1} \quad (2.7)$$

*is independent of $\theta$ and B (with the stated properties) and equals*

$$\frac{(2\pi i)^n}{\epsilon^{2n}} \int_{Z\cap B_\epsilon} \omega(x)^n \quad (2.8)$$

*for any sufficiently small $\epsilon$.*

The proof of the lemma is an exercise with the *equivariant Stokes' Theorem.* It will be clearest to explain the procedure in some greater generality. (The method is not new: it originates in a paper of BOTT [4] and was further developed by Berline-Vergne [2] and others.)

Let v be a $C^\infty$ vector field on X. (Here X need only be a real $C^\infty$ manifold.) Introduce the *equivariant exterior derivative* operator $d_v$ (generally inhomogeneous) $C^\infty$ differential forms $\omega$ on X by the formula

$$d_v\omega = d\omega + i(v)\omega \quad .$$

It satisfies $(d_v)^2 = d\circ i(v) + i(v)\circ d = L_v$,the Lie derivative, and $d_v(\alpha\beta) = (d_v\alpha)\,\beta + (-1)^a\alpha\;(d\beta)$ if $\alpha$ is homogeneous of degree a. (All products of forms are exterior products.) The equivariant Stokes's Theorem says : if $\Gamma$ is a piecewise $C^\infty$, finite m-chain on X which is tangential to v, then

$$\int_\Gamma d_v\omega \;=\; \int_{\partial\Gamma}\,\omega$$

for every $C^\infty$ form $\omega$. (Proof: the second summand of $d_v\omega \;= d\omega + i(v)\omega$ vanishes in degree $m = \dim\Gamma$ on $\Gamma$, in view of the "tangential" condition. $-$ The integral of an inhomogeneous form is the integral of its component in the appropriate degree.)

We now turn to the proof of the lemma. Fix $x \in h$ with $\alpha_k(x) \neq\; 0$ for all k. Assume given $\theta \;=\; \theta(x)$ with $L_x\theta \;=\; 0$ and $i(x)\theta \;=\; 1$. The integral (2.7) may be written as

$$\int_{\partial(Z\cap B)} \theta(d\theta)^{n-1} \;= (-1)^{n-1} \int_{\partial(Z\cap B)} \theta(1 + d\theta)^{-1}$$

where the inverse is taken in the exterior algebra:

$$(1 + d\theta)^{-1} \;=\; \sum\,(-1)^k(d\theta)^k \qquad .$$

Observe that $1 + d\theta \;\;= d_x\theta$ and $d_x(\theta(1 + d\theta)^{-1}) \;\;= d_x(\theta(d_x\theta)^{-1}) \;\equiv\; 1$. So $d_x(\theta(1 + d\theta)^{-1})$ is 0 except in degree 0. The independence of B of the integral (2.7) is therefore immediate from the equivariant Stokes' Theorem.

To see the independence of $\theta$, suppose $\theta_1$ and $\theta_2$ are two forms with the required properties. Choose coordinate balls $B_1 \subset B_2$. Construct a third such form $\theta$ so that

$$\theta = \begin{cases} \theta_1 & \text{on } \partial B_1 \\ \theta_2 & \text{on } \partial B_2 \end{cases}$$

This is possible: Since $\alpha_k(x)$ is imaginary for all k, the real one-parameter group $\exp(\mathbb{R}x)$ generated by x is a circle and the form $\theta$ may be taken as $\theta \;= c_1\theta_1 + c_2\theta_2$ where $c_1$ is an $\exp(\mathbb{R}x)$-invariant $C^\infty$ function which $\equiv\; 1$ on $\partial B_1$ and $\equiv\; 0$ on $\partial B_2$. $c_2$ is defined similarly, and $c_1 + c_2 \;\equiv\; 1$. Because of the independence of B:

$$\int_{\partial(Z\cap B)} \theta(d\theta)^{n-1} \;=\; \int_{\partial(Z\cap B_j)} \theta_j(d\theta_j)^{n-1} \qquad , \quad j = 1,2.$$

It remains to prove the last assertion of the lemma. For that purpose we construct a particular form $\theta$ as follows. Set $a_k \;=\; \alpha_k(x)$. Write $\xi$ for the holomorphic vector field on X corresponding to $x \in h$:

$$\xi \;=\; \sum a_k\, z_k\, \frac{\partial}{\partial z_k}$$

and define $(1,0)$-form $\varphi \;=\; \varphi(x)$ by

$$\varphi \;=\; \sum \frac{1}{a_k}\, \overline{z}_k\, dz_k \qquad .$$

Then $\theta \;=\; \theta(x) = \;\; \varphi/\|z\|^2$ is defined except at p and has the required properties:

$$L_\xi\theta \;=\; 0, \quad i(\xi)\theta \;\equiv 1$$

Observe that $d\varphi \;= 2\pi i\,\omega$ where $\omega \;=\; \omega(x)$ is the form defined earlier. Furthermore, $\theta \;\equiv\; \varphi/\epsilon^2$ on $\partial B_\epsilon$ and $d\theta \;= d\varphi/\epsilon^2 \;=\; 2\pi i\,\omega/\epsilon^2$ there. Thus

$$\begin{aligned} &\int_{\partial(Z\cap B)} \theta(d\theta)^{n-1} \\ &\quad= \frac{1}{\epsilon^{2n}} \int_{\partial(Z\cap B_\epsilon)} \varphi(\,d\varphi)^{n-1} \\ &\quad= \frac{1}{\epsilon^{2n}} \int_{Z\cap B_\epsilon} (\,d\varphi)^n \\ &\quad= \frac{(2\pi i)^n}{\epsilon^{2n}} \int_{Z\cap B_\epsilon} \omega^n \;. \end{aligned}$$

This finishes the proof of the lemma.

We now turn to the proof of the theorem. To prove the equivalence of the the two definitions we may replace the variety Z by its tangent cone at p: For the algebraic definition this is evident because the tangent cone is exactly spec of the graded ring

$$\mathrm{gr}\,\mathcal{O}_{Z,p} \;=\; \sum_{k=0}^{\infty} \mathcal{M}_{Z,p}^k/\mathcal{M}_{Z,p}^{k+1} \tag{2.9}$$

where $\mathcal{M}_{Z,p} \;=\; \mathcal{M}_{X,p}\mathcal{O}_{Z,p}$ is the maximal ideal of the local ring $\mathcal{O}_{Z,p}$ (Mumford [11], p.302 or Fulton [5], p.435). For the analytic definition the corresponding passage to the tangent cone was already mentioned.

We shall therefore now assume that Z, as scheme, is a cone in $X = \mathbb{C}^N$ defined locally at $p = 0$ by a homogeneous ideal J of $R = \mathrm{gr}\,\mathcal{O}_{X,p}$. To prove the equivalence of the algebraic and analytic definitions we have to show that

$$\int_{[Z\cap B1]} \omega(x)^n = \frac{e(x)}{\pi(x)} . \tag{2.10}$$

e(x) is the multiplicity of the $\mathcal{O}_N$,H - module (2.9); $[Z\cap B_1]$ cycle associated to $Z\cap B_1$ (Fulton [5]). $\omega(x)$ is the form:

$$\omega(x) = -\frac{1}{2\pi i} \sum_k \frac{1}{\alpha_k(x)} \, dz_k \, d\,\overline{z}_k. \tag{2.4}$$

Both sides of (2.10) are rational in x; it therefore suffices to prove (2.10) for x in a Zariski dense subset of h.

Since the cone Z is invariant under the action of $\mathbb{C}^x$, we may now assume that the action of H on $X = \mathbb{C}^N$ contains the scalar multiplications: otherwise we replace H by $H \times \mathbb{C}^x$. Denote by $x_1$ an element of h which generates the scalar multiplications: $\alpha_k(x_1) = 1$ for $k = 1, 2, \ldots, N$. The elements $x \in h$ satisfying

$$\alpha_k(x) \in \mathbb{N} \quad \text{for } k = 1, 2, \ldots, N \tag{2.11}$$

are now Zariski-dense in h: If a polynomial vanishes on all of these points, then it vanishes at $x + sx_1$ whenever $\alpha_k(x) \in \mathbb{Z}$ for all k and $s \in \mathbb{N}$ is sufficiently large. Hence it vanishes identically on $x + sx_1$, $s \in \mathbb{C}$, hence at all such x, hence identically.
Fix $x \in h$ satisfying (2.11) and set $\alpha_k(x) = a_k$, $a_k \in \mathbb{N}$. Introduce new variables $w_j$ and define a map f: $w \to z$ by setting

$$z_k = w_k^{a_k}. \tag{2.12}$$

The map f: $\mathbb{C}^N \to \mathbb{C}^N$ is finite of degree $p = \Pi_k a_k$ ( $= \pi(x)$ for the fixed x). Under this map the action of the $\mathbb{C}^x$ on the z through the one-parameter group generated by $2\pi ix$,

$$e^{2\pi it}\cdot z = \exp(2\pi itx)z = (e^{2\pi i a_1 t} z_1, \ \ldots, e^{2\pi i a_N t} z_N),$$

corresponds equivariantly to the action of $\mathbb{C}^x$ on the w by scalar multiplication.
Choose a form $\theta = \theta(2\pi ix)$ as in the lemma for the action of $\mathbb{C}^x$ on the z and set $e = e(x)$, $p = \pi(x)$. After cancelling a factor $(1/2\pi i)^n$, the equation (2.10) to be proved becomes

$$\int_{\partial[Z\cap B]} \theta(d\theta)^{n-1} = \frac{e}{p} . \tag{2.13}$$

Let $\widetilde{\theta} = f^*\theta$ be the pull-back of $\theta$. $\widetilde{\theta}$ is then a form of the type required by the lemma for the multiplication action of $\mathbb{C}^x$ on the w. Let B be a sufficiently small neighbourhood of $z = 0$, $\widetilde{B}$ its inverse image. Let $\widetilde{Z} = f^{-1}Z$ denote the inverse image of Z under (2.13) as scheme (Fulton [5] ).

The ring $\mathbb{C}[w]$ is free over $f^*\mathbb{C}[z] = \mathbb{C}[w^a]$ (with basis consisting of the monomials $w_1^{m_1} \ldots w_N^{m_N}$ with $m_1 < a_1, \ldots, m_N < a_N$ ). The cycle $[\widetilde{Z}\cap\widetilde{B}] = [f^{-1}(Z\cap B)]$ is therefore the flat pull back $[\widetilde{Z}\cap\widetilde{B}] = f^*[Z\cap B]$ and

$$f_*f^*[Z\cap B] = (\deg f)\,[Z\cap B] = p\,[Z\cap B]$$

(Fulton [5], Lemma 1.7.1, p.18 and Proposition 8.3 (c), p.140.) Thus

$$\begin{aligned}
&\int_{\partial[Z\cap B]} \theta(d\theta)^{n-1} \\
&\quad = \frac{1}{p} \int_{\partial f_* f^*[Z\cap B]} \theta(d\theta)^{n-1} \\
&\quad = \frac{1}{p} \int_{\partial[\widetilde{Z}\cap\widetilde{B}]} f^*(\theta(d\theta)^{n-1}) \\
&\quad = \frac{1}{p} \int_{\partial[\widetilde{Z}\cap\widetilde{B}]} \widetilde{\theta}(d\widetilde{\theta})^{n-1}
\end{aligned} \tag{2.14}$$

The lemma now applies to both sides of this equation and gives

$$\begin{aligned}\int_{[Z\cap B_1]} \omega(x)^n &= \int_{\partial[Z\cap B]} \theta(d\theta)^{n-1} \\ &= \frac{1}{p}\int_{\partial[\widetilde{Z}\cap\widetilde{B}]} \widetilde{\theta}(d\widetilde{\theta})^{n-1} \\ &= \frac{1}{p}\int_{[\widetilde{Z}\cap\widetilde{B}_1]} \widetilde{\omega}(x)^n \end{aligned} \tag{2.15}$$

where $\widetilde{B}_1$ is the ball $\{\|w\| \leq 1\}$ and $\widetilde{\omega}$ the form (2.4) corresponding to the multiplication action of $\mathbb{C}^{\times}$ on the w. Thus

$$\widetilde{\omega} = -\frac{1}{2\pi i}\sum_k dw_k\, d\overline{w}_k$$

is the form entering into the analytic definition of the classical multiplicity, and we get

$$\int_{[Z\cap B_1]} \omega(x)^n = \widetilde{e}, \tag{2.16}$$

the classical multiplicity of 0 on the cone $\widetilde{Z}$. It remains to calculate $\widetilde{e}$.

Let J be the ideal in R = {polynomials in z} which defines Z locally at 0, $\widetilde{J} = J\widetilde{R}$ the ideal it generates $\widetilde{R}$ = {polynomials in w}. (R is considered a subring of $\widetilde{R}$ via $z = w^a$). $\widetilde{J}$ is the ideal of definition of $\widetilde{Z}$, and $\widetilde{e}$ is the (classical = equivariant) multiplicity of $\widetilde{R}/\widetilde{J}$. $\widetilde{J}$ is a direct sum

$$\widetilde{J} = \sum_{m_j<a_j} J\, w_1^{m_1}\dots w_N^{m_N}$$

Thus the $\widetilde{R},\mathbb{C}^{x}$-module character ( = Poincaré series) of $\widetilde{J}$ is

$$ch_{\widetilde{J}}(h) = ch_J(h)\sum_{m_j<a_j} h^{m_1+\cdots+m_N} = ch_{\widetilde{J}}(h)\,\frac{\Pi_k(1-h^{a_k})}{(1-h)^N} .$$

Therefore

$$ch_{\widetilde{R}/\widetilde{J}}(h) = ch_{\widetilde{R}}(h) - ch_{\widetilde{J}}(h) = \frac{1-\; ch_{\widetilde{J}}(h)\,\Pi_k(1-h^{a_k})}{(1-h)^N} . \tag{2.17}$$

Write $R,\mathbb{C}^{x}$-module characters in the form $ch_M = f/D$ as in Lemma (1.2), and $\widetilde{R},\mathbb{C}^{x}$-module characters similarly as $ch_{\widetilde{M}} = \widetilde{f}/\widetilde{D}$. Here $D(h) = \Pi_k(1-h^{a_k})$ and $\widetilde{D}(h) = (1-h)^N$. One finds from (2.17)

$$ch_{\widetilde{R}/\widetilde{J}}(h) = \frac{1-f_J(h)}{\widetilde{D}(h)} = \frac{f_{R/J}(h)}{\widetilde{D}(h)}.$$

Hence $f_{\widetilde{R}/\widetilde{J}} = f_{R/J}$ . From Lemma 2 of §1 one finds that the $\widetilde{R},\mathbb{C}^{x}$-multiplicity $\widetilde{e}$ of $\widetilde{R}/\widetilde{J}$ equals the $R,\mathbb{C}^{\times}$-multiplicity e of R/J:

$$e = \widetilde{e} \tag{2.18}$$

From (2.14) − (2.18) follows the desired formula (2.11).
This proves the equivalence of the algebraic and analytic definitions of $e_p(x)$ and completes the proof of the theorem.

**3. The Localization Formula.** The Localization Formula of equivariant cohomology may be stated as follows (Berline-Vergne [2]).

Let T be a real torus acting on a compact, oriented manifold M of dimension 2n. Assume all fixed-points of T are non-degenerate. Let $x \in t$ be a regular element and $\mu(x)$ a $C^{\infty}$ form on M satisfying $d_x\mu(x) = 0$. Then

$$\left(\frac{-1}{2\pi \mathrm{i}}\right)^{\mathrm{n}} \int_{\mathrm{M}} \mu(\mathrm{x}) = \sum_{\mathrm{p}} \frac{1}{\pi_{\mathrm{p}}(\mathrm{x})} \mu_{\mathrm{p}}(\mathrm{x}), \tag{3.1}$$

sum over all fixed-points p of T.

Explanation. Around a fixed-point p of T one may introduce positively oriented coordinates $x_1, \ldots, x_{2n}$ around p on M so that $\exp x \in T$ acts by the rotation

$$\begin{bmatrix} \cos\theta_k(x) & -\sin\theta_k(x) \\ \sin\theta_k(x) & \cos\theta_k(x) \end{bmatrix}$$

in the $x_k, x_{n+k}$ – plane. The $\alpha_k = i\theta_k$ are the weights of T on the tangent space at p, and the fixed point p is non-degenerate if these weights $\alpha_k$ are all non-zero (for every fixed-point p). An element $x \in t$ is regular if $\alpha_k(x) \neq 0$ for every $\alpha_k$. $d_x = d + i(x)$ is the equivariant exterior derivative, as explained in connection with the Lemma (2.2). $\mu_p(x)$ is the value at p of the degree-zero component of the (inhomogeneous) form $\mu(x)$.

Remark. The formula concerns only one vector field at a time; T can therefore be replaced by any compact Lie group, since any one-parameter subgroup is then contained in a torus.

We shall prove an analogous localization formula when the smooth manifold M is replaced by a possibly singular complex variety. For this purpose we have to consider integrals over chains Γ, namely the chains [Z∩B] cut out from the fundamental cycle of a complex variety by a coordinate ball B. Such chains satisfy the following regularity condition. For every $C^\infty$ form $\psi$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^k} \int_{\Gamma\cap B_\epsilon} \psi = 0 \quad \text{for } k < \dim\Gamma. \tag{3.2}$$

Here $B_\epsilon$ denotes the $\epsilon$-ball in an arbitrarily chosen coordinate system about an arbitrarily chosen point of X. The intersection $\Gamma\cap B_\epsilon$ may be defined using a subdivision of Γ. That the condition (3.2) is satisfied for the fundamental cycles of complex analytic varieties is clear from the usual proof of local integrability over such cycles (Griffiths-Harris [6], p.32). (It is in fact more generally satisfied when Γ is a subanalytic chain on a real analytic manifold, as one can see in Hardt [7])

**(3.1) Localization Formula**. *Let H be a complex torus acting holomorphically on a complex manifold X, Z a compact subvariety of X of dimension n. Assume all fixed-points of H are non-degenerate. Let $x \in h$ be a regular element and $\mu(x)$ a $C^\infty$ form on X depending holomorphically on $x \in h$ and satisfying $d_x\mu(x) = 0$. Then*

$$\left(\frac{-1}{2\pi i}\right)^{n} \int_{Z} \mu(x) = \sum_{p} \frac{e_p(x)}{\pi_p(x)} \mu_p(x), \tag{3.3}$$

*sum over all fixed-points p of H on Z. $e_p$ is the equivariant multiplicity of p and $\pi_p = \Pi_k\alpha_k$ the product of the weights of H on the tangent space of X at p.*

Proof. The method (which goes back to Bott [4]) is the same as for the formula (3.1). We give the argument here in order to indicate how the regularity property (3.3) is used and how $e_p(x)$ comes in.

Let $\mu(x)$ be a form on X of the prescribed kind. It is enough to prove the formula (3.3) when $\alpha_k(x)$ is imaginary and non-zero for all weights $\alpha_k$ at all fixed-points p. Fix such an $x \in h$. Around each fixed point p one can then find a $C^\infty$ one-form $\theta = \theta(x)$ which has the properties of the Lemma (2.2) : $L_x\theta = 0$, $i(x)\theta = 1$. These local $\theta$ may be patched together with the help of an $\exp(\mathbb{R}x)$-invariant partition of unity to obtain a globally defined $\theta$ with the same properties. As noted before, $d_x\theta = (1 + d\theta)$ has the exterior inverse

$$(1 + d\theta)^{-1} = \sum (-1)^k (d\theta)^k \quad .$$

Set $\mu = \mu(\mathrm{x})$ for the fixed x. Since $\mathrm{d_x}\mu = 0$ one finds that

$$\mathrm{d_x}(\theta(\mathrm{d_x}\theta)^{-1}\mu) = \mu .$$

The equivariant Stokes' Theorem now gives

$$\begin{aligned}
&\int_{\mathrm{Z}} \mu \\
&\quad = \sum\nolimits_{\mathrm{p}} \lim_{\epsilon \to 0} \int_{\mathrm{Z}-\mathrm{Z}\cap \mathrm{B}_\epsilon(\mathrm{p})} \mathrm{d_x}(\theta(\mathrm{d_x}\theta)^{-1}\mu) \\
&\quad = -\lim_{\epsilon \to 0} \sum\nolimits_{\mathrm{p}} \int_{\partial(\mathrm{Z}\cap \mathrm{B}_\epsilon(\mathrm{p}))} \theta(\mathrm{d_x}\theta)^{-1}\mu \\
&\quad = \sum\nolimits_{\mathrm{p,k}}(-1)^{\mathrm{k}+1} \lim_{\epsilon \to 0} \int_{\partial(\mathrm{Z}\cap \mathrm{B}_\epsilon(\mathrm{p}))} \theta(\mathrm{d}\theta)^{\mathrm{k}}\mu \, .
\end{aligned}$$

$\mathrm{B}_\epsilon(\mathrm{p})$ is a coordinate $\epsilon$-ball around p. We take the coordinates $\mathrm{z}_1, \ldots, \mathrm{z}_{\mathrm{N}}$ around p to be linearizing and we assume that there the form $\theta$ is the one constructed in the proof of the Lemma (2.2): $\theta = \varphi/\|\mathrm{z}\|^2$ where

$$\varphi = \sum \frac{1}{\mathrm{a_k}} \overline{\mathrm{z}}_{\mathrm{k}} \, \mathrm{dz_k} .$$

Then the above integral becomes

$$\begin{aligned}
\int_{\mathrm{Z}} \mu &= \sum\nolimits_{\mathrm{p,k}} (-1)^{\mathrm{k}+1} \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2(\mathrm{k}+1)}} \int_{\partial(\mathrm{Z}\cap \mathrm{B}_\epsilon(\mathrm{p}))} \varphi(\mathrm{d}\varphi)^{\mathrm{k}}\mu \\
&= \sum\nolimits_{\mathrm{p,k}} (-1)^{\mathrm{k}+1} \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2(\mathrm{k}+1)}} \int_{\mathrm{Z}\cap \mathrm{B}_\epsilon(\mathrm{p})} \mathrm{d}\Big(\varphi(\mathrm{d}\varphi)^{\mathrm{k}}\mu\Big) \\
&= \sum\nolimits_{\mathrm{p}} (-1)^{\mathrm{k}+1} \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2\mathrm{n}}} \int_{\mathrm{Z}\cap \mathrm{B}_\epsilon(\mathrm{p})} \mathrm{d}\Big(\varphi(\mathrm{d}\varphi)^{\mathrm{n}-1}\mu\Big)
\end{aligned}$$

because of the regularity property (3.2). The only component of $\mu$ which contributes to the last integral is the component in degree zero; in the limit, it may be evaluated at p and taken out from the integral. This gives

$$\int_{\mathrm{Z}} \mu = \sum\nolimits_{\mathrm{p}}(-1)^{\mathrm{n}} \mu_{\mathrm{p}} \lim_{\epsilon\to 0} \frac{1}{\epsilon^{2\mathrm{n}}} \int_{\mathrm{Z}\cap \mathrm{B}_\epsilon(\mathrm{p})} (\mathrm{d}\varphi)^{\mathrm{n}} .$$

In the notation of §2, $\mathrm{d}\varphi = (1/2\pi \mathrm{i})\,\omega$. Thus the last equation is exactly the desired formula (3.3)

**(3.2) Example: Schubert varieties**. Let $\mathrm{X} = \mathrm{G/B}$ be the flag manifold of a semisimple complex algebraic group G. Let H be a Cartan subgroup of G contained in the Borel subgroup B. For each element $\mathrm{w} \in \mathrm{W}$ (the Weyl group of G, H) denote by $\mathrm{Z_w}$ corresponding Schubert variety. Let $\mu(\mathrm{x})$ be a form on X as in the Localization Formula: $\mu(\mathrm{x})$ depends holomorphically on $\mathrm{x} \in \mathrm{H}$ and satisfies $\mathrm{d_x}\mu(\mathrm{x}) = 0$. Assume in addition that $\mu(\mathrm{x})$ is invariant under the action of W on X and h:

$$\mathrm{w}\cdot\mu(\mathrm{x}) = \mu(\mathrm{w}\cdot\mathrm{x}).$$

(As usual, the action of W on X depends on the choice of a compact form K of G: $\mathrm{w}\cdot(\mathrm{kB}) = \mathrm{kwB}$ for $\mathrm{k} \in \mathrm{K}$ and $\mathrm{w} \in \mathrm{W}$.)
In this situation there is an explicit formula for the integral of $\mu(\mathrm{x})$ over $\mathrm{Z_w}$, due to Arabia [1]:

$$\left(\frac{-1}{2\pi \mathrm{i}}\right)^{\mathrm{n}} \int_{\mathrm{Z_w}} \mu(\mathrm{x}) = \mathrm{A_w}\,\mu_{\mathrm{y}}(\mathrm{x}) \tag{3.4}$$

where $\mathrm{n} = \mathrm{l(w)} = \dim \mathrm{Z_w}$.

**Explanation**. $\mathrm{A_w}$ is the operator on holomorphic functions on h introduced by Bernstein-Gelfand-Gelfand [3]: for a reflection $\mathrm{s}_\alpha$ in a simple root $\alpha$, $\mathrm{A}_{\mathrm{s}_\alpha} = \mathrm{A}_\alpha$ is defined by

$$\mathrm{A}_\alpha = \frac{1}{\alpha}\Big(\mathrm{s}_\alpha - 1\Big) . \tag{3.5}$$

(Weyl group elements are here considered as operators on functions on h: $\mathrm{w}\cdot\mathrm{f(x)} = \mathrm{f}(\mathrm{w}^{-1}\mathrm{x})$.) For general $\mathrm{w} \in \mathrm{W}$,

$$\mathrm{A_w} = \mathrm{A}_{\alpha_1} \cdots \mathrm{A}_{\alpha_n} \tag{3.6}$$

where $\mathrm{w} = \mathrm{s}_{\alpha_1} \cdots \mathrm{s}_{\alpha_n}$ is any reduced expression for w as a product of simple reflections. $\mu_{\mathrm{y}}(\mathrm{x})$ is the component of degree zero of the form $\mu(\mathrm{x})$ at the point $\mathrm{p_y} = \mathrm{yB}$ of $\mathrm{X = G/B}$. Written out explicitly, the formula (3.4) reads

$$\left(\frac{-1}{2\pi \mathrm{i}}\right)^{\mathrm{n}} \int_{\mathrm{Z_w}} \mu(\mathrm{x}) = \sum\nolimits_{\mathrm{y}} \mathrm{q_{w,y}(x)}\ \mu_{\mathrm{y}}(\mathrm{x}) \tag{3.7}$$

where

$$\mathrm{q_{w,y}} = \sum\nolimits_{\mathrm{s_1 \cdots s_n}} \frac{1}{\Pi^{\mathrm{n}}_{\mathrm{k=1}} \mathrm{s_1 \cdots s_k}(\alpha_{\mathrm{k}})}, \tag{3.8}$$

sum over all sequences $(\mathrm{s_1}, \cdots, \mathrm{s_n})$ with $\mathrm{s_j} = \mathrm{s}_{\alpha_{\mathrm{j}}}$ or 1 and $\mathrm{s_1} \cdots \mathrm{s_n} = \mathrm{y}$. The corresponding points $\mathrm{p_y} = \mathrm{yB}$ which occur in the sum (3.7) are precisely the fixed-points of H on $\mathrm{Z_w}$. If one compares (3.7) with (3.3) one comes to the conclusion that

$$\mathrm{q_{y,w}} = (-1)^{\mathrm{l(y)}} \frac{\mathrm{e_y}}{\pi} \tag{3.9}$$

where $\mathrm{e_y}$ is the equivariant multiplicity at $\mathrm{p_y}$ and $\pi$ the product of the positive roots. This conclusion presupposes that there are enough form $\mu(\mathrm{x})$ of the required kind so that the rational functions $\mathrm{q_{w,y}}$ on the right side of (3.7) are uniquely determined when the left side is known for all such $\mu(\mathrm{x})$. This is indeed the case: If f(x) is a holomorphic function on h one can construct a form $\mu^{\mathrm{f}}$of the required type by the equivariant Chern-Weil homomorphism of Berline-Vergne [2]) with the property $\mu^{\mathrm{f}}_{\mathrm{y}}(\mathrm{x}) = \mathrm{f}(\mathrm{y}^{-1} \cdot \mathrm{x})$, as explained by Arabia [1].

The formula (3.9) shows that the rational functions $\mathrm{q_{y,w}}$ can be written with denominator $\pi$, a property which can be proved in other ways, but is not evident from their definition (3.8).

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