Matrix Groups: An Introduction to Lie Group Theory. By Andrew Baker. Springer-Verlag, 2002, (Springer Undergraduate Mathematics Series), xi + 330 pp., ISBN 1-85233-470-3, $39.95.

Lie Groups: An Introduction Through Linear Groups. By Wulf Rossmann. Oxford University Press, 2002, (Oxford Graduate Texts in Mathematics vol. 5), x + 265 pp., ISBN 0-19-859683-9, $60.00.

Reviewed by **A. W. Knapp**

The elementary theory of Lie[1] groups has something in common with precalculus: One wants to get past it in order to get on with the beautiful and powerful theory that follows, as well as the applications. With precalculus the theory that follows is built around calculus and its fundamental theorem, extending into calculus of several variables and having profound applications through the subject of differential equations. With Lie groups the theory that follows is due to Élie Cartan (1869–1951) and Hermann Weyl (1885–1955) and concerns compact Lie groups and their representations, as well as real and complex semisimple Lie algebras and Lie groups; the Cartan–Weyl theory introduces one to the exceptional Lie algebras and their remarkable manifestations throughout mathematics, it extends via the work of Harish-Chandra (1923–1983) and others into representation theory and harmonic analysis, and it has applications in many branches of mathematics.

Another thing in common for precalculus and elementary Lie theory is that the beautiful and powerful mathematics that follows—calculus in the first case and Cartan–Weyl theory in the second case—brings together several branches of mathematics, serving as a reminder that mathematics is a unifying force rather than a springboard to greater and greater specialization. In fact, because of its great beauty, its wide applicability, and its unifying effect, Cartan–Weyl theory is on my own personal list of What Every Young Mathematician Should Know.

If Cartan–Weyl theory is to be accessible to every young mathematician, one has to include it earlier in the curriculum than is done at present in the United States, where it might occur as an elective for second-year graduate students. In turn, some provision has to be made to teach elementary Lie theory earlier and more rapidly than now, with fewer prerequisites. A number of people who share my sentiments have given considerable thought to how this miracle might be accomplished, and the two books under review are the authors' contributions to this effort. Both books purport to be for "undergraduates," and we return to them in a moment.

There is another difficulty with teaching elementary Lie theory at its current normal pace. For a point of reference, consider the effect of lingering too long over precalculus. We know what happens. Large numbers of students end up with a feeling of having seen a jumble of unrelated topics that are going nowhere. Many who once visualized that they would be taking calculus shortly afterward instead make precalculus into their final mathematics course. And potentially they carry a bad attitude toward mathematics with them throughout their lives.

I have the sense that, in a similar way but on a smaller scale, lingering too long over the elementary theory of Lie groups can dampen the spirit and prevent people from ever getting to the beautiful and powerful Cartan–Weyl theory that comes

[1]Pronounced "Lee."

afterward.

ELEMENTARY LIE THEORY. Let us understand, then, what elementary Lie theory is. The subject has evolved considerably since the days of Sophus Lie (1842–1899), but its basic shape stabilized for the most part upon the publication in 1946 of the pioneering book [4] of Claude Chevalley (1909–1984). A short 1930 book [3] by Cartan, equally pioneering, had paved the way. If we are willing to ignore a number of details, the subject has two ingredients and three correspondences. The two ingredients are Lie groups and Lie algebras.

A Lie group is first of all a group, and in addition it has the structure of a smooth manifold; these two structures are related in that multiplication and inversion are required to be smooth mappings. To understand this definition fully, one needs to know something about abstract groups, point-set topology (at least for metric spaces), topological groups, and smooth manifolds and maps (including the use of the Inverse and Implicit Function Theorems). There are some easily accessible examples, such as Euclidean space under addition and the real general linear group of a particular size, i.e., the space of nonsingular real square matrices with matrix multiplication as group operation. But these are not so illuminating, and the construction of illuminating examples requires hard work. To a student it may seem that meaningful examples are being withheld for what feels like too long a time.

The Lie algebras of interest are finite-dimensional real vector spaces with a multiplication law that satisfies certain properties. A simple but important example is the vector space of all square matrices of a particular size with multiplication law given by $[A, B] = AB - BA$. Moreover, any vector subspace that is closed under the product operation is again a Lie algebra; for instance the subspace of skew-symmetric square matrices of a particular size has this closure property. To understand this definition fully requires only a little linear algebra, including acquaintance with linear transformations. Examples abound.

For each of the three correspondences, there is a direct part and an inverse part. The first direct correspondence is that to each Lie group corresponds a Lie algebra, specifically by passage to "left-invariant vector fields." To understand this notion fully requires understanding tangent spaces, computing with vector fields, and working with bracket products of vector fields. Examples of the first correspondence are limited by how many examples one knows of Lie groups. One can at least see that the Lie algebra of the general linear group of a particular size can be identified with the vector space of all square matrices of that size with bracket product $[A, B] = AB - BA$.

The second direct correspondence is that to each subgroup of a certain kind corresponds a Lie subalgebra. The subgroups in question are themselves Lie groups, and one is again hindered in giving examples by not knowing many specific Lie groups. However, one can construct one-parameter subgroups with less than full-strength pain, and these provide examples; what is needed for their construction is the standard existence-uniqueness theorem for systems of ordinary differential equations. Once this work has been done, a little more effort will allow one to prove that a closed subgroup of a Lie group is a Lie group; this is a theorem of John von Neumann (1903–1957) for subgroups of general linear groups and is due to Cartan [3] for arbitrary Lie groups. At this stage one has an extensive supply of examples—rotation groups, for example, and many others.

The third direct correspondence is that to each smooth homomorphism of Lie groups, there corresponds a homomorphism of Lie algebras. To understand this definition fully requires only a knowledge of abstract homomorphisms, the notion of smoothness of a map, and the notion of the differential of a map at a point; these are likely to be learned at the same time as abstract groups, smooth manifolds, and tangent spaces. Examples are not too hard to come by if the theorem of von Neumann and Cartan is known; one can readily construct many examples of interesting homomorphisms between Lie groups of matrices.

The inverse part for each correspondence tells the extent to which the direct part is one-to-one and is onto something easy to describe. The identity component of a Lie group is always an open set, as well as a subgroup, and the Lie algebra cannot distinguish between the whole group and the identity component. Thus connectivity of the group or subgroup always has to be assumed in discussing the inverse correspondences.

For the first correspondence two connected Lie groups with isomorphic Lie algebras are not necessarily isomorphic, but they must have covering groups that are isomorphic. To understand this statement fully, one has to know something about covering groups or covering spaces. The statement that the correspondence is onto something easy to describe asserts that every (finite-dimensional real) Lie algebra is isomorphic to the Lie algebra of some Lie group; since 1930 this result has been known as Lie's Third Theorem, and its proof is beyond the scope of elementary Lie theory.

The second correspondence is the key one, and this is the big new result in Chevalley's book. The theorem of von Neumann and Cartan shows that connected closed subgroups of a given Lie group correspond to subalgebras of the Lie algebra, and they do so in one-to-one fashion. But the well-known example of a two-dimensional torus shows that one should consider a wider class of connected subgroups than the closed ones. The torus can be regarded as the set of pairs $(e^{i\theta_1}, e^{i\theta_2})$ with θ_1 and θ_2 real. The group operation is multiplication in each coordinate, which can be viewed as addition modulo 2π of the exponent in each coordinate. The Lie algebra as a vector space can be identified with the tangent space at the identity element of the group. The tangent space at the identity for the torus can be identified in turn with the set of all pairs (θ_1, θ_2), and the bracket operation is identically 0. The closed one-dimensional subgroups of the torus are given by the sets where $\theta_1 = 0$ and where $\theta_2 = r\theta_1$, r being any rational number. These correspond to the lines $\{(0, \theta)\}$ and $\{(\theta, r\theta)\}$, i.e., the lines with rational slope. To get a good correspondence, one wants to realize lines of arbitrary slope in the tangent space as in the image, and this means that one needs to consider the set with $\theta_2 = r\theta_1$, r being irrational, as an allowable subgroup. This subgroup is dense and not closed. So the "right" definition of allowable subgroup has to include certain nonclosed subgroups as well as the closed ones. Chevalley found the right definition. The connected subgroups that he worked with were called "analytic subgroups"; their definition will not concern us. With the definition in hand, Chevalley proved that the correspondence of analytic subgroups to subalgebras of the Lie algebra is one-to-one and onto. This is a difficult theorem. It has a local part involving partial differential equations that comes down to a theorem of Frobenius, and it has a global part involving tricky point-set topology.

The third correspondence associates homomorphisms of Lie algebras to homomorphisms of Lie groups, and it does so in a one-to-one fashion if the domain group is connected. The “onto” statement is more subtle: a homomorphism of Lie algebras lifts to a homomorphism of Lie groups provided the domain is connected *and simply connected*. A clever idea in [4] largely reduces this theorem to the result that attaches an analytic subgroup to a Lie subalgebra. Namely, let G and H be the given groups. The Lie algebra of $G \times H$ is the direct sum of the Lie algebras of G and H, and one forms in it the graph of the given homomorphism; this is a Lie subalgebra. Let $\widetilde{G}$ be the corresponding analytic subgroup of $G \times H$. The projection of $\widetilde{G}$ to the first coordinate of $G \times H$ is shown to be a covering map and has to be an isomorphism since G is simply connected. Thus G maps into $\widetilde{G}$; the composition of this map with the projection of $\widetilde{G}$ to the second coordinate of $G \times H$ is the desired homomorphism of groups.

The preceding description of the content of elementary Lie theory makes clear that the subject involves extensive prerequisites from a variety of areas. On the one hand, students who learn elementary Lie theory in this traditional way get to see relationships among diverse areas of mathematics. On the other hand, it takes them a long time, and they may get discouraged in the process. In that case they may never get to the beautiful mathematics that follows.

CARTAN–WEYL THEORY. The purpose of trying to move elementary Lie theory to an earlier place in the curriculum is to make room for Cartan–Weyl theory, which is on that list of What Every Young Mathematician Should Know.

To understand a little about Cartan–Weyl theory, it is helpful to have some background. We continue to insist that our Lie algebras are finite-dimensional, but we shall now allow them to have real or complex scalars. In either case an *ideal* in a Lie algebra is a Lie subalgebra such that $[X, Y]$ is in the ideal if X is in the ideal and Y is in the whole Lie algebra. A *simple* Lie algebra is a nonzero Lie algebra whose only ideals are 0 and the whole Lie algebra, except that, by convention, a one-dimensional Lie algebra is not considered to be simple. In the study of Lie algebras one's attention soon focuses on the simple ones and their direct sums, which are called *semisimple*.

In his 1894 thesis, Cartan, correcting and improving earlier work of Wilhelm Killing (1847–1923), classified the simple Lie algebras when the scalars are complex. Among them, there are some infinite classes: the complex square matrices of trace 0 of each size ≥ 2, the complex skew-symmetric square matrices of each size ≥ 3 except 4, and the complex square matrices X of size $2n$ satisfying $X^{\mathrm{tr}} J + JX = 0$, where J is the $2n$-by-$2n$ matrix given in block form by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and where $n \geq 2$. These were all known to Lie. The subject of Lie groups would be comparatively easy but for the existence of five exceptional simple Lie algebras, of respective dimensions 14, 52, 78, 133, and 248. These five Lie algebras keep popping up in unexpected places in mathematics and have intrigued people ever since Cartan first constructed them.

A *representation* of a Lie algebra is a homomorphism φ into the Lie algebra of all linear transformations on a complex vector space V, the bracket rule being $[A, B] = AB - BA$. The representation is irreducible if it is not zero and there is no nontrivial subspace of V left invariant by all the transformations $\varphi(X)$ for

X in the Lie algebra. There is a natural definition of isomorphism. In a 1913 paper Cartan classified, up to isomorphism, the irreducible representations of any complex semisimple Lie algebra. The relevant theorem is known as the Theorem of the Highest Weight. As might be expected, the tools were completely algebraic.

In view of the correspondence between Lie groups and Lie algebras, it is natural to define a *representation* of a Lie group to be a smooth homomorphism of the group into the Lie group of all invertible linear transformations on a complex vector space. Oddly, however, representations of (infinite) Lie groups were not considered by anyone until the 1920s. Issai Schur (1875–1941) began the study of representations of the rotation and unitary groups, and in short order, Weyl classified, up to isomorphism, the irreducible representations of all compact connected Lie groups. His methods were direct and analytic, and the relevant theorem is a version of the Theorem of the Highest Weight. Weyl went on to establish a character formula and a dimension formula for his representations.

Cartan discovered the relationship between the two versions of the Theorem of the Highest Weight. Here it is, in part: Any compact connected Lie group is the commuting product of the commutator subgroup and the identity component of the center. The commutator subgroup is compact and connected, and its Lie algebra is semisimple (with real scalars). The identity component of the center has only a minor effect on matters, and so one might as well assume that the given compact connected Lie group coincides with its commutator subgroup and therefore is semisimple in the sense of having a semisimple Lie algebra. The complexification of this semisimple Lie algebra will be one to which the Cartan theorem applies, and then the two versions of the theorem come to roughly the same thing if one takes into account the Lie correspondence between homomorphisms of groups and homomorphisms of their Lie algebras. Conversely, for any complex semisimple Lie algebra V of complex dimension n, there is a real subspace U of real dimension n such that $V = U + iU$ and U is the Lie algebra of some compact connected semisimple Lie group. Thus Weyl's theorem applies, and again the two versions of the theorem come roughly to the same thing. To get a precise match, one invokes a further theorem of Weyl—that if U is a compact connected semisimple Lie group, then any connected Lie group having an isomorphic Lie algebra is itself compact; otherwise said, the universal covering group of a compact connected semisimple group is compact. For a simply connected compact connected Lie group, the irreducible representations correspond exactly, via the Lie correspondence, to the irreducible representations of the complexified Lie algebra.

This is the essence of Cartan–Weyl theory. It has applications in real and complex analysis, algebraic number theory, algebraic geometry, topology, differential geometry, differential equations, and mathematical physics.

BOOKS INTRODUCING LIE THEORY VIA LINEAR GROUPS. A way to get into elementary Lie theory more quickly with fewer prerequisites is to concentrate on *linear groups*, by which is meant groups of nonsingular real or complex square matrices. Or at least one can introduce linear groups first and continue with more general Lie groups later.

Most authors who follow this approach work with closed linear groups, which we know are going to be Lie groups because of von Neumann's theorem. If G is such a group, the Lie algebra of G can be defined immediately and concretely as the set $\mathfrak{g}$ of all matrices $c'(0)$, where $t \mapsto c(t)$ is a smooth curve of matrices with $c(t)$ in G

for all t and with $c(0)$ equal to the identity. It is fairly easy to see that $\mathfrak{g}$ is a Lie algebra, and in special cases such as when G is a rotation group, one can readily compute $\mathfrak{g}$. For any square matrix X, if e^X denotes the usual power series for the exponential function, $e^X = \sum_{n=0}^{\infty} X^n/n!$, then $t \mapsto e^{tX}$ is a smooth curve whose value at $t = 0$ is the identity matrix and whose derivative at $t = 0$ is equal to X. From the fact that G is closed, it takes only a page to see that if X is in $\mathfrak{g}$, then e^{tX} is in G for all real t. Consequently $\mathfrak{g}$ may also be defined as

$$\{X \mid e^{tX} \text{ is in } G \text{ for all real } t\}.$$

In turn, it is then not very hard, using only sophomore mathematics and the definition of smooth manifold, to prove von Neumann's theorem that G is indeed a Lie group. Godement [7] and Howe [9] give proofs along these lines, and a shorter proof, worked out with D. A. Vogan, appears in Chapter I of [11]. An improved version of the latter proof appears in the Introduction of the second edition of [12]. At any rate, once one has this theorem, many examples of Lie groups are at hand, and it is easy to establish the direct parts of the Lie correspondence to the extent that they apply to closed linear groups.

At least twenty-five books on elementary Lie theory have been written since Chevalley's in 1946, and many of them treat some or all of Cartan–Weyl theory; in addition, there are a number of other books that skip elementary Lie theory and begin with Cartan–Weyl theory. Some of the twenty-five—including ones by Freudenthal and de Vries [7], M. L. Curtis [5], Godement [8], Sattinger and Weaver [13], and Ise and Takeuchi [10]—treat closed linear groups before (if ever) defining general Lie groups. The books by Baker and Rossmann under review are in this category.

The competitors all have disadvantages from the point of view of the present review: Freudenthal–de Vries [7] is too hard, and its notation is difficult to absorb. Curtis [5] is too easy, written more as a book to exercise one's undergraduate skills than to teach elementary Lie theory. Godement [8] is nice and is thorough, but one has to go through two lengthy chapters of prerequisites before coming to von Neumann's theorem; in addition, American undergraduates may be less than pleased that it is in French. Sattinger–Weaver [13] is written with physicists in mind, and its standards of precision probably will not please mathematicians. Part I of Ise–Takeuchi [10] contains material about elementary Lie theory, some parts of Cartan–Weyl theory, and some material beyond those topics, but it is only about a hundred pages total and is not sufficiently self contained.

The advertising for the books under review suggests that the books are at a level suitable for advanced undergraduates, but "undergraduates" is to be understood in the British or Canadian sense. For students in the United States, the level is more at the first-year graduate level.

BAKER'S BOOK. Baker is a topologist, and the preface indicates that his book is an outgrowth of a course he taught while on leave one semester. He says that the beginning of the book is influenced by Curtis [5], and it appears that some later material is influenced by the wonderful book [1] by the late topologist J. F. Adams. These influences are noticeable but not large.

Baker's preface mentions three chains of chapters running through the text. The main one from the point of view of this review consists of Chapters 1, 2, 3, and 7 and is the elementary course in Lie theory. Closed linear groups (called "matrix groups" in the text) are defined in Chapter 1, the exponential of a matrix is in Chapter 2, and the direct parts of the three Lie correspondences are in Chapter 3. Largely, however, the first three chapters are reviewing prerequisite material and providing exercises for the reader in working with matrices. The proof of the von Neumann theorem following Howe [9] appears in Chapter 7 shortly after the definitions of manifolds and Lie groups. The direct parts of the three Lie correspondences for Lie groups and their closed subgroups are in Chapter 7 as well. The chapter contains one particularly nice example, namely, the quotient of the group of real matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

by the discrete subgroup in which $x = y = 0$ and z is any integer, and he proves that this Lie group is not isomorphic to a group of matrices.

Elementary Lie theory has a trap for topologists who like to define submanifolds to be closed, and Baker fell into the trap. In his definition of continuous homomorphism between closed linear groups, he insists that such a homomorphism have closed image, and therefore the inclusion into the torus of one of the lines of irrational slope is not for him a continuous homomorphism. He will end up with trouble when he confronts analytic subgroups. Adams did not fall into this trap.

The best of the three chains of chapters is the middle one, consisting of Chapters 4, 5, 6, 8, and 9. For the most part these chapters do computations with specific examples, establishing canonical forms and other structure theorems for certain classes of groups. They also introduce certain important homogeneous spaces, such as Grassmannians. Chapter 6, which is about Lorentz groups, is marred by the fact that its canonical-form theorem about Lorentz groups is false. If the theorem were valid, a Lorentz group could contain no matrix other than the identity whose only eigenvalue is 1. However, $\mathrm{Lor}(2,1)$ contains the element

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1/2 & 1/2 \\ -1 & -1/2 & 3/2 \end{pmatrix},$$

which has 1 as its only eigenvalue.

The other chain of chapters is 10, 11, 12 and concerns a part of Cartan–Weyl theory. The writing in these chapters appears to be a sketchy summary of part of Adams [1], and I think that a person would be better off by reading this material in [1].

I do not like Baker's book. A serious problem is that there is no mention at all of the inverse Lie correspondences. A lecturer could not readily fix this problem by adding supplementary material because continuous homomorphisms have been defined too narrowly.

There are other nonstandard definitions as well, such as for "inner product," and one cannot expect to remember some of the nonstandard multiple-letter symbols for groups. A blunder occurs on page 182 when Baker wants to define separability of a topological space as referring to a countable base but instead says, "A topological space X is *separable* if it has a countable open covering."

The prerequisite material at the beginning is uneven, unclear, and occasionally sloppy. For example, after defining "compact" in a Euclidean space as closed and bounded, he transfers this statement to matrix space in Proposition 1.19 but precedes it with the comment, "Our next result is standard for metric spaces." What he apparently means is that the equivalence of closed and bounded with the properties in Proposition 1.19 extends to all metric spaces; one is not to infer anything about general compactness, which is defined a few lines later. He refers to a particular situation on page 19 as a group action but does not define group action until page 37. He defines "curve" but not "path" and uses both. His definition of "hyperplane reflection" on page 23 is internally inconsistent unless he means something unusual by "represents." And so on.

The book has enough errors, misleading statements, and typographical errors that it would be really hard to read the book for independent study. Even a lecturer using the book would not have an easy time. It is an important job for mathematicians to make Lie theory more widely accessible, and Baker is to be applauded for trying. But this book does not represent progress toward that goal.

ROSSMANN'S BOOK. Rossmann's book is a gem. The book, as he says in the preface, results from a "trail of lecture notes" left from teaching elementary Lie theory a number of times. These notes were combined and edited, and they eventually became the book. For the students he was teaching, he was quite interested in reducing the list of prerequisites considerably, and working with linear groups enabled him to get by with only linear algebra, advanced calculus, and the rudiments of group theory. He adds, "a desire to shorten the list of prerequisites is not the only reason for the point of view taken; the restriction to linear groups seems desirable to me, even if the prerequisites are available; for it puts into focus from the beginning the essential aspects of the theory, free of technicalities."

Rossmann's treatment of elementary Lie theory for linear groups occupies the first two chapters and takes ninety pages. In those chapters he establishes, within the context of linear groups, all the Lie correspondences and their inverses except for Lie's Third Theorem. Only after those ninety pages does he introduce the general definition of Lie group. His development has some elements that are new to me, and I regard those elements as a breakthrough in making elementary Lie theory more widely accessible.

Chapter 1 is about thirty pages long. It works at length with the exponential for matrices, giving examples, and then computes the derivative for all t of $\exp X(t)$ for an arbitrary curve $X(t)$ of matrices. Armed with this, Rossmann derives the Campbell-Baker-Hausdorff formula, following the argument in [6]; this formula expresses a matrix Z in terms of iterated brackets of X and Y when X, Y, and Z are small and $e^Z = e^X e^Y$.

In Chapter 2 Rossmann works with absolutely arbitrary linear groups G, with no restriction on the topology. He defines the Lie algebra $\mathfrak{g}$ of G, as was done earlier in this review, to be the set of derivatives at $t = 0$ of smooth curves of matrices that take values in G for all t and that are the identity at $t = 0$. As always, $\mathfrak{g}$ is indeed a Lie algebra. On page 46 comes the remarkable result that e^X is in G whenever X is in $\mathfrak{g}$, without any topological assumption on G.

The proof of this result is correct, but this is one of two places in the book where there are undefined terms and confusing misprints. Because of the importance of this result to Rossmann's approach, I shall list the corrections: In the second paragraph of the proof, $a_k(\tau)$ is a smooth curve with $a_k(0) = 1$ and $a'_k(0) = X_k$, and a recurring notation of the form "$h : \mathsf{s} \cdots \to M$" indicates what would usually be written as "$h : \mathsf{s} \to M$." In the line after the display, insert "and" before "$dg_0 X = X$." In the next line, insert "to" after "complementary." In the next-to-last line on the page, change "defined on" to "defined and." On line 4 of page 47, change "is a neighborhood" to "in a neighborhood."

Next Rossmann introduces an intrinsic topology on G, often finer than the relative topology. The identity component will be open, and there may be uncountably many components. A neighborhood basis of a in G is defined to consist of the sets $\{ae^X \mid X \in \mathsf{g} \text{ and } \|X\| < \epsilon\}$ as ϵ varies. Using this definition, he introduces a notion of coordinates and does everything with them except call them a manifold.

The "analytic subgroups" in the sense of Chevalley will be the subgroups that are connected in their intrinsic topology. The reader has a few specific examples at this stage because Rossmann has shown that the image of the exponential map generates the whole group for some particular linear groups. The von Neumann theorem would show that any closed subgroup that is connected in its ordinary topology is connected in its intrinsic topology. That theorem could well be proved now, in order to provide further examples, and I would have preferred that; but it does not appear until twenty-five pages later.

The final step of this construction is to prove the direct and inverse part of the second Lie correspondence, and the argument is beautiful. The way that the assumption of closure under brackets enters for g is through the Campbell-Baker-Hausdorff formula, and the main analytic tool is the Baire Category Theorem; the latter is proved in the text after the proof of the second correspondence. The proof as a whole of the correspondence is correct, but this is the other of the two places in the book where misprints cloud what is happening.

Most of the misprints involve missing auxiliary symbols. On page 68 in the display before (1), change the left side to "$(\exp \bar{U})^k$." In (2), change $\exp \bar{U}$ to $\exp U$. Insert a tilde over "U" on the right side of the display after (2) and in the definition on the next line, also on the ϵ that occurs twice on that line. Insert a tilde once on the left side on (3), twice on the line of text afterward, and once on the left side of (4). At this point the Campbell-Baker-Hausdorff formula on the ambient group can be invoked to conclude (5) but with "$\widetilde{X}$" on the right side in place of "X." Insert tildes twice on "X" in the next-to-last line of page 68 and once on the left side of the bottom display, as well as on the left side of the first two displays on page 69. In the last line of the proof, change "U" to "$\widetilde{U}$."

In treating the third Lie correspondence, Rossmann follows Chevalley [4] in avoiding the introduction of the fundamental group and the construction of covering spaces, but Rossmann's argument for the rest is tidier. A covering $\widetilde{G} \to G$, in the context of Lie groups, is for Rossmann a smooth homomorphism of $\widetilde{G}$ onto G with discrete kernel. To pass to a homomorphism between analytic groups G and H when a homomorphism is given between their Lie algebras, Rossmann begins with the clever idea from [4] that was mentioned earlier in this review: he forms a subgroup $\widetilde{G}$ of $G \times H$ and obtains, by restricting the projection maps on $G \times H$ to $\widetilde{G}$, smooth homomorphisms $\widetilde{G} \to G$ and $\widetilde{G} \to H$. The first of these projection maps exhibits $\widetilde{G}$ as a covering of G, and the second is the required smooth homomorphism of groups. The group $\widetilde{G}$ is a linear group, being a subgroup of the linear group $G \times H$.

In response to an inquiry, Rossmann explained how he discovered all this. He was reading deep inside the Bourbaki treatment of Lie groups and came upon a result[2] that seemed to provide him with new structure that he had not seen before. He wondered what the structure would be like if specialized to linear groups and was led to the foregoing approach.

Chapter 3 is about the classical complex semisimple Lie groups and does the Cartan part of some of Cartan–Weyl theory for them; the reader gets to see a full-fledged definition of fundamental group at this point, together with computations of fundamental groups.

Chapter 4 is about manifolds, homogeneous spaces, and general Lie groups. The general exponential map is constructed from integral curves in a standard way, and Rossmann shows how the general Campbell-Baker-Hausdorff formula and similar results can be proved by easy adaptations of his arguments for linear groups.

The last hundred pages do the Weyl part of the Cartan–Weyl theory for unitary groups, developing all the necessary additional tools. Toward the end, the book proves the Borel–Weil Theorem for unitary groups, providing a concrete realization of each of the irreducible representations.

Rossmann's book is a pioneering treatment of elementary Lie theory, in the same sense as the books of Cartan and Chevalley. It takes a big step toward making elementary Lie theory widely accessible to mathematicians.

References

1. J. F. Adams, *Lectures on Lie Groups*, W. A. Benjamin, New York, 1969; reprinted by University of Chicago Press, Chicago, 1982.
2. N. Bourbaki, *Eléments de Mathématique, Groupes et Algèbres de Lie: Chapitres II et III*, Actualités Scientifiques et Industrielles, vol. 1349, Hermann, Paris, 1972.
3. É. Cartan, *La Théorie des Groupes Finis et Continus et L'Analysis Situs*, Mémorial des Sciences Mathématiques, vol. 42, Gauthiers-Villars, Paris, 1930. (= *Œuvres Complètes*, vol. I, pp. 1165–1225).
4. C. Chevalley, *Theory of Lie Groups, I*, Princeton University Press, Princeton, 1946; 15th printing, 1999.
5. M. L. Curtis, *Matrix Groups*, Springer-Verlag, New York, 1979; 2nd ed., 1984.
6. J. J. Duistermaat and J. A. C. Kolk, *Lie Groups*, Springer-Verlag, Berlin, 2000.
7. H. Freudenthal and H. de Vries, *Linear Lie Groups*, Academic Press, New York, 1969.
8. R. Godement, *Introduction à la Théorie des Groupes de Lie, I, II*, Université de Paris VII, Paris, 1982.
9. R. Howe, Very basic Lie theory, this MONTHLY **90** (1983) 600–623; correction, ibid. **91** (1984) 247.
10. M. Ise and M. Takeuchi, *Lie Groups, I, II*, Translations of Mathematical Monographs, vol. 85, American Mathematical Society, Providence, 1991 (English translation of 1984 edition of first volume by both authors in Japanese and of 1978 edition of second volume by second author, the original Japanese edition of the first volume by the first author alone being dated 1977).
11. A. W. Knapp, *Lie Groups, Lie Algebras, and Cohomology*, Princeton University Press, Princeton, 1988.
12. A. W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996; 2nd ed., 2002.
13. D. H. Sattinger and O. L. Weaver, *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*, Springer-Verlag, New York, 1986; reprinted with corrections, 1993.

[2]Proposition 9 of [2, chap. III, sec. 4, no. 5]. The result in Rossmann's book that is closest to this is Proposition 1 on page 55.

State University of New York at Stony Brook
Current address: 81 Upper Sheep Pasture Road
East Setauket, NY 11733, USA
aknapp@math.sunysb.edu