CONTOUR PATH INTEGRALS: DEFINTIONS AND EXAMPLES

W.ROSSMANN

ABSTRACT. The paper proposes a construction of representations of Lie groups by contour path integrals and discusses some examples of this construction.[1]

…"Is it unitary?"…"I'll explain it to you," Feynman said, "and then you can tell me if it is unitary." He went on and from time to time he thought he could still hear Dirac muttering, "Is it unitary?" [J. Gleick, 1992]

1. Introduction

Some thirty years ago Bott [1965] pointed out that Weyl's Character Formula is closely related to the Atiyah–Singer Index theorem. It is not evident how this relation can be seen directly from the representations themselves, apart from character theory. There is however an intriguing heuristic derivation of the Index Theorem, due to Witten [1982] and discussed by Atiyah [1985], which provides some insight into this relation. It is based on the Feynman's path integral formula, one version of which leads to an expression for the kernels of the representation operators of the type

$$K(\exp X, z'', z') = \int_{z_\cdot} \exp[\int_{z'}^{z''} \{\alpha(\dot{z}_t) - H_X(z_t)\} dt] \Pi \sigma^n(dz_\cdot); \tag{I}$$

the "integral" is over all paths $z_\cdot = \{z_t\}$ from z' to z''. It is not necessary to go into details in order to explain the relation to the Index Theorem: it amounts here to the assertion that the trace of the operator K is given by an analogous finite–dimensional integral

$$\mathrm{tr} K(\exp X) = \int_z \exp[-H_X(z)]\ j^{-1/2}(X) \sigma^n(dz). \tag{II}$$

The form σ in (I) and (II) is a symplectic form on a real manifold, the phase space, from which the representation is constructed by a method of quantization, possibly in cohomology; the function H_X is the Hamiltonian for X, α is a 1–form satisfying $d\alpha = \sigma$, and j is a universal function. The passage from (I) to (II) is accomplished by a formal application of the Localization Formula of Duistermaat–Heckman [1983] and Berline–Vergne [1982], a procedure which has by been justified by Bismut [1984].

All of this is still in the context of representations of compact groups. But a formula of the type (II) also exists for non–compact groups. For those unitary representations which can be constructed by geometric quantization it amounts to

[1] Archiv version of the paper published in the CMS Conference Proceedings v16, 1995, 288-314 with an appendix added January 2004.

a version of Kirillov's Character Formula. For arbitrary (admissible, possibly non–unitary) irreducible representations of semisimple Lie groups there is a formula analogous to (II), but a new phenomenon arises: the integral becomes a contour integral in a complex symplectic manifold, and the real phase space fades into a contour of integration, which has significance only as a homology class [Rossmann, 1990]. The question then arises how to make sense of (I) in this setting, which would seem to require a notion of contour path integrals. I hasten to admit that I do not have a complete answer to this question, which seems to be very difficult, if one is to judge by the literature on "ordinary" path integrals; but I shall try to indicate a preliminary and tentative framework in which contour path integrals of the type (I) might be understood, and to work this out explicitly for some examples. This is all I hope to do here.

2. Some prequantum mechanics

2.1 Hamilton's principle. Consider a point object moving in some space with position q and momentum p. The set of all a priori possible position–momentum pairs $z = (q, p)$ forms what is called the *phase–space* of the dynamical system under consideration. To specify an actual motion, one needs to specify a Hamiltonian $H(z)$ which gives the total energy as a function of position and momentum. *Hamilton's principle* in phase space asserts that an actual trajectory makes stationary the *action*

$$S = \int_{t'}^{t''} pdq + d\phi - Hdt$$

for variations of the path which fix its endpoints z' and z''. The the endpoints cannot both be specified arbitrarily, but the function ϕ is arbitrary: the term $d\phi$ only adds the constant $\phi(z'') - \phi(z')$ to the action S as a function of the paths with the given endpoints. It has been included explicitly to emphasize this arbitrariness in the integrand, which will play an important role later. An important feature of the integrand is that it is linear in the tangent vector of the path, which enters through the 1–form $pdq + d\phi$.

We now take the point of view that phase–space Z is a manifold equipped with a class of locally defined differential 1–forms α, any two of which differ by an exact form $d\phi$ where defined. The differential 2–form $\sigma = d\alpha$ is therefore globally well–defined; it is called the *canonical* 2*–form*, and we shall assume that it is everywhere non–degenerate. This means that (Z, σ) is symplectic manifold. Its dimension is necessarily even, say $2n$. The choice of a local representative α of the class of one–forms is called the choice of a *gauge* and α itself is called a *connection 1–form* for σ.

2.2 The quantum line bundle. Imagine we have attached to every point z of Z a complex line (one–dimensional complex vector space) Φ_z ; these Φ_z's make up what will be called the *quantum line bundle* on (Z, σ). As part of its definition we require that it come with an extra piece of structure, a *connection* with curvature σ. This amounts to having for any choice α of a local gauge a local identification $\iota_z : \mathrm{C} \xrightarrow{\approx} \Phi_z$, determined up to multiplication by a constant independent of z, which satisfies the following condition. Given a path $z_{.} = \{z_t \mid t' \leq t \leq t''\}$ from z' to z''

the map $A[z_.] : \Phi_{z'} \to \Phi_{z''}$ defined by

$$A[z_.] = \iota_{z''} \circ \exp[\int_{z'}^{z''} \alpha] \circ \iota_{z'}^{-1}$$

is independent of the choice of α. If α is replaced by $\alpha' = \alpha + d\phi$, then ι is replaced by $\iota' = \exp(-\phi)\iota$ (or some constant multiple thereof). It will be convenient to introduce the notation

$$A[z_.] = \exp_\iota[\int_{z'}^{z''} \alpha].$$

The identification ι is incorporated as a subscript to exp. Throughout we use the notation $z_.$ for a path $\{z_t\}$.

The map $A[z_.]$ is called the *(covariant) transport* along the path $z_.$ and the path functional $z_. \to A[z_.]$ may be taken to be the connection, although this is an unusual convention. The equation $\sigma = d\alpha$ expresses the condition that σ be the curvature of the connection. A geometric interpretation is provided by Stokes's theorem: the integral of α over the boundary of a parametrized 2–surface in Z equals the integral of $d\alpha = \sigma$ over its interior. For there to exist a line bundle with a connection of curvature σ the exponential integral $\exp[\int \sigma]$ of σ over any two oriented 2–surfaces with the same boundary must therefore be the same. Equivalently, the integral of σ over any closed, oriented 2–surface must be an integral multiple of $2\pi i$. Conversely if σ satisfies this condition, then such a line bundle with connection exists; it is unique up to isomorphism if Z is simply connected [Woodhouse, 1991, §8.3]. Thus a quantum line bundle for (Z, σ) always exists and is unique locally, but not necessarily globally. For this reason, and others, our point of view will be local throughout: in our constructions we may have to replace Z by a sufficiently small open subset.

2.3 Lagrangian fibrations. Let (Z, σ) be a $2n$–dimensional symplectic manifold. The transport within a submanifold of Z on which the canonical 2–form σ vanishes is independent of the path, so that the Φ_z's on it may be identified with each other (locally). Such a submanifold has dimension $\leq n$; if it has dimension$= n$ then it is called Lagrangian. For example, the position fibres $\{q = q_o\}$ and the momentum fibres $\{p = p_o\}$ are Lagrangian for $dp \wedge dq = \sum dp_i \wedge dq_i$.

We define a *Lagrangian fibration* to be any map $\pi : Z \to R$ with a differential of rank $n = \dim R$ everywhere, whose fibres $\pi^{-1}(x)$ are Lagrangian submanifolds. A second Lagrangian fibration $\tilde{\pi} : Z \to \tilde{R}$ is *transverse* to $\pi : Z \to R$ if the map $\tilde{\pi} \times \pi : Z \to \tilde{R} \times R$ is a local diffeomorphism. For example, the position–momentum maps q, p above are transverse Lagrangian fibrations for $\sigma = dp \wedge dq$. The fibres of a Lagrangian fibration $\pi : Z \to R$ are in a natural way affine spaces, at least locally, so that one has the concept of a *straight line* between any two sufficiently close points in a given fibre [Woodhouse, 1991, p.67].

2.4 Action of canonical transformations on Φ. Let H be a smooth function on Z, $X = X_H$ the corresponding *Hamiltonian vector field*: this is by definition the vector field corresponding to the 1–form dH under σ, i.e. $dH(\cdot) = \sigma(X, \cdot)$. Write $\exp(tX)$ for the flow generated by X, i.e. the 1–parameter group of transformations of Z defined (locally in z and t) by

$$\frac{d}{dt}\exp(tX)z = X(\exp(tX)z).$$

The defininig relation $\dot{z}_t = X(z_t)$ of the path $z_t = \exp(tX)z_0$ is is the Euler–Lagrange equation for Hamilton's principle, so that this path is the phase–space trajectory of H through z_0.

The flow of a Hamiltonian vector field consists of *canonical transformations* of (Z, σ), i.e. it leaves the canonical 2–form σ invariant. These transformations of Z extend to transformations of line bundle Φ in the following way. For $z \in Z$ define $A(\exp X,\, z) : \Phi_z \to \Phi_{\exp(X)z}$ by

$$A(\exp X, z) = \exp_\iota \int_z^{\exp(X)z} [\alpha - H dt]$$

where the integral is taken along the trajectory $\exp(tX)z$, $0 \leq t \leq 1$. We note that H is constant along the trajectories of X, because along a trajectory

$$\frac{d}{dt}H = dH(X) = \sigma(X, X) = 0.$$

Thus H can be replaced by the constant $H(z)$ and taken outside of the integral.

By a section of the line bundle Φ we mean a function φ which associates to each $z \in Z$ and element $\varphi(z) \in \Phi_z$. If we choose a gauge α then we can write $\varphi = \iota f$ where f is scalar–valued. The maps $A(\exp X)$ induce an action of $\exp(X)$ on sections of Φ, denoted $\varphi \to A(\exp X)\varphi$ and defined by

$$[A(\exp X)\varphi](z) = A(\exp X,\ \exp(-X)z)\varphi(\exp(-X)z).$$

3. Some quantum mechanics

3.1 Wave functions. Fix a phase space (Z, σ) with a quantum line bundle Φ and a Lagrangian fibration $\pi : Z \to R$. According to the method of geometric quantization [Woodhouse, 1991], the quantum–mechanical state space $H(Z, \sigma, \Phi, \pi)$ of a mechanical system with phase space (Z, σ) consists of sections φ of Φ which are *(covariantly) constant* along the fibres. (This has a meaning, since the Φ_z's along a fibre may be identified.) Such a section φ will be called a wave function. In a gauge given by a connection 1–form α which vanishes on vectors tangential to the fibres a wave function on Z is represented by scalar–valued function f which is constant along the fibres $\pi^{-1}(x)$. Such an α is said to be *adapted* to π. If we chose another gauge $\alpha + d\phi$ then f is replaced by $(\exp\phi)f$, and in particular will generally not be constant along the fibres. Wave functions are usually further required to be square integrable, but we shall ignore this condition; it does not make sense in the present setup.

3.2 Complex phase space and contours. Up to this point the phase space Z was understood to be a real manifold, but it could equally well have been complex. From now on (Z, σ) will be a $2n$–dimensional complex symplectic manifold: a complex manifold Z with a non–degenerate, holomorphic, closed 2–form σ. The connection 1–form α and the Lagrangian fibration π are required to be holomorphic as well.

We shall have to consider what might be called “contour integrals“ in the complex manifold Z. For this purpose we define a contour in Z to be a locally finite $2n$–chain in the sense of integration theory. For example, Γ could be specified by a oriented *real* submanifold of dimension $2n$ in Z. The n–the exterior power σ^n , known as *Liouville form*, is a holomorphic form of degree $2n$, and it makes sense to consider integrals of the type

$$\int_\Gamma f\sigma^n$$

if f is scalar function defined (at least) on Γ for which the integral exists. We shall drop the form σ^n from the notation, if there is no risk of confusion. A contour Γ will now remain fixed, in addition to the data (Z, σ, Φ, π) introduced earlier.

3.3 Pairing and kernels. Let Φ^* be the line bundle dual to Φ, whose fibre Φ^*_z at $z \in Z$ is the one–dimensional space of complex linear functionals on Φ_z. We write $\tilde\varphi(z)\varphi(z)$ for the value of $\tilde\varphi(z) \in \Phi^*_z$ on $\varphi(z) \in \Phi_z$. If $\tilde\varphi$ and φ are sections of Φ^* and Φ, then $\tilde\varphi\varphi$ is a complex scalar function on Z. The line bundle Φ^* on Z carries a natural connection characterized by the property that $\tilde\varphi\varphi$ is invariant under the transport. If α is a connection 1–form for Φ, then $-\alpha$ is a connection 1–form for Φ^*. The corresponding curvature form for Φ^* is therefore $-\sigma$ rather than σ.

We now fix two Lagrangian fibrations, $\pi : Z \to R$ and $\tilde\pi : Z \to \tilde R$. If $\tilde\varphi \in H(\Phi^*, \tilde\pi)$ and $\varphi \in H(\Phi, \pi)$ we define

$$\langle\tilde\varphi, \varphi\rangle = \int_\Gamma \tilde\varphi\varphi.$$

The integral is with respect to a constant multiple σ^n/C of the Liouville form; the normalization factor C, will play a role later. It is understood that the line bundle and the sections φ and $\tilde\varphi$ are defined along the contour Γ. (This brings a global element into our generally local discussion.) Thus $\langle\tilde\varphi, \varphi\rangle$ is a partially defined *pairing* on $H(\Phi^*, \tilde\pi) \times H(\Phi, \pi)$. Since φ is constant along the fibres of π and $\tilde\varphi$ is constant along the fibres of $\tilde\pi$, the function $\varphi\tilde\varphi$ is constant on their intersections. We shall assume that the fibres of π and $\tilde\pi$ intersect locally in a single point, i.e. that *π and $\tilde\pi$ are transverse.*

Given the data $(Z, \sigma, \Phi, \pi)$and $(Z, -\sigma, \Phi^*, \tilde\pi)$, we consider $(Z \times Z,\ \sigma \otimes -\sigma,\ \Phi \otimes \Phi^*,\ \pi \times \tilde\pi)$, which is a datum of the same type. An element $F(z'', z')$ of $\Phi_{z''} \otimes \Phi^*_{z'}$ can be considered as a linear transformation $\Phi_{z'} \to \Phi_{z''}$ written $\varphi(z') \to F(z'', z')\varphi(z')$. An element F of $H(\Phi^* \otimes \Phi, \tilde\pi \times \pi)$ gives a partially defined operator on $H(\Phi, \pi)$, defined by the formula

$$F\varphi(z'') = \int_{z'\in\Gamma} F(z'', z')\varphi(z')$$

whenever the integral exists. Thus $F\varphi(z'')$ is the pairing of $\varphi(z')$ with $\tilde\varphi(z') = F(z'', z')$ for fixed z''. The section $F(z'', z')$ will be called the *kernel* of the operator $F\varphi$. We have the *composition rule*

$$F'' \circ F'(z'', z') = \int_{z\in\Gamma} F''(z'', z)F'(z, z')$$

with the usual proviso concerning the existence of the integral.

4. Path integral operators

It will be clearest to divide the definition of path integral operators into two parts: a first part, which amounts to an unusual notation for matrix products, and a second part, which makes use of phase space with its additional data.

4.1 Matrix products. Path integrals arise from the formula for the N–th composition power $K^N(z'',z')$ of a kernel $K_\epsilon(z'',z')$:

$$K^N(z'',z') = \int \prod_{k=1}^{N} K_\epsilon(z_{k-1},z_k).$$

The integral goes over $(N-1)$–tuples $(z_1,\cdots,z_{N-1})$ from some set Γ and we set $z_0=z', z_N=z''$. The nature of the set Γ, of the kernels K, and of the integral is not important at this stage. The kernel K_ϵ is required to be of the form

$$K_\epsilon(z'',z') = \exp\, S_\epsilon(z'',z')$$

for some function $S_\epsilon(z'',z')$, which is formally written as an integral

$$S_\epsilon(z'',z') = \int_{z'}^{z''} L.$$

One can imagine that this might be an integral along some specified *elementary path* from z' to z'' and take the up to now purely symbolic subscript ϵ to stand for the duration (parameter interval) of the path. The formula for K^N then becomes

$$K^N(z'',z') = \int \exp[\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} L].$$

If one could take the limit as $N\to\infty$ this could be conceived of as the integral over all paths $z_.$ which can be approximated by chains of elementary paths (for $\epsilon = 1/N$) and written as a *path integral*

$$K(z'',z') = \int_{z_.} \exp[\int_{z'}^{z''} L].$$

Physicists call the kernel $K(z'',z')$ the *propagator* generated by the *approximate* (*short–time*) *propagator* $K_\epsilon(z'',z')$. It is important to keep in mind that the approximate propagator is not uniquely determined by K.

As first step toward converting this scheme into something involving true paths and integrals, we will need a set Z on which there is defined an integral over certain subsets Γ so that the composition formula for kernels makes sense. The kernels $F(z'',z')$ we take momentarily to be scalar valued. We fix a functional on these paths, written with slight modification of previous notation as

$$S[z.] = \int_{z't'}^{z''t''} L[z.].$$

Finally we assume that there has been specified a class of *elementary paths* in Z with the following property. Given any interval $t'\le t\le t''$ and any two points z',z'' in Z there is a unique elementary path $\{z_t \mid t'\le t\le t''\}$ from $z't'$ to $z''t''$.

Given an integration domain Γ, two points z', z'' on Γ, and an interval $[t', t'']$, we consider

$$F^N(z''t'', z't') = \int_{z_k \in \Gamma} \exp[\sum_{k=1}^{N} \int_{z_{k-1}t_{k-1}}^{z_k t_k} L[z_.]]$$

where the outer integral is over all $(N-1)$–tuples $(z_1, \cdots, z_{N-1})$ from Γ. In the exponent we have set $z_0 = z', z_N = z''$ and take for $\{t_k\}$ the partition of $[t', t'']$ with N equal steps of size $\epsilon = 1/N$. The integrals in the exponents are over the elementary path in the sense just explained.

It is clear that F^N is the composite of the kernels F_k associated to the N elementary paths, taken in the order of increasing t. If the path functional $\int L[z_.]$ is time translation invariant, i.e. depends only on $t'' - t'$, then all F_k are equal and F^N is the N–th composition power of this elementary kernel F. If F^N has a limit as $N \to \infty$, we call it a *path integral* and denote it by the symbol

$$F(z''t'', z't') = \int_{z_. \in \Gamma} \exp[\int_{z't'}^{z''t''} L[z_.]]. \tag{3.1}$$

The notation is suggestive, but can be misleading: the endpoints z_{k-1} and z_k of the elementary paths entering into the limit can be arbitrarily far apart, so that the "approximation" of paths from z' to z'' by piecewise elementary paths does not provide a Riemann sum approximation of the integral in the exponent.

If one uses subdivisions of the time interval $t' \le t \le t''$ which contain a given $t^* = t_{k^*}$, then the multiple integral over the z_k can be written as an integral over z_{k^*} of a product of two integrals, one over the z_k for $t' \le t_k \le t^*$ the other over the z_k for $t^* \le t_k \le t''$. This gives the *composition formula* for path integrals:

$$F(z''t'', z't') = \int_{z^* \in \Gamma} F(z''t'', z^*t^*) F(z^*t^*, z't').$$

If F depends only on $s = t'' - t'$, then one can write $F_s(z'', z')$ instead of $F(z''t'', z't)$ and finds the semigroup property

$$F_{s'+s''} = F_{s'} \circ F_{s''}.$$

4.2 Contour path integrals. We now return to the complex symplectic manifold (Z, σ), equipped with the additional data Φ, π, $\tilde{\pi}$, and Γ. We choose a gauge α and consider a path functional of the form

$$S_\alpha[z_.] = \int_{z't'}^{z''t''} \{\alpha(\dot{z}_t) - H(z_t) - \tau(\dot{z}_t, \dot{z}_t)\} dt$$

where $H(z)$ is defined and holomorphic in a neighbourhood of Γ and τ is a complex holomorphic quadratic form on the tangent spaces of Z. The integrand is therefore an inhomogeneous quadratic function of the velocity. It is the *Lagrangian* of the *action* functional S_α. The classical phase–space action functional of 2.1 is of this form, but with $\tau = 0$. It will arise as a limit as $\tau \to 0$ (in a sense which will be specified) of quadratic action functionals of the above type.

We now assume fixed such an action functional S_α. The map

$$\exp_\iota S_\alpha[z_.] \; : \Psi_{z'} \to \Psi_{z''}$$

is independent of the choice of the gauge α. The above construction gives a *path integral*

$$\int_{z_.\in\Gamma} \exp_\iota[\int_{z't'}^{z''t''} \{\alpha(\dot z_t) - H(z_t) - \tau(\dot z_t, \dot z_t)\}dt]\ \Pi C^{-1}\sigma^n(dz_.)$$

as a limit of integrals over N–step piecewise elementary paths

$$z't' \to z_1t_1 \to \cdots \to z_{N-1}t_{N-1} \to z''t''$$

with $z_k \in \Gamma$. The approximating paths need not lie on Γ, only the step points z_k do. The symbolic product $\Pi C^{-1}\sigma^n(dz_t)$ is a reminder of the Liouville forms in the approximating integrals, and will again be omitted. The path integral itself is then a map $\Phi_{z'} \to \Phi_{z''}$, independent of the choice of α. There is no guarantee that the limit exists and there is furthermore a question of the nature of these limits: the kernels $K(z'', z')$ should define operators on a space wave functions by the formula (3.1), and the limits should exist in this sense, which depends on a more precise specification of the space of wave functions, e.g by growth conditions along Γ. We shall ingore this point for now, but come back to it later in the examples.

To complete the definition it remains to specify the class of elementary paths, which is an essential part of their definition. We shall consider two possibilities.

5. First definition of contour path integrals

5.1 Holomorphic trajectories. We denote by T the tangent space of Z at a general point $z \in Z$, a complex vector space equipped with the complex symplectic form σ, and by $V, \tilde V$ the complex subspaces of T tangential to the fibres of $\pi, \tilde\pi$ through z. The form σ vanishes on V and $\tilde V$ and $T = V \oplus \tilde V$. Define a symmetric bilinear form τ on T by the requirement that

$$\tau(\bar v, \bar v) = \sigma(v, \tilde v)$$

if $\bar v = v + \tilde v$ with $v \in V$ and $\tilde v \in \tilde V$.

We fix a holomorphic function H on Z, a gauge α, two points z', z'' in Z, and two points t', t'' in the complex t–plane. Let $\{z_t : t \in D\}$ be a *holomorphic arc* in Z, i.e. a holomorphic map $D \to Z$, $t \to z_t$ defined on some simply connected domain D in the complex plane, so that $t', t'' \in D$ and $z_{t'} = z'$, $z_{t''} = z''$. Consider the integral

$$S_\alpha[z_.] = \int_{t'}^{t''} \{\alpha(\dot z_t) - H(z_t) - \frac{1}{2\nu}\tau(\dot z_t,, \dot z_t)\}dt$$

taken along any path C in D; we may consider it as an integral along the image path $\{z_t \mid t \in C\}$ on the holomorphic arc itself. Because of the holomorphicity of the integrand, the integral is independent of the path from t' to t'' and therefore defines a functional on the holomorphic arcs. The map $\exp_\iota S_\alpha[z_.] : \Phi_{z'} \to \Phi_{z''}$ is furthermore independent of the choice of gauge.

The *holomorphic trajectory* for S_α from $z't'$ to $z''t''$ is by definition the critical holomorphic arc for the functional S_α (or equivalently for $\exp_\iota S_\alpha$), i.e. the solution of the Euler–Lagrange equation with boundary values $z't'$, $z''t''$. The existence and uniqueness theorem for second order, holomorphic differential equations, together with the non–degeneracy of the form τ, guarantees the existence and uniqueness of such holomorphic trajectories *locally*. Our constructions are therefore local, as usual.

Adapting the definition of Feynman [1948] to the complex domain, we take as the elementary path $z't' \to z''t''$ entering into the construction of the path integral any path on the holomorphic trajectory of S_α between these points.

5.2 Definition I. $K(\exp X; z'', z') :=$

$$\lim_{\nu\to\infty}\lim_{\epsilon\to 0}\int_{z_k\in\Gamma}\exp_\iota\ [\sum_{k=1}^{N}\int_{z_{k-1}t_{k-1}}^{z_k t_k}\{\alpha(\dot z_t)-H(z_t)-\frac{1}{2\nu}\tau(\dot z_t,\dot z_t)\}dt].$$

We have set $z_0 = z'$ and $z_N = z''$. The outer integral is over $(N-1)$–tuples $(z_1, \cdots, z_{N-1})$ from Γ, taken with respect to a normalized Liouville form σ^n/C. We take $t_k - t_{k-1} = \epsilon$ with $\epsilon = 1/N$ so that $t'' - t' = 1$. The normalization factor C, which may depend on ν, ϵ and z_k, will be specified presently. The integral in the exponent is taken along any path on the holomorphic trajectory from $z_{k-1}t_{k-1}$ to $z_k t_k$. This is written in path integral notation as

(5.1) $$K(\exp X;\, z'', z') = \lim_{\nu\to\infty}\int_{z_\cdot}\exp_\iota\ [\int_{t=0}^{1}\{\alpha(\dot z_t)-H(z_t)-\frac{1}{2\nu}\tau(\dot z_t,\dot z_t)\}dt].$$

Feynman remarks in a footnote [1948, footnote (11)] that for action functionals which are quadratic, but perhaps inhomogeneous, in the velocities $\dot z_t$, such as the the integrand above, one can equally well take as elementary paths the critical paths of the free action functional

$$\int_{t'}^{t''}\tau(\dot z_t,\dot z_t)dt$$

defined by the leading homogeneous form $\tau/2\nu$. The form τ there is positive definite and ν is purely imaginary on Γ. In the case of a real manifold and positive definite form $\tau/2\nu$, these critical paths are the geodesics of τ as Riemann metric. The path integral then has an interpretation in terms of stochastic integrals for the Wiener measure. In that context the definition of path integrals as limits of the type (6.1) is due to Daubechies and Klauder [1985], who evaluated them explicitly in several specific cases.

5.3 The normalization factor. The determination of the normalization factors for path integrals is a delicate matter, in general. If one extrapolates from Feynman's reasoning [1948, equation (28)] one arrives at the Gaussian integral

$$C = \int_{\bar v\in T_\Gamma}\exp[\alpha(\bar v)-\frac{1}{2\nu\epsilon}\tau(\bar v,\bar v)]\sigma^n(d\bar v).$$

The integral is taken over the tangent space T_Γ to Γ at point $z\in\Gamma$ and converges if $\mathrm{Re}(\tau/\nu)$ is positive–definite on T_Γ. It then has the value

(5.2) $$C = e^{\nu\tau^{-1}(\alpha,\alpha)/2}(2\pi\nu\epsilon)^n$$

where τ^{-1} is the inverse of τ, a quadratic form on the dual space of T. In general C may depend on z as well as on ν and ϵ. The formula (6.2) for the nomalization constant is correct for the examples considered by considerd by Daubechies and Klauder [1985] and Klauder [1988].

6. Second definition of contour path integrals

Path integrals according to definition I are hard to evaluate. There is however a recipe for their evaluation, which produces the correct result in a number of special cases. Since I do not have a proof of this recipe (which may in fact apply only under additional hypotheses), I shall simply take it as another definition and I shall only give a heuristic argument for its equivalence with the previous definition. There is no problem as far as the examples to be discusses later are concerned: they can be understood entirely on the basis of the second definition.

6.1 Zig–zag paths. For z', z'' in Z sufficiently close there is a unique point z^* satisfying $\pi(z^*) = \pi(z'')$, $\tilde\pi(z^*) = \tilde\pi(z')$. By the *zig–zag* from z' to z'' we mean the composite path $z' \to z^* \to z''$ consisting of the line segment from z' to z^* in the fibre of $\tilde\pi$ containing these points (an affine space) followed by the line segment from z^* to z'' in the fibre of π.

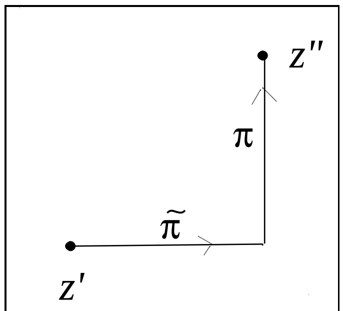

Figure (6.1). A zig-zag

These line segments have natural parametrizations, defined up to an affine change of parameter $t \to at + b$. Up to a time translation, the parametrization is therefore uniquely determined by the time interval, which can be taken to be $0 \le t \le 1$, although this is immaterial here. We shall use these zig–zag paths as elementary paths in the formal construction of the path integral operators outlined above. To complete their definition as operators on wave functions we add some remarks concerning transport along zig–zags.

6.2 Zig–zag rectangles and generating function. Let z', z'' be a pair of points in Z. By a *zig–zag rectangle* $[z'z'']$ we mean any parametrized rectangle $\{z_{t\tilde t}| 0 \le t,\tilde t \le 1\}$ in Z with sides $t = 0$ and1, $\tilde t = 0, 1$ in the fibres of π and $\tilde\pi$ and with z', z'' as two opposite vertices z_{00}, z_{11}. For example, choose curves $\{x_t\}$ from $\pi(z')$ to $\pi(z'')$ and $\{\tilde x_{\tilde t}\}$ from $\tilde\pi(z')$ to $\tilde\pi(z'')$ and take $z_{t\tilde t} \leftrightarrow (x_t, \tilde x_{\tilde t})$. Let

$$A(z'', z') = \exp[\int_{[z'z'']} \sigma].$$

Since σ is closed and vanishes on the fibres of π and $\tilde\pi$, the integral depends only on (z'', z'). For any fixed $z_0 \in Z$ one has

$$\sigma = \partial\tilde\partial \log A(z_0, z) \tag{6.1}$$

where $d = \partial + \tilde{\partial}$ is the splitting of the exterior derivatives provided by $\pi \times \tilde{\pi} : Z \to R \times \tilde{R}$. This means that the function

$$\log A(z_0, z) = \int_{[z_0 z]} \sigma$$

considered as function of $(x, \tilde{x}) = (\pi(z), \tilde{\pi}(z))$ is a *generating function* for the Lagrangian fibrations π and $\tilde{\pi}$ [Woodhouse, 1991 §6.9].

The formula (6.1) provides a construction of connection 1–forms for σ. The simplest are $-\partial \log A(z_0, z)$ and $\tilde{\partial} \log A(z_0, z)$, and from these one can built

$$\alpha = -\partial \log \Pi + \tilde{\partial} \log \tilde{\Pi}$$

where $\Pi = \prod A(z_k, z)^{\mu_k}$, $\tilde{\Pi} = \prod A(\tilde{z}_{\tilde{k}}, z)^{\tilde{\mu}_{\tilde{k}}}$, the products run over any finite set $\{z_k\}$, $\{\tilde{z}_{\tilde{k}}\}$ of points in Z, and the complex coefficients are subject to $\sum \mu_k + \tilde{\mu}_k = 1$. The transport along a zig–zag $z' \to z^* \to z''$ in the gauge α is

$$F(z'', z') = \exp[\int_{z'}^{z^*} + \int_{z^*}^{z''} \alpha].$$

On the first segment only the first component of α contributes and on the second segment only the second component. The integral is

$$F(z'', z') = \frac{\tilde{\Pi}(z'')\Pi(z^*)}{\tilde{\Pi}(z^*)\Pi(z')}. \tag{6.2}$$

6.3 Zig–zag triangles. Let $\gamma = \{z_t^* \mid 0 \le t \le 1\}$ be a (parametrized) path in Z, $\{x_t^*\}$ and $\{\tilde{x}_t^*\}$ its image in R and $\tilde{R}$. Let $z_{t\tilde{t}} \leftrightarrow (x_t^*, \tilde{x}_{\tilde{t}}^*)$ under $Z \to R \times \tilde{R}$. Then $\{z_{t\tilde{t}} \mid 0 \le t, \tilde{t} \le 1\}$ is a rectangle in Z whose vertical and horizontal sections $t =$const and $\tilde{t} =$const lie in the fibres of π and $\tilde{\pi}$, and whose diagonal $t = \tilde{t}$ is the path γ. By the *zig–zag triangle* over γ we mean the parametrized triangle

$$\Delta = \{z_{t\tilde{t}} : 0 \le t, \tilde{t} \le 1 \text{ and } t \ge \tilde{t}\}.$$

The boundary $\partial\Delta$ of Δ consists of γ, followed by the zig–zag β from the initial point of γ to its endpoint, but traversed in the opposite direction. Thus we write

$$\partial\Delta = \gamma - \beta.$$

We recall that a connection 1–form α satisfies $d\alpha = \sigma$. Hence by Stokes's theorem

$$\int_\Delta \sigma = \int_{\gamma - \beta} \alpha,$$

which implies that the transport along a path γ is related to the transport along the zig–zag β over γ by the relation

$$\exp_\iota[\int_\gamma \alpha] = \exp_\iota[\int_\beta \alpha] \exp[\int_\Delta \sigma].$$

This relation between the transport along an arbitrary path γ and the transport along the zig–zag joining its endpoints will be useful later.

6.4 Free propagators. Given z', z'' in Z, define the a*pproximate free propagator*

$$F(z'', z') = \exp_\iota[\int_{z'}^{z''} \alpha] : \Phi_{z'} \to \Phi_{z''}$$

to be the transport along the zig–zag from z' to z''. As far as this definition is concerned, the straight line segments $z \to z^*$ and $z^* \to z$ in the fibres of $\tilde{\pi}$ and π could be replaced by any paths between these points, since the transport within these fibres depends only on the endpoints. As a function of (z'', z'), the kernel is constant along the fibres of $\pi \times \tilde{\pi}$, i.e.

(6.3) $F(z'', z')$ *belongs to* $H(\Phi \otimes \Phi^*, \pi \times \tilde{\pi})$,

as it should. The formula (6.2) gives an expression for it in terms of generating functions.

The *free propagator*, denoted $E(z'', z')$, is defined as a path integral:

$$E(z'', z') = \int_{z_\cdot \in \Gamma} \exp_\iota[\int_t \alpha(dz_t)].$$

By definition, this means that

$$E(z'', z') = \lim_{N \to \infty} \int_{z_k \in \Gamma} \exp_\iota[\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \alpha].$$

The sum of the integrals in the exponent can be written as the integral along the composite of the zig–zags $z' \to z_1 \to \cdots \to z_{N-1} \to z''$, which will be called the *zig–zag path* through these points. The zig–zag paths will of course not lie in Γ.

The free propagator is necessarily a *projection* in the sense that $E \circ E = E$. If the approximate propagator F is itself a projection then $E = F$ and we say that the approximate propagator F is *exact*. In that case there is no problem of convergence of F^N, since the limit collapses, but E can then hardly be considered a genuine path integral operator.

6.5 The propagator for an arbitrary Hamiltonian. Let H be a smooth function on Z, X the corresponding Hamiltonian vector field. For any z', z'' and any ϵ (which may be considered as another variable) we define the *approximate propagator for* X as the linear transformation $\Phi_{z'} \to \Phi_{z''}$

$$\exp_\iota[\int_{z't'}^{z''t''} \alpha(dz) - H(z'')dt]$$

where the integral is along the zig–zag path $z' \to \exp(-\epsilon X)z'' \to z''$ of total duration $t'' - t' = \epsilon$. The value of H has been frozen at the endpoint and the Hamiltonian vector field X enters into the definition of the path. This is artificial in general, since the path integral involves points z_{k-1}, z_k, which may be arbitrarily far apart, but is appropriate in the special case considered below. We shall return to this point later. We now define the *propagator* $K(\exp X;\, z'', z')$ for X along Γ:

6.6 Definition II. $K(\exp X;\, z'', z') :=$

$$\lim_{N \to \infty} \int_{z_k \in \Gamma} \exp_\iota[\sum_{k=1}^{N} \int_{z_{k-1}t_{k-1}}^{z_k t_k} \alpha(dz) - H(z_k)dt].$$

The exponential term is the composite of the approximate propagators for X with $\epsilon = t_k - t_{k-1} = 1/N$ and $H_k = H(z_k)$. The outer integral is over $(N-1)$–tuples

$(z_1, \cdots, z_{N-1})$ from Γ, taken with respect to a normalized Liouville form σ^n/C. The formula is written as a *path integral*,

$$K(\exp X;\, z'', z') = \int_{z_\cdot \in \Gamma} \exp_\iota[\int_{t=0}^{1} \alpha(dz_t) - H(z_t)dt].$$

But this notation has to be interpreted with care, again because H is frozen at the endpoint z_k of each approximating path $z_{k-1} \to \exp(-\epsilon X)z_k \to z_k$.

6.7 Hamiltonians leaving the fibrations invariant. Assume that $\exp(tX)$ leaves the fibrations π and $\tilde{\pi}$ *invariant*, in the sense that $\exp(tX)$ maps fibres to fibres. Then

$$K(\exp X) = A(\exp X) \circ E \tag{6.3}$$

as operators.

This is seen as follows. By definition, E is the limit of the N–th composition power F^N of the approximate free kernel. The invariance of the fibrations under $\exp(tX)$ implies that $A(\exp X) \circ F = F \circ A(\exp X)$ and therefore

$$A(\exp X) \circ E \approx A(\exp X) \circ F^N = A(\exp X/N) \circ F^N.$$

In the notation of (6.4) the kernel of $A(\exp X) \circ E$ is obtained by applying the operator $A(\exp X)$ to $F(z_k, z_{k-1})$ as function of z_k. This means that the kernel of $K(\exp X)$ is given by the same path integral that of $A(\exp X) \circ E$ but with different elementary paths: for $A(\exp X) \circ E$ the integration from z_{k-1} to z_k is taken along the composites of the zig–zags $z_{k-1} \to \exp(-\epsilon X)z_k$ and $\beta : \exp(-\epsilon X)z_k \to z_k$; for $K(\exp X)$ the second zig–zag β is replaced by the trajectory $\gamma = \{\exp(tX)z_k \mid -\epsilon \le t \le 0\}$. The two path integrals therefore differ by the integral $\int_\Delta \sigma$ over the zig–zag triangle Δ over γ. Because of the invariance of the fibrations under $\exp(tX)$, this integral is independent of the choice of z_k. Hence we can replace z_k by some fixed $z_o \in \Gamma$. The integral $\int_\Delta \sigma$ is a function of ϵ. Its leading term as $\epsilon \to 0$ is the integral of σ over the triangle in the tangent space at z_o spanned by the components of the vector ϵX along the fibres of π and $\tilde{\pi}$ through z_o. Therefore $\int_\Delta \sigma$ is $O(\epsilon^2)$, independently of z_k. The contribution of such terms to the sum in the exponent of of the path integral for $K(\exp X)$ is then $NO(1/N^2)$ and does not contribute to the limit.

The formula (6.3) gives a realization of the propagators $K(\exp X) = A(\exp X) \circ E$ representing the canonical transformations $\exp(tX)$ which preserve the fibrations π, $\tilde{\pi}$ by path integral operators. These are generally genuine path integral operators in the sense that they are not equal to the approximate propagators, even if the approximate free propagator itself is exact. Nevertheless, in the exact case the path–integral formalism can be avoided for Hamiltonians leaving the fibrations invariant, since the formula (6.3) gives the propagators directly.

6.8 Comparison of the two definitions. We compare the two definitions of contour path integrals, but we restrict attention to the free propagators. Fix z', z'' in Γ and $\epsilon > 0$. The two approximate free propagators are $\exp_\iota$ of

$$\text{(I)} \int_\gamma [\alpha(\dot{z}_t) - \frac{1}{2\nu}\tau(\dot{z}_t, \dot{z}_t)dt] \qquad \text{(II)} \int_\beta \alpha(\dot{z}_t)dt,$$

where γ a path on the holomorphic trajectory and β is the zig–zag from z' to z'', both of parametrized on $[0, \epsilon]$, although the first integral is of course independent of the parametrization. In accordance with Feynman's remark quoted above we replace the holomorphic trajectory of $\int \alpha - \tau/2\nu$ by that of τ, which we call the holomorphic geodesic. Thus $\gamma = \{z_t \mid 0 \le t \le \epsilon\}$ will now denote a path on this holomorphic geodesic.

Let $\bar{z}$ be any point on γ. The tangent vector of γ at $\bar{z}$ is of the form $(1/\epsilon)\bar{v}$ where $\bar{v}$ is independent of ν and ϵ, but does depend on z', z''. We approximate the zig–zag triangle in Z by the triangle in the tangent space at $\bar{z}$ with base $\bar{v} = v + \tilde{v}$ and sides $v, \tilde{v}$ in the fibres of π and $\tilde{\pi}$. Since $\sigma(v, \tilde{v}) = \tau(\bar{v}, \bar{v})$ this gives

$$\int_\Delta \sigma = \frac{1}{2}\tau(\bar{v}, \bar{v}) + o(\delta^2).$$

where $o(\delta^2)$ denotes a function which vanishes to order > 2 along $z' = z''$. On the other hand

$$\int_\gamma \tau(\dot{z}_t, \dot{z}_t)dt = \frac{1}{\epsilon}\tau(\bar{v}, \bar{v})$$

because $\tau(\dot{z}_t, \dot{z}_t)$ is constant along the complex geodesic, and γ has duration ϵ. Hence

$$\int_\Delta \sigma = \int_\gamma \frac{\epsilon}{2}\tau(\dot{z}_t, \dot{z}_t)dt + o(\delta^2).$$

As in 6.3, Stokes's theorem applied to $\partial\Delta = \gamma - \beta$ gives

$$\int_\gamma \{\alpha(\dot{z}_t) - \frac{1}{2\nu}\tau(\dot{z}_t, \dot{z}_t)dt\} = \int_\beta \alpha(\dot{z}_t)dt + o(\delta^2), \tag{6.4}$$

where we have put $\nu = 1/\epsilon$. The free propagator according to Definition I is

$$\lim_{\nu\to\infty}\lim_{\epsilon\to 0}\int_{z_k\in\Gamma} \exp_\iota[\sum_{k=1}^{N}\{\int_{\gamma_k} \alpha(\dot{z}_t) - \frac{1}{2\nu}\tau(\dot{z}_t, \dot{z}_t)dt\}].$$

where in the nation introduced above, $\epsilon = 1/N$ and $\gamma = \gamma_k$, a path from z_{k-1} to z_k on the complex geodesic. We set $\nu = N$, $\epsilon = 1/N$ and replace the double limit by the limit as $N \to \infty$. In view of (6.4), the last expression becomes

$$\lim_{N\to\infty}\int_{z_k\in\Gamma} \exp_\iota[\sum_{k=1}^{N}\{\int_{\beta_k} \alpha(\dot{z}_t)dt\} + o(\delta_k^2)] \tag{6.5}$$

If we drop the terms $o(\delta_k^2)$ from this limit, we get exactly the free propagator according to Definition II,

$$\lim_{N\to\infty}\int_{z_k\in\Gamma} \exp_\iota[\sum_{k=1}^{N}\{\int_{\beta_k} \alpha(\dot{z}_t)dt] \tag{6.6}$$

The last step is crucial, and I do not know how to justify it. A similar step occurs in Feynman's discussion [1948, p.375]. The justification given there amounts to a stationary phase argument to the effect that terms vanishing to order > 2 along $z_{k-1} = z_k$ may be dropped. This is Feynman's rule that "$x_{i+1} - x_i$ is of order $\epsilon^{1/2}$". The analogous statement for the Wiener measure is a theorem concerning stochastic integrals.

In the context of the Wiener integral on a Riemannian manifold, the path integral of Definition I represents the kernel for $\exp(\nu\Delta)$ where Δ is the Laplace operator on L^2 functions on Γ (Feynman–Kac Formula). Since $\Delta \le 0$, the limit as $\nu \to +\infty$

gives the kernel for the projection onto the space of L^2 harmonics. This is the McKean–Singer argument, which, adapted to Δ acting on forms rather than on functions, is the starting point of one approach to the Index Theorem [McKean and Singer, 1967].

7. Path integral representations of SL(2, ℝ)

The integral (I) in the introduction now has a meaning (and in fact two). We shall examine some simple examples, namely the coadjoint orbits for $\mathrm{SL}(2,\mathbb{C})$, to see how it can be evaluated and what its relation to (II) might be. In these examples the Hamiltonians leave the fibrations π and $\tilde{\pi}$ invariant, and the strategy will be to verify that the approximate free propagator is exact, so that the formula $K(\exp X) = A(\exp X) \circ E$ of 6.7 applies. Throughout we let $G = \mathrm{SL}(2,\mathbb{C})$ and $\mathfrak{g}=\mathrm{sl}(2,\mathbb{C})$, its Lie algebra. We shall specify certain contours of integration on G–orbits in $\mathfrak{g}^*$ by conditions on their behaviour at infinity. For this reason we start with some observations on certain completions of these orbits.

7.1 Coadjoint orbits. Let $\widehat{\mathfrak{g}}^* = P(\mathfrak{g}^* \times \mathbb{C})$ be the projective completion of $\mathfrak{g}^*$: the elements of $\widehat{\mathfrak{g}}^*$ are of the form ξ/τ, $\xi \in \mathfrak{g}^*$, $\tau \in \mathbb{C}$, not both zero, with the understanding that $\xi'/\tau' = \xi/\tau$ if $\xi'\tau = \xi\tau'$. The space $\mathfrak{g}^*$ is embedded in $\widehat{\mathfrak{g}}^*$ as elements of the form $\xi/1$; its complement is the plane at infinity, consisting of elements $\xi/0$, and denoted $\mathfrak{g}^\infty$. It is naturally isomorphic with the projective space $P\mathfrak{g}^* \approx \mathbb{CP}^2$ associated to the vector space $\mathfrak{g}^*$.

Whenever convenient, we identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by the form $\langle X, Y\rangle{:=}\frac{1}{2}\mathrm{tr}(XY)$, but we continue to use the notation X for elements of $\mathfrak{g}$ and ξ for elements of $\mathfrak{g}^*$. We write $q(\xi) = \frac{1}{2}\mathrm{tr}(\xi^2)$ for the invariant quadratic form on $\mathfrak{g}^*$. For a fixed $\lambda \in \mathbb{C}$, we have quadric surfaces in $\mathfrak{g}$ and in $\widehat{\mathfrak{g}}^*$:

$$\Omega_\lambda = \{\xi \in \mathfrak{g}^* \mid q(\xi) = \lambda^2\} \quad \widehat{\Omega}_\lambda = \{\xi/\tau \in \widehat{\mathfrak{g}}^* \mid q(\xi) = (\tau\lambda)^2\}.$$

If $\lambda \neq 0$ then Ω_λ is an orbit of G in $\mathfrak{g}^*$ for the coadjoint action $g \cdot \xi = \xi \circ \mathrm{Ad}(g^{-1})$; if $\lambda = 0$ then Ω_λ degenerates into the nilpotent cone Ω_0. All Ω_λ intersect at infinity in a variety $\Omega^\infty = \widehat{\Omega}_\lambda \bigcap \mathfrak{g}^\infty$, isomorphic with the projective nilpotent variety $P\Omega_0 = \{q(\xi) = 0\}$ in $P\mathfrak{g}^*$.

There is another way of looking at $\widehat{\mathfrak{g}}^*$ which will be useful. Let M be the space of complex 2×2 matrices. The inclusion $\mathfrak{g} \to M$ gives a projection $M^* \to \mathfrak{g}^*$, $\zeta \to \zeta \mid \mathfrak{g}$, which extends to a G–equivariant isomorphism $M^* \xrightarrow{\approx} \mathfrak{g}^* \times \mathbb{C}$, $\zeta \to (\zeta \mid \mathfrak{g}, \lambda^{-1}\mathrm{tr}(\zeta))$ for any $\lambda \neq 0$. It induces an isomorphism of projective spaces, which sends the projective variety $\widehat{Z}$ of of rank one matrices in PM^* onto Ω_λ:

$$p_\lambda : PM^* \xrightarrow{\approx} \widehat{\mathfrak{g}}^*, \quad \widehat{Z} \to \widehat{\Omega}_\lambda.$$

Whenever convenient we identify M^* with M by the form $\frac{1}{2}\mathrm{tr}(XY)$ and both with $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$ so that $x \otimes y$ becomes the linear transformation $v \to \langle v, y\rangle x$ of $\mathbb{C}^2$, $\langle v, y\rangle$ being the natural pairing. Then $\widehat{Z} = P\{x \otimes y \mid x \neq 0,\ y \neq 0\}$ and the map $p_\lambda : \widehat{Z} \to \widehat{\Omega}_\lambda$ is given by

$$\xi/\tau = \lambda(x \otimes y - \langle x, y\rangle t)/\langle x, y\rangle \tag{7.1}$$

where $t = \frac{1}{2}(e \otimes \tilde{e} + f \otimes \tilde{f})$ for any pair of dual bases e, f and $\tilde{e}, \tilde{f}$ and the notation ξ/τ for elements of $\widehat{\mathfrak{g}}^*$ has been used.

In terms of the structure of the group G, the situation is this. The natural action of G on $\mathbb{C}^2$ induces an action on the flag manifold (Riemann sphere) $R = P(\mathbb{C}^2)$ and on the dual projective space $\tilde{R} = P(\mathbb{C}^2)^*$. Let R_∞ denote the subvariety of points (x, y) in $R \times \tilde{R}$where $\langle x, y\rangle = 0$, another copy of $\mathbb{CP}^1$. There is a G–equivariant isomorphism

$$\pi_\lambda \times \tilde{\pi}_\lambda : \widehat{\Omega}_\lambda \to R \times \tilde{R}$$

given by (7.1) where (x, y) now represents a point of $R \times \tilde{R}$. On the part at infinity in $\widehat{\Omega}_\lambda$ this map induces the isomorphism $\Omega^\infty \to R_\infty$, $x \otimes y \to x \otimes y/0$ and on the finite part $\{\langle x, y\rangle \neq 0\}$ it induces the isomorphism

$$\Omega_\lambda \to R \times \tilde{R} - R_\infty, \quad \xi{:=}\lambda\Big(\frac{x \otimes y}{\langle x, y\rangle} - t\Big) \to (x, y).$$

We shall consider $Z \subset \widehat{Z}$ together with the maps $p_\lambda : \widehat{Z} \to \widehat{\Omega}_\lambda$, and $\pi \times \tilde{\pi} : \widehat{Z} \to R \times \tilde{R}$ as a *standard orbit* for the family of orbits Ω_λ, $\lambda \neq 0$, recorded in the diagram

$$\begin{array}{ccc} \widehat{Z} & \xrightarrow{p_\lambda} & \widehat{\Omega}_\lambda \\ \pi \times \tilde{\pi} \searrow & & \swarrow \pi_\lambda \times \tilde{\pi}_\lambda \\ & R \times \tilde{R} & \end{array}$$

The particular realization of Z is not important; what is important is that it come equipped with isomorphisms p_λ and $\pi \times \tilde{\pi}$ which fit into this diagram.

7.2 Symplectic form and connection form. As a coadjoint orbit, Ω_λ carries a natural symplectic form σ_λ defined by $\sigma_\lambda(X \cdot \xi, Y \cdot \xi) = \langle \xi, [X, Y]\rangle$ where $X \in \mathfrak{g}$ acts by the coadjoint action $X \cdot \xi = -\mathrm{ad}(X)^*\xi$ on $\xi \in \Omega_\lambda$. The pull–back of σ_λ by $p_\lambda : Z \to \Omega_\lambda$ can be written as $\sigma_\lambda = \lambda\sigma$ where σ is a symplectic form on Z. The map $\widehat{Z} \to R \times \tilde{R}$ provides Lagrangian fibrations $\pi : \widehat{Z} \to R$ and $\tilde{\pi} : \widehat{Z} \to \tilde{R}$ for σ. In the notation of 6.2, the exponential integral $A(z_0, z)$ of σ over the the zig–zag rectangle $[z, z_0]$ is

$$A(z_0, z) = \frac{\langle x_0, \tilde{x}\rangle\langle x_0, \tilde{x}_0\rangle}{\langle x, \tilde{x}\rangle\langle x, \tilde{x}_0\rangle}$$

where $(x, \tilde{x}; x_0, \tilde{x}_0)$ corresponds to (z, z_0) under $\widehat{Z} \to R \times \tilde{R}$. This is the classical cross–ratio of the four image points in $\mathbb{CP}^1$ if we identify identify $\mathbb{C}^2$ and $(\mathbb{C}^2)^*$ so that the pairing $\langle x, \tilde{x}\rangle$ becomes the $G = \mathrm{SL}(2, \mathbb{C})$ invariant form $\langle x, \tilde{x}\rangle = x_1\tilde{x}_2 - x_2\tilde{x}_1$ on $\mathbb{C}^2$. The form σ_λ is given by the formula:

$$\sigma_\lambda = \partial\tilde{\partial}\log A^\lambda(z_0, z).$$

(We keep the parameter λ in sight, for the moment). As mentioned in 6.2, this formula provides a construction of connection 1–forms α for σ. A simple choice is

$$\alpha = \partial\log A^{-\lambda}(z_0, z).$$

The approximate free propagator then is

$$F(z'', z') = A^{-\lambda}(z_0, z^*\rangle A^\lambda(z_0, z')$$

where z^* is the corner of the zig–zag $z' \to z^* \to z''$: under $\widehat{Z} \to R \times \tilde{R}$, z', z^*, z'' correspond to $(x', \tilde{x}')$, $(x'', \tilde{x}')$, $(x'', \tilde{x}'')$ respectively.

7.3 Contour integrals. We now bring in a contour Γ on Z. We recall the significance of Γ for path integral operators comes from the composition rule

$$F'' \circ F'(z'', z') = \int_{z\in\Gamma} F''(z'', z) F'(z, z') C^{-1} \sigma(dz).$$

We shall take for F' and F'' the approximate free propagator F according to Definition II, in the gauge provided by the connection form α above. We shall write such integrals as

$$I = \int_{\Gamma} \Psi$$

where $\Psi = F(z'', z) F(z, z') C^{-1} \sigma(dz)$. In order to make sense of the integral I we shall impose three conditions on the contour Γ:

(a) *The line bundle Φ is globally defined along Γ.*
(b) Γ *is a relative cycle on $Z \subset \widehat{Z}$, i.e. $\partial\Gamma \subset Z^\infty$.*
(c) $\partial\Gamma$ *is invariant under a real form $G_\mathbb{R}$ of G.*

The real form $G_\mathbb{R}$ will remain fixed throughout. Up to conjugacy, there are only two choices for $G_\mathbb{R}$, namely $G_\mathbb{R} = \mathrm{SL}(2, \mathbb{R})$ or $G_\mathbb{R} = \mathrm{SU}(2)$. In accordance with its interpretation as a kernel, we shall consider the integral I as a generalized function in the variables z' and z''. As its space of test function we take Fourier transforms of compactly supported C^∞ functions on the Lie algebra $\mathfrak{g}_\mathbb{R}$ of $G_\mathbb{R}$. These test functions are naturally functions on $\mathfrak{g}^*$, and can therefore be integrated against generalized functions on an orbit Ω_λ in $\mathfrak{g}^*$.

The integrals I may further have to be interpreted by regularization procedure. For the examples we have in mind the following will be sufficient. We assume that the integrand Ψ exists for fixed z as a generalized function of (z'', z') depending on the parameter λ. If the integral exists in the same sense, then we take this as its definition; we extend the definition by analytic continuation in λ whenever possible.

7.4 Invariance properties and projection formula. We shall be interested in contours Γ for which the integral I defines a G–invariant functional on the integrands Ψ, in so far as this makes sense. To discuss the invariance properties of the integral I as function of (z'', z') we shall assume that the integral I converges pointwise in (z'', z') for suitable forms λ; it then represents a holomorphic functions of these λ.

The function $F(z', z'') \in \mathrm{Hom}(\Phi_{z'}, \Phi_{z''})$ has the following invariance property under the action of G:

$$F(gz'', gz') = A(g, z'') \circ F(z'', z') \circ A(g, z')^{-1} \tag{7.2}$$

for (z'', z'', g) in a neighbourhood of $\Gamma \times \Gamma \times \{1\}$ in $Z \times Z \times G$. It follows that the integrand $\Psi(z'', z', dz)$ satisfies

$$\Psi(gz'', gz', gdz) = A(g, z'') \circ \Psi(z'', z', dz) \circ A(g, z')^{-1}$$

in the same sense. Consider the integral of both sides of this equation over $z \in \Gamma$. If $g \in G$ leaves $\partial\Gamma \subset Z^\infty$ invariant, then one can change variables $z \to g^{-1}z$ on the left–hand side and finds that

$$\int_{z\in g^{-1}\Gamma} \Psi(gz'', gz', dz) = A(g, z'') \circ \left\{ \int_{z\in\Gamma} \Psi(z'', z', dz) \right\} \circ A(g, z')^{-1}.$$

This means that the integral $I(z'', z')$ satisfies (7.2) as long as $g \in G$ fixes $\partial\Gamma$ and remains in a neighbourhood of 1, which may depend on (z'', z'). By condition (c) on Γ, this applies to $g \in G_{\mathbb{R}}$, hence I satisfies (7.2) for g in a neighbourhood of 1 in the real group $G_{\mathbb{R}}$; because of holomorphicity in then holds for g in a neighbourhood of 1 in the complex group G as well.

The complement of R_∞ in $R \times \tilde{R}$ is a single G–orbit. It follows that on $Z \times Z$ a function $F(z'', z')$ which satisfies (7.2) and which is covariantly constant along the fibres of $\pi \times \tilde{\pi}$ is uniquely determined locally by its value at a single point. This holds in particular for the integrals I and implies that for these integrands and for connected contours we have $I = CF$ were C is a constant, depending holomorphically on the parameter λ. If C is non–zero, then we can take it to be the constant entering into the definition of the composition rule and the discussion above then shows that then $I = F \circ F$ coincides with F. This amounts to what we shall call the projection formula:

$$F \circ F = F. \tag{7.3}$$

So in this case the approximate free propagator F is *exact*. We can then apply the general construction in 6.7 to elements of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of the real form $G_{\mathbb{R}}$ of G which fixes $\partial\Gamma$. The result is what we call a *path integral representation* of $G_{\mathbb{R}}$, which associates to each $X \in \mathfrak{g}_{\mathbb{R}}$ the kernel $K(\exp X, z'', z')$ given by Definition II, or equivalently by the formula (7.3). It corresponds to a group representation in the usual sense only if the kernels $K(\exp X; z'', z')$ define genuine integral operators on the state space $H(Z, \Phi, \pi)$, which need not be the case, even if the wave functions are subject to further restrictions, e.g. L^2.

7.5 The contours. We now examine the contours Γ. We recall that $\partial\Gamma \subset R_\infty \approx \mathbb{CP}^1$ must be invariant by a real form $G_{\mathbb{R}}$ of $G = \mathrm{SL}(2, \mathbb{C})$. We shall take $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$. To get an overview of the possibilities for Γ it is useful to consider the Lagrangian fibration $\pi : Z \to R$. It is G–equivariant and its fibres are one–dimensional affine spaces. This fibration admits a continuous section

$$R \to Z,\, x \to z = \frac{x \otimes x^*}{\langle x, x^* \rangle} - t$$

here $x \to x^*$, $\mathbb{C}^2 \to (\mathbb{C}^2)^*$ is the conjugate–linear map for which $\langle x, x^* \rangle = x_1\bar{x}_1 + x_2\bar{x}_2$ is the positive–definite SU(2)–invariant form. As an affine bundle with a section, the fibration $\pi : Z \to R$ can be considered a vector bundle over R, namely the cotangent bundle $R^* \to R$. Thus we can identify $Z \approx R^*$. (This identification is only topological and not holomorphic nor G–equivariant.)

The group $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$ has three orbits on the Riemann sphere R: the upper hemisphere, the lower hemisphere and the equator. Let S be the subset of R^* consisting of cotangent vectors orthogonal to these orbits; it is the union of the conormal bundles of the $G_{\mathbb{R}}$–orbits, a real analytic variety, the *conormal variety* or the $G_{\mathbb{R}}$–action on R. The map $p_\lambda : Z \to \Omega_\lambda$ maps S to the orbit Ω_λ, which degenerates to the nilpotent cone Ω_0 as $\lambda \to 0$. The *limit map* $p_0 : Z \to \Omega_0$ is still well–defined.

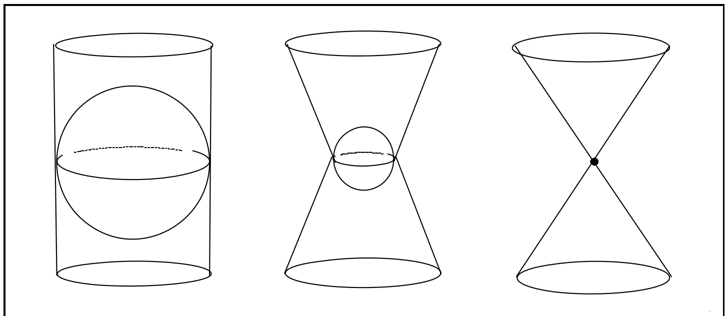

Figure (7.1). The variety $\mathcal{S}$ and its degeneration

The condition that $\partial\Gamma \subset Z^\infty$ be invariant under $G_\mathbb{R}$ means that Γ has to asymptotically approach the conormal variety S. It may then be deformed into a cycle which actually lies on S and then represents an element of the top homology group $H_2(S)$ of 2–cycles with arbitrary support. Four such cycles are shown below.

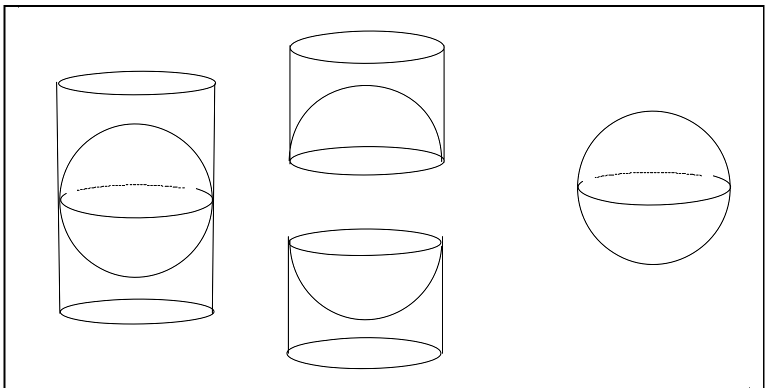

Figure (7.2). Four cycles on $\mathcal{S} : \Gamma_0 = \Gamma_+ + \Gamma_- + \Delta$.

(For more on the variety S and its homology, see [Rossmann, 1994], from where the pictures are taken.) We shall discuss the path integral representations corresponding to these cycles. For that purpose we shall specify equivalent cycles directly on $R \times \tilde{R} - R_\infty \approx Z$, rather than on the variety S. We also replace $\mathrm{SL}(2,\mathbb{R})$ by its conjugate $\mathrm{SU}(1,1)$, when this is more convenient.

7.6 Coordinate formulas. To work out the integrals we need coordinates. We introduce linear coordinates (x_1, x_2) on $\mathbb{C}^2$ and $(\tilde{x}_1, \tilde{x}_2)$ on $(\mathbb{C}^2)^*$ so that the pairing becomes $x_1\tilde{x}_2 - x_2\tilde{x}_1$, and we use $x = x_2/x_1$,and $\tilde{x} = \tilde{x}_2/\tilde{x}_1$ as coordinates on R and $\tilde{R}$. We consider $(x, \tilde{x})$ as coordinates on Z and Ω_λ by the maps $Z, \Omega_\lambda \to R \times \tilde{R}$. The symplectic form on Ω_λ is then

$$\sigma_\lambda = \frac{\lambda dx \wedge d\tilde{x}}{(1 - x\tilde{x})^2}.$$

In order to write σ_λ in the form $\partial\tilde{\partial}\log A(z_0, z)$ we choose a base–point z_0 for Z, corresponding to a point $(x_0, \tilde{x}_0)$ in $R \times \tilde{R}$, say with coordinates $(x, \tilde{x}) = (0, 0)$. We use the constructions and the notation of 6.2. The function A, the symplectic

2-form σ_λ, and the gauge 1-form α_λ are

$$A(z_0, z) = (1 - x\tilde{x})^{-\lambda}, \quad \sigma_\lambda = \partial\tilde{\partial}\log(1 - x\tilde{x})^{-\lambda}, \quad \alpha = \partial\log(1 - x\tilde{x})^{\lambda}.$$

The approximate free propagator according to Definition II is

$$F(z'', z') = (1 - x''\tilde{x}')^{\lambda}(1 - x'\tilde{x}')^{-\lambda}.$$

The integral $I = F \circ F$ to be evaluated is

$$I = (1 - x'\tilde{x}')^{-\lambda} \int_\Gamma (1 - x''\tilde{x})^{\lambda}(1 - x\tilde{x})^{-\lambda-2}(1 - x\tilde{x}')^{\lambda}\ dx \wedge d\tilde{x}$$

7.7 Evaluation of the integral. We consider separately the integral I over the contours Δ, $\Gamma_\pm, \Gamma_0$.

(A) $\Delta = \{z = (x, \tilde{x}) \in Z \mid \tilde{x} = -\bar{x}\}$. In this case $\partial\Delta = 0$ hence trivially invariant under any real form of G. The cycle Δ itself is only SU(2)–invariant. A quantum line bundle with connection exists globally on Δ iff $\lambda \in$ Z, is then unique, and extends to all of Z. The locally defined transformations $A(\exp X)$, $X \in \mathrm{sl}(2, \mathbb{C})$, induce a globally defined action of $\mathrm{SL}(2, \mathbb{C})$ on this line bundle. The integral I converges pointwise in (z'', z') for $\lambda + 1 > 0$. Because of the invariance property it suffices to take $(z'', z') = (z_0, z_0)$, and one arrives the value $I = CF$ with $C = \pi/(\lambda + 1)$. Alternatively, the integral I itself can be evaluated directly (e.g. by power series expansion of the terms $(\cdots)^{\lambda}$), which leads to the same result. If $\lambda = 0, 1, 2, \cdots$ then the path integral representation of $G_{\mathbb{R}}$ gives a representation in the usual sense and we get a realization of the finite–dimensional irreducible representations of $\mathrm{SL}(2, \mathbb{C})$ by path integral operators.

(B) $\Gamma_\pm = \{z = (x, \tilde{x}) \in Z \mid \tilde{x} = \bar{x},\ \pm(1 - x\bar{x}) > 0\}$. In this case $\partial\Gamma_\pm$ is the circle $\{(x, \bar{x}) \mid x\bar{x} = 1\}$ in $P \approx \mathbb{CP}^1$; it is invariant under $G_{\mathbb{R}} = \mathrm{SU}(1, 1)$, which leaves $\Gamma_\pm$ itself invariant. A quantum line bundle with connection exists globally on $\Gamma_\pm$ for all $\lambda \in \mathbb{C}$, and is unique. The locally defined transformations $A(\exp X)$, $X \in \mathrm{su}(1, 1)$, induce a globally defined action of the universal covering group $\tilde{\mathrm{SU}}(1, 1)$ on this line bundle, which passes to $\mathrm{SU}(1, 1)$ itself if $\lambda \in$ Z. The integral I converges for $\lambda + 1 < 0$; as above one finds the value $I = CF$ with $C = \pm\pi/(\lambda + 1)$. If $\lambda + 1 < 0$ then the path integral representation of $G_{\mathbb{R}}$ gives a representation in the usual sense of the universal covering group $\tilde{\mathrm{SU}}(1, 1)$ and we get a realization of the "discrete" series of $\tilde{\mathrm{SU}}(1, 1)$ by path integral operators; for integral $\lambda + 1 = -1, -2, \cdots$ we get the discrete series of SU(1, 1) itself.

(C) $\Gamma_0 = \{z = (x, \tilde{x}) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid x,\ \tilde{x} \in \mathbb{R},\ 1 - x\tilde{x} \neq 0\}$. In this case $\partial\Gamma_0 = \{(x, x^{-1}) \mid x \in \mathbb{RP}^1\}$ is a circle on $R_\infty \approx \mathbb{CP}^1$; it is invariant under $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$, which leaves Γ_0 itself invariant. In this case a quantum line bundle with connection exists globally on Γ_0 for all $\lambda \in \mathbb{C}$, but is not unique: for a given $\lambda \in \mathbb{C}$, the inequivalent quantum line bundles on Γ correspond to complex–valued characters χ of the stabilizer of a base–point z_0 on Γ_0 in the universal covering group $\tilde{\mathrm{SL}}(2, \mathbb{R})$. The locally defined transformations $A(\exp X)$, $X \in \mathrm{sl}(2, \mathbb{R})$, induce a globally defined action of $\tilde{\mathrm{SL}}(2, \mathbb{R})$ on this line bundle, which passes to $\mathrm{SL}(2, \mathbb{R})$ itself if the character χ passes to the stabilizer of z_0 in $\mathrm{SL}(2, \mathbb{R})$. It is then of the form $\mathrm{sgn}^{\epsilon}(t) \mid t \mid^{\lambda}$, as we shall now assume.

The integral I over $\Gamma = \Gamma_0$ does not converge pointwise in (z'', z'), but can be evaluated as a generalized function as follows. Set $y = \tilde{x}^{-1}$. Then I becomes

$$I' = (y' - x')^{-\lambda} \int_{\Gamma_0} (y - x'')^{\lambda} (y - x)^{-\lambda - 2} (y' - x)^{\lambda} \, dx \wedge dy.$$

The integrand along Γ_0 is specified by interpreting the three factors $(y_j)^{\mu_j}$ in it as $\mathrm{sgn}^{\epsilon}(y_j) \mid y_j \mid^{\mu_j}$ according to the choice of the line bundle. In these factors, $y_j = y_j(x, y)$ is an affine function of the integration variables $(x,\ y)$ with with coefficients depending on $(x', y'; x'', y'')$. For $(y_j)^{\mu_j}$ to be considered as a generalized function we have to require that $\lambda \notin \mathrm{Z}$. Interpreted in this way the integral can be evaluated by the duality theorem of Gel'fand and Graev [1987]. For this purpose it is preferable to rewrite the integral I' in homogeneous coordinates (x, y, z) with Γ_0 given by $\{z = 1\}$:

$$I'' = (y' - x')^{-\lambda} \int_{\{z=1\}} (z)^{-\lambda} (zy - x'')^{\lambda} (y - x)^{-\lambda - 2} (zy' - x)^{\lambda} \ (zdx \wedge dy).$$

The integrand now contains four factors $(x_j)^{\mu_j}$ in which $x_j = x_j(x, y, z)$ is a linear form in (x, y, z). According to [Gel'fand and Graev, 1987], the value of the integral is the product of the Fourier transforms of the $(x_j)^{\mu_j}$ as generalized functions of one variable, evaluated at any point (ξ_j) on the line $\sum x_j(x, y, z)\xi_j \equiv 0$ in the dual space of the (x_j). According to this recipe one finds the value $C(y' - x'')^{\lambda}$ for the integral over $\{z = 1\}$ where C equals $\prod \sin(\pi\mu_i/2)\Gamma(\mu_i + 1)$ or $\prod \cos(\pi\mu_j/2)\Gamma(\mu_j + 1)$ depending on the ϵ in $\mathrm{sgn}^{\epsilon}(t) \mid t \mid^{\lambda}$, apart from a non–zero constant. We find

$$I'' = C(y' - x')^{-\lambda} (y' - x'')^{\lambda}$$

which is just $CF(z'', z')$. Thus we again have the projection formula $F \circ F = F$. For $\lambda \notin \mathrm{Z}$, the path integral representation of $G_{\mathbb{R}}$ gives a representation in the usual sense and we get a realization of the principal series representations of $\mathrm{SL}(2, \mathbb{R})$ by path integral operators.

In all of the cases considered, the representations by path integral operators are realized by means of the projection formula (7.3), based on the Definition II of path integrals. For the cases (A) and (B) the path integrals have also been worked out according to Definition I, see [Daubechies and Klauder 1985; Klauder, 1988]. The results agree with those above.

It may be appropriate to add a comment on path integrals in general, since they are often regarded with suspicion, and rightly so. Their definition along the lines of Feynman [1948], a version of which is explained in section 4, is in principle quite acceptable; but the question then arises whether these limits exist and in what sense. This is a very difficult problem, and it may indeed be the case that other definitions preferable to deal with it, for example along the lines of stochastic integrals for the Wiener measure. In simple cases as above, however, where the approximate propagator is exact, the question of convergence does not arise, or can be dealt with directly.

7.8 Characters. We can now state in precise fashion the character formula (II) of the introduction. It says that whenever the path–integral representation corresponding to one of the contours Γ above gives a genuine representation of $G_{\mathbb{R}}$, then its character is given by

$$\operatorname{tr}K(\exp X) = j^{-1/2}(X) \int_{p_\lambda(\Gamma)} e^{\langle \xi, X\rangle}\ \sigma(d\xi) \tag{7.4}$$

where $j(X) = \det[\sinh \operatorname{ad}(X/2)/\operatorname{ad}(X/2)]$. The formula is understood as an identity of distributions in the variable $X \in \mathfrak{g}_{\mathbb{R}}$, representing the group element $\exp(X)$ of $G_{\mathbb{R}}$ in exponential coordinates. (Thus (7.4) determines the character only on the image of exp.) Such a formula is valid for any irreducible character of a semisimple real Lie group [Rossmann, 1990]. The proof of the formula (7.4) in general makes no use of path integral operators, but is based on Harish–Chandra's theory of characters.

Homology relations between contours become relations between characters. For example, the relation $\Gamma_0 \sim \Gamma_+ + \Gamma_- + \Delta$, visible in Figure 2, becomes the familiar decomposition formula for principal series characters. It holds as a character identity on the image of the exponential map whenever all four contours correspond to genuine representations. The homology relation between the contours is best understood as a monodromy relation in the sense of Picard and Lefschetz [Rossmann, 1994]. By means of the formula (7.4) it translates directly into a relation between characters, but it might be interesting to see this relation in the kernels of the representations themselves, presumably through contour path integrals.

Of course it now transpires that there still is no real explanation of the relation between formulas (I) and (II) in the introduction. My excuse is that one first has to make sense out of (I) for contour integrals. There also remains the question of how to extend (I) to kernels operating in cohomology, which is not addressed here at all. It seems that the passage from (I) to (II), though difficult, should still proceed by a version of the Localization Formula, adapted to contour integrals, perhaps along the lines of [Rossmann, 1991]. Needless to say, apart from the the case when the form σ is purely imaginary on the contour Γ, questions of unitarity of path integral operators remain entirely unexplored.

7.9 Notes. Many examples of path integrals, in its various forms, can be found in the literature. I mention only a few which are relevant to the discussion here. Examples of constructions of representations by path integrals are discussed in [Hashimoto et al., 1991]. Path integrals in the sense of Wiener integrals have been studied by Daubechies and Klauder [1985] in a number of cases; see [Klauder, 1988] for a survey. A different approach to representations via path integrals is that of [Alekseev, 1989]. As mentioned in the introduction, the relation of path integrals to the Index Theorem is discussed by Witten [1982] and Atiyah [1985] and is treated in detail by Bismut [1984].

REFERENCES

A. Alekseev, L. Faddeev, S. Shatashvili. *Quantization of symplectic orbits of compact Lie groups by means of the functional integral.* J. Geo. and Phy. 5 (1989), 391–406.

M. F. Atiyah. *Circular symmetry and stationary phase approximation. In Proceedings of the conferece in honor of L. Schwartz.* Astérisque, 131 (1985), 43–59. Reprinted in Collected Works, Clarendon Press, Oxford, 1988.

N. Berline and M. Vergne. *Classes charactéristiques équivariantes. Formule de localization en cohomoogie équivariante.* C. R. Acad. Sci. Paris 295 (1982), 539–541.

J.–L. Bismut. *The Atiyah–Singer theorems: a probabilistic approach, I, II.* J. Funct. Anal. 57 (1984), 59–99, 329–348.

R. Bott. *The Index Theorem for homogenous differential operators, in Differential and Cominatorial Topology,* Princeton U. Press, 1965, 167–185.

I. Daubechies and J. R. Klauder. *Quantum–mechanical path integrals with Wiener measure for all polynomial Hamiltonians, II.* J. Math. Phys. 26 (1985), 2239–2256.

J. J. Duistermaat and G. Heckman. *On the variation in the cohomology of the symplectic form of the reduced phase space.* Invent. Math. 69 (1982), 259–268. Addendum, 72 (1983), 153–158.

R. P. Feynman. *A space–time approach to non–relativistic quantum mechanics,* Rev. of Mod. Phys. 20 (1948), 267–238. Reprinted in Quantum Electrodynamics, J. Schwinger, Ed., Dover, 1958.

I. M. Gel'fand and I. Graev. *A duality theorem for general hypergeometric functions.* Soviet Math. Dokl. 34 (1987) 9–13.

T. Hashimoto, K. Ogura, K. Okamoto, R. Sawae, and H. Yasunaga. *Kirillov–Kostant theory for coadjoint orbits I.* Hokkaido Math. J. 20 (1991), 355–405.

J. R. Klauder. Quantization is geometry, after all. Annals of Phys. 188 (1988), 120–141.

H. McKean and I. M. Singer. *Curvature and the eigenvalues of the Laplace operator.* J. Diff. Geo. 1 (1967), 85–110.

W. Rossmann. *Nilpotent orbital integrals in a real semisimple Lie algebra and representations of Weyl groups.* In *Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Actes du colloque en l'honneur de Jacques Dixmier.* A. Connes, et al., eds. Progess in Math. 92, Birkhäuser 1990, 263–287.*)

W. Rossmann. I*nvariant eigendistributions on a semisimple Lie algebra and homology classes on the conormal variety I: an integral formula; II: representations of Weyl groups*. J. Func. Anal., 96 (1991) 130–154, 155–192.*)

W. Rossmann. *Picard–Lefschetz theory for the coadjoint quotient of a semisimple Lie algebra.* (1994) Inventiones Math., to appear.*)

W. Rossmann. *Picard–Lefschetz theory and characters of a semisimple Lie algebra.* (1994) Inventiones Math., to appear.*)

E. Witten. Supersymmetry and Morse theory. J. Diff. Geo. 17 (1982) 661–692.

N. M. J. Woodhouse. Geometric Quantization. Second Edition. Clarendon Press, Oxford 1991.

*) On line at http://www.mathstat.uottawa.ca/Profs/Rossmann/

Department of Mathematics, University of Ottawa, Ottawa, Cananda
E-mail address: rossmann@uottawa.ca