Corrections and comments (posted Aug 2004)

Corrections labeled (2002) refer to the first printing, already incorporated in the reprinting of 2003. I am grateful for all correction and comments I receive!

p2, line 7 (2002). Replace $p_k = \frac{1}{k!}X(p_0)$ by $p_k = \frac{1}{k!}X^k(p_0)$.

p9, line 1. (Comment on exercise 11.) The formula is Weyl's group theoretic formulation of the Canonical Commutation Relations in quantum mechanics. (Weyl, 1950, §14). As he emphasizes (p.95), such matrices exist only in infinite dimensions (if $k \neq 0$). The formula is a formal consequence of the properties of the matrix exponential listed in Proposition 1

p11, line -3. Add equation number (A.8).

p19, lines -4, -2. Replace $a \neq -1$ by $a = -1$ and $0 < \alpha \neq 1$ by $-1 \neq \alpha < 0$.

p22, line 13. For 12(b) add the hypothesis "Suppose $LL \subset L$".

p24, line 14. Replace $\sum_{k=0}^{\infty}$ by $\sum_{k=1}^{\infty}$.

p24, line -4. Eq. (7) is better written as $\frac{dZ}{d\tau} = \frac{-\text{ad}\,Z}{\exp(\text{ad}\,Z)-1} + \frac{\text{ad}\,Z}{\exp(\text{ad}\,Z)-1}\exp(\text{ad}Z)Y$.

p25, line 3,4. Expand the explanation to read: "Apply (9) to the numerator $A = \text{ad}\,Z$ of the fractions in (7), divide by the denominator, and apply (8) to write $\exp(\text{ad}\,Z)Y = \exp(\tau X)Y$ in (7). Then compute as in the derivation of (2):"

p25, line 8. In (10) replace $i_k + 1!$ by $i_{k+1}!$

p28, line 6. Replace "onto" by "into".

p31, line 2 (2002). Replace $\mathbf{sl}(3, \mathbb{R})$ by $\mathbf{so}(3, \mathbb{R})$.

p32, eq.(9) (2002). Replace aY by $a\vec{Y}$.

p32, line -3. Replace $E \rightarrow$ by $\vec{E}$

p33, line -7 (2002). Replace $g \exp(X)c^{-1}$ by $c \exp(X)c^{-1}$

p33-35. Example 2 (up to QED p35) is done below in a somewhat different way.

p34, line 9 (2002). After "tr $X = 0$" add "and $X^* = -X$"

p35, line 1 (2002). Replace $p(c)e_{=}(a\overrightarrow{E_3 a^{-1}})$ by $p(c \exp(a\overrightarrow{E_3 a^{-1}})\, c^{-1})$

p37, line 7. Replace $\left[\begin{smallmatrix}\alpha\\1\end{smallmatrix}\right]$ by $\left[\begin{smallmatrix}\alpha\\0\end{smallmatrix}\right]$.

p44, line 7. Replace (a) by 11. (a).

p46-47. (2002) In the second paragraph of the proof, $a_k(\tau)$ is a smooth curve with $a_k(0) = 1$ and $a_k'(0) = X_k$ and a recurring notation of the form "$h :$s$\cdots \rightarrow M$" indicates [a partially defined map, as explained in the footnote on p12]. In the line after the display insert "and" before "$dg_0X = X$". In the next line, insert "to" after "complementary". In the next-to-last line on page 46, change "defined on" to "defined and". Online 4 on page 47, change "is a neighborhood" to "in a neighborhood".

p48 line 12. Change "lies" to "lie" and "$ga(\tau)$" to "$\mathfrak{g}a(\tau)$".

p50, line -3 (2002). Replace ad:g→g/(g) by ad:g→**gl**(g).

p52, #3 (2002). Replace $\{a \in G \mid$ by $\{a \in M \mid$.

p57 line -3.Change "Lemma 4a" to "Lemma 5".

p61 line 11. Change (b) to (a) twice.

p61 line -4. Replace $cd \exp X_k$ by $\cdots \exp(X_k)$.

p68-69. (2002; *from A. Knapp's review.*) On page 68 in the display before (1), change the left side to "$(\exp(\bar{U}))^k$". in (2), change $\exp \bar{U}$ to $\exp U$. Insert a tilde over "U" on the right side of the display after (2) and in the definition on the next line, also on the ϵ that occurs twice on that line. Insert a tilde once on the left side on (3), twice on the line of text afterward, and once on the left side of (4). At this point the Campbell-Baker-Hausdroff formula on the ambient group can be invoked to conclude (5) but with "$\tilde{X}$" on the right side in place of "X". Insert tildes twice on "X" in the next-to-last line page 68 and once on the left side of the bottom display, as well as on the left side of the first two displays on p69. In the last line of the proof, change "U" to "$\tilde{U}$".

p74 line 5. Change exercise 6 to read: "The only connected complex abelian group which is a compact subset of the matrix space is the trivial group $\{1\}$". [The assertion is correct as it stands, but harder to prove.]

p75 line 14. Change "problem 10" to "problem 9".

p86 line -6. For problem 6 (b) insert: Let SU(1,1) be the group of all complex 2×2 matrices of the form $\begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}$, $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$.

p86 line -1. For problem 8, insert: Assume known that $\mathrm{SO}(4, \mathbb{C})$ is connected.

p87 line 18. For problem 10, insert: See §3.1 for the definition of $\mathrm{Sp}(2, \mathbb{C})$.

p91, line -7 (2002). Omit the repeated "with respect to φ".

p132, line -6,-5. Replace "$U \cap U$" by "$U \cap \tilde{U}$", "(U, x) and (U, x)" by "(U, x) and $(\tilde{U}, \tilde{x})$" , "$x(U \cap U$" by $x(U \cap \tilde{U})$.
p133, line -11 Replace "MFLD 3" by "MFLD 2".

p142, line -10. Replace $\sum_j$ by $\sum_{jk}$.

p142-143. An alternative, more algorithmic, formulation of Problem 13 is the following.
Fix a basis $(e_1, \cdots, e_n)$ for E to represent its elements as column n vectors. Represent $P \in \mathrm{Gr}_m(E)$ by an $n \times m$ matrix $p = [p_1, \cdots, p_m]$ whose columns form a basis for P, unique up to right multiplication $p \mapsto pa$ by an invertible $m \times m$ matrix a. Show:
(a)Given $P \in \mathrm{Gr}(P)$ there is a permutation matrix $se_i = e_{s(i)}$ so that first m rows of sp are linearly independent. Then sp can be uniquely written in the form $sp = [{}^1_x]a$ where $x = x(P)$ is an $(n-m) \times m$ matrix, which is independent of the basis p chosen.
(b) As s runs over runs over all permutation of $(e_1, \cdots, e_n)$, the partially defined maps $P \mapsto x(P)$ from $\mathrm{Gr}(E)$ to $\mathbb{R}^{(n-m)\times m}$ or $\mathbb{C}^{(n-m)\times m}$ satisfy the axioms MFLD 1-4. (Specify the coordinate domains. Explain why it suffices to take permutations of the form $(e_{i_1}, \cdots, e_{i_m}, \cdots)$ where $i_1 < \cdots < i_m$ and the dots indicate the remaining e_is in their proper order as well.)

p147, line -9. Replace "$\alpha\beta, \gamma, \delta$" by "$\alpha, \beta$".

p148, line 18. Replace "$\{\exp X \cdots$" by "$\exp\{X \cdots$".

p157, line 6. Replace $\exp(0) = 0$ by $\exp(0) = 1$.

p157, line 17. Replace $(\exp X/pq)^p q$ by $(\exp X/pq)^{pq}$.

p160, line 4. Replace SU(2) by SU(1,1).

p160, line -3. Replace $2\pi\mathrm{i}$ by $\pi\mathrm{i}$ twice.

p160, line -2. Replace su(2) by su(1,1).

p192, line -13. Replace $\tilde{e}_j \otimes \tilde{f}_k$ by $\tilde{e}_j \tilde{\otimes} \tilde{f}_k$.

p201, line 8. Replace $g(ba)d$ by $g(ba)$.

§2.1 Example 2. (The special unitary group SU(2)).

This is the group

$$\mathrm{SU}(2) = \{a \in M_2(\mathbb{C}) \mid aa^* = 1\}$$

where a^* is the Hermitian adjoint (conjugate transpose) of a. Explicitly, the elements of SU(2) are of the form

$$a = \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}, \quad \alpha\bar{\alpha} + \beta\bar{\beta} = 1.$$

The group SU(2) is therefore just the 3–sphere S^3 in $M_2(\mathbb{C}) \approx \mathbb{R}^4$. We also set

$$\mathfrak{su}(2) = \{X \in M_2(\mathbb{C}) \mid X^* = -X, \, \mathrm{tr}X = 0\}.$$

This is a *Lie algebra*, i.e. real vector space matrices closed under the bracket operation. (It is not a complex vector space.)

Lemma 2.A. (a)Let $X \in M_2(\mathbb{C})$. $\exp(\tau X) \in$SU(2) for all $\tau \in \mathbb{R}$ if and only if $X \in \mathfrak{su}(2)$.
(b)exp : $\mathfrak{su}(2) \to$SU(2) is surjective.

Proof. (a) $(\exp X)^* = \exp(X^*)$ and $\det(\exp X) = e^{\mathrm{tr}X}$.
(b)By consideration of eigenvectors and eigenvalues one can check that any $a \in$SU(2) is conjugate to a matrix of the form

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \exp \begin{bmatrix} i\theta & 0 \\ 0 & -i\theta \end{bmatrix}.$$

This implies (b) . QED

The three dimensional real vector space $E = \mathfrak{su}(2)$ comes equipped with a positive definite inner product defined by $(X, X) = \frac{1}{2}\mathrm{tr}(X^*X)$. Explicitly, for $X \in \mathfrak{su}(2)$

$$X = \begin{bmatrix} i\xi_3 & -\xi_1 + i\xi_2 \\ \xi_1 + i\xi_2 & -i\xi_3 \end{bmatrix}, \quad \frac{1}{2}\mathrm{tr}(X^*X) = \xi_1^2 + \xi_2^2 + \xi_3^2.$$

We now take SO(3) to be the rotation group of this space $\mathfrak{su}(2)$. If $a \in$SU(2), then Ad(a) gives a linear transformation of $\mathfrak{su}(2)$ which belongs to SO(3), still denoted Ad(a). The map Ad:SU(2) $\to$SO(3) is a homomorphism of groups. i.e. preserves products: Ad(ab) =Ad(a)Ad(b). Similarly, $X \in \mathfrak{su}(2)$ gives ad$(X) \in \mathfrak{so}(3)$. The map ad: $\mathfrak{su}(2) \to \mathfrak{so}(3)$ is a *homomorphism of Lie algebras*, i.e. preserves brackets: ad$[X, Y]$ = [ad$X$,ad$Y$].

Lemma 2.B. (a)ad: $\mathfrak{su}(2) \to \mathfrak{so}(3)$ is bijective.
(b)Ad:SU(2) $\to$SO(3) is surjective with kernel $\{\pm 1\}$.

Proof. (a)Since ad: $\mathfrak{su}(2) \to \mathfrak{so}(3)$ is a linear map between spaces of the same dimension (namely 3), it suffices to show that its kernel is zero. Suppose $X \in \mathfrak{su}(2)$ satisfies ad$(X)\mathfrak{su}(2) = 0$. Any $Z \in M_2(\mathbb{C})$ is of the form $Z = U + iV$ where U, V are skew Hermitian, namely $U = \frac{1}{2}(Z - Z^*)$, $V = \frac{1}{2i}(Z - Z^*)$. Hence $M_2(\mathbb{C}) = \mathfrak{su}(2) + i\mathfrak{su}(2) + \mathbb{C}1$ and ad$(X)\mathfrak{su}(2) = 0$ implies ad$(X)M_2(\mathbb{C}) = 0$, i.e. X commutes with all matrices. Hence X is a scalar matrix and tr$X = 0$ implies $X = 0$.
(b)The surjectivity of Ad:SU(2) $\to$SO(3) follows from the surjectivity of the exponential maps and of ad: $\mathfrak{su}(2) \to \mathfrak{so}(3)$:
Ad$(\exp \mathfrak{su}(2)) = \exp(\mathrm{ad}\, \mathfrak{su}(2) = \exp \mathfrak{so}(3)$ =SO(3).
Suppose $a \in$SU(2) belongs to the kernel of Ad:SU(2) $\to$SO(3), i.e. Ad$(a)X = X$ for all $X \in \mathfrak{su}(2)$. As in the proof of (a) this implies that $a \in$SU(2) commutes with all $Z \in M_3(\mathbb{C})$. Hence a is a scalar matrix and $\det(a) = 1$ implies $a = \pm 1$. QED

Part (b) of the lemma says that the group homomorphism SU(2) $\to$SO(3) is a *double covering* in the sense that each element a of SO(3) has exactly two preimages $\pm\tilde{a}$ in SU(2). The 'inverse map' SO(3) $\rightsquigarrow$SU(2) is double-valued; it associates *two* unitary transformations $\psi \mapsto \pm\tilde{a}\psi$ of $\mathbb{C}^2$ to each $a \in$SO(3). This is the famous *spin representation* of SO(3). The elements ψ of $\mathbb{C}^2$ are referred to as *spinors* in this context, as in Weyl's *"Theory of Groups and Quantum Mechanics"* of 1931.