2.5 The Lie Correspondence[1]

We shall prove in this section that the passage from a linear group to its Lie algebra sets up a one-to-one correspondence between *connected* linear groups and linear Lie algebras, the inverse being the passage from a linear Lie algebra to the group generated by its exponentials. This is the essence of *Lie Theory*, as Lie must have understood it, even though Lie's conception was on the one hand broader, in that he considered groups of transformations which were not necessarily linear, but on the other hand less complete, in that he took a local point of view. The global Lie correspondence is a refinement that is rather routine once the appropriate topological notions are available. The version of the Lie correspondence stated above follows from Satz 1 of Freudenthal (1941) and can be found in Bourbaki (1960) in the context of general Lie groups.

The essence of the Lie correspondence, in turn, is the Campbell-Baker-Hausdorff formula in its qualitative form, saying that in exponential coordinates the group multiplication is given by a bracket series and therefore completely determined by the Lie algebra, at least in a neighbourhood of the identity. (Actually, Lie himself might object, if he could: he was not fond of any such algebraic formulation of his theory, which he conceived of as being essentially geometric and analytic. Even today the Lie correspondence is often established without Campbell-Baker-Hausdorff; but the principle that "the Lie algebra determines the group" is certainly most simply and forcefully expressed by this formula.)

To succinctly state the Lie correspondence we use the following notation. The Lie algebra of a linear group G will be denoted $L(G)$ rather than g when it is necessary to bring out its dependence on G. Furthermore, we shall use the characterization of $L(G)$ in terms of exp:

$$L(G) = \{X \in M \,|\, \exp \tau X \in G \text{ for all } \tau \in \mathbb{R}\}.$$

On the other hand, given a linear Lie algebra g, we denote by $\Gamma(\mathrm{g})$ the linear group generated by expg:

$$\Gamma(\mathrm{g}) = \{\exp X_1 \cdot \exp X_2 \cdots \exp X_k | X_1, X_2, \cdots, X_k \in \mathrm{g}\}.$$

$\Gamma(\mathrm{g})$ is simply called the *linear group generated by* g.

Theorem 1. (The Lie Correspondence). *There is a one-to-one correspondence between connected linear groups G and linear Lie algebras g given by*

$$G \leftrightarrow g$$
$$\textit{if}$$
$$g = L(G) \textit{ or equivalently } G = \Gamma(g).$$

Proof $\Gamma(L(G)) = G$. This says that G is generated by $\exp L(G)$, which is (d) of Proposition 1, §2.4.

[1] This excerpt comes from my own manuscript, not the printed version. The formatting is primitive, but it should be free of at least those typographical errors pointed out by A. Knapp in his review.

Proof g=L(Γ(g)). Let g be a linear Lie algebra. Then Γ(g) is connected because any element $\exp X_1 \exp X_2 \cdots \exp X_k$ of Γ(g) can be joined to 1 by the continuous path $\exp \tau X_1 \exp \tau X_2 \cdots \exp \tau X_k$, $\tau \in \mathbb{R}$.

g$\subset$L(Γ(g)) is clear since $\exp(\tau X) \in \Gamma(\mathrm{g})$ for all $X \in$g. The point is to show that L(Γ(g)) $\subset$g. Let

$$U = \{X \in \mathrm{g} |\ \| X \| < \epsilon\} \text{ and } \bar{U} = \{X \in \mathrm{g} |\ \| X \| \leq \epsilon\}$$

for small $\epsilon > 0$. From Campbell-Baker-Hausdorff we know that the equation

$$\exp Z = \exp X \exp Y$$

defines a map $Z = C(X,Y)$ from $\bar{U} \times \bar{U}$ to a neighbourhood of 0 in g : $C(X,Y)$ is the Campbell-Baker-Hausdorff series. We set $V = C(U,U)$, $\bar{V} = C(\bar{U},\bar{U})$. Thus $\exp(V) = \exp(U)\exp(U)$. Since $C(X,Y)$ reduces to $C(0,Y) = Y$ for $X = 0$, the map $U \to$g, $Y \to C(X,Y)$ (X fixed) has a differential of rank = dimg at $Y = 0$, as is obvious if $X = 0$ and remains true for X in a neighbourhood of 0 by continuity. The Inverse Function Theorem implies that $C(X,U)$ is an open neighbourhood of X in g provided X is sufficiently close to 0 in g (which we may assume to be the case for $X \in \bar{V}$) and provided the ϵ defining U is sufficiently small. This we assume to be so.

The set $\bar{V} = C(\bar{U},\bar{U})$ is covered by the open sets $C(X,U)$, $X \in \bar{V}$ (because certainly $X \in C(X,U)$). Since $\bar{V}$ is a compact subset of g (being a continuous image of the compact set $\bar{U} \times \bar{U}$) already finitely many $C(X,U)$ cover $\bar{V}$, say $C(X_j,U)$, $j = 1,\cdots,$N, $X_j \in \bar{V}$. Write $\exp X_j = a'_j a''_j$ with a'_j, a''_j $\in \exp\bar{U}$ and apply exp to $\bar{V} \subset \bigcup_j C(X_j, U)$ to find that

$$\exp\bar{U}\exp\bar{U} \subset \bigcup_j a'_j a''_j \exp\bar{U},$$

even with $\bar{U}$ replaced by U on the right. Let $\{b_j : j = 1,2,\cdots\}$ be the (countable) set of all products of finite sequences from $\{a'_j, a''_j : j = 1, \cdots, N\}$ and write $(\exp\bar{U})^k$ for the set of k-fold products of elements of $\exp\bar{U}$. From the above inclusion one gets inductively that

$$(\exp \bar{U})^k \subset \bigcup_{j=1}^{\infty} b_j(\exp \bar{U})$$

for all $k \geq 1$. Hence $\Gamma(\mathrm{g}) = \bigcup_{k=1}^{\infty}(\exp\bar{U})^k$ is expressible as a countable union

$$\Gamma(\mathrm{g}) = \bigcup_{j=1}^{\infty} b_j(\exp \bar{U}) \tag{1}$$

for appropriate $b_j \in \Gamma(\mathrm{g})$. By Baire's Covering Lemma (proved below; see also the comment (b) thereafter):

some $b_j \exp U$ contains a neighbourhood of some point a_o in $\Gamma(g)$. (2)

Say

$$b_j \exp U \supset a_{\mathrm{o}} \exp \tilde{U}$$

where $\tilde{U} = \{\tilde{\mathrm{X}} \in \mathrm{L}(\Gamma(\mathrm{g})) : \| \tilde{\mathrm{X}} \| < \tilde{\epsilon}\}$ for some $\tilde{\epsilon} > 0$. Then

$$\exp \tilde{U} \subset c \exp U \tag{3}$$

where $c = a_{\mathrm{o}}^{-1} b_j$. This implies that for all $\tilde{X} \in \tilde{U}$

$$\exp \tilde{X} = c \exp X \tag{4}$$

with $X \in U$. Furthermore, for ϵ and $\tilde{\epsilon}$ sufficiently small, $X \in U$ and $\tilde{X} \in \tilde{U}$ will be arbitrarily close to 0, hence $c = \exp(-X)\exp(\tilde{X})$ will be close to 1 and the unique solution of (4) for $X \in U$ is

$$X = \log(c^{-1}\exp\tilde{X}). \tag{5}$$

Replacing $\tilde{X}$ by $\tau\tilde{X}$ with τ close to 0 in $\mathbb{R}$ we see that

$$\exp \tau\tilde{X} = c \exp X(\tau)$$

with $\tilde{X}(\tau) \in \tilde{U}$ depending differentiably on τ. Setting $\tau = 0$ we find $c = \exp(-X(0))$ and therefore

$$\exp \tau \tilde{X} = \exp(-X(0)) \exp X(\tau).$$

Differentiating this equation at $\tau = 0$ we obtain

$$\tilde{X} = \tfrac{1-\exp -\mathrm{ad}(X(0))}{\mathrm{ad}(X(0))} X'(0),$$

which lies in g since $X(0)$, $X'(0) \in$g and g is a Lie algebra. Thus the neighbourhood $\tilde{U}$ of 0 in L($\mathit{\Gamma}$(g)) is contained in g, hence L($\mathit{\Gamma}$(g)) $\subset$g as required.

It remains to show that (1) implies (2). This will follow from the following general lemma, which will be useful more than once.

Lemma 2: Baire's Covering Lemma. *Let $\{A_j\}$ be a countable family of subsets of G that cover G:*

$$G = \bigcup_{j=1}^{\infty} A_j \ . \tag{6}$$

Then the closure $\bar{A}_j$ of some A_j contains an open subset of G.

Comments. (a) One could as well take the A_j to be closed in the first place.

(b) If A_j contains an open, dense subset of its closure $\bar{A}_j$, as does a ball, for example, then some A_j itself will have to contain an open subset of G.

(c) The lemma (and its proof) hold in more general spaces, in particular in manifolds, to be defined later.

Proof (by contradiction). Assume *no* $\bar{A}_j$ contains an open subset of G. The part of G outside of $\bar{A}_1$ is then certainly non-empty, hence (being open) contains a closed coordinate-ball $\bar{B}_1$. For the same reason, the part of B_1 outside of $\bar{A}_2$ contains a closed coordinate-ball $\bar{B}_2$. Continuing in this way one gets a nested sequence of closed coordinate-balls in G :

$$\bar{B}_1 \supset \bar{B}_2 \supset \cdots \ .$$

The intersection of the $\bar{B}_j$ is non$-$empty, but lies outside of all $\bar{A}_j$, in contradiction to (6). (To see that $\bigcap_j \bar{B}_j$ is non$-$empty, one may consider any limit point of a sequence whose $j-$th term is taken from $\bar{B}_j$. Such a limit point always exists: since $\bar{B}_1$ may identified with a ball in $\mathbb{R}^m$, we are essentially dealing with a bounded sequence in $\mathbb{R}^m$.) QED.