Trimmed sums of long-range dependent moving averages

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Abstract

In this paper we establish asymptotic normality of trimmed sums for long range dependent moving averages. Our results extend those of Ho and Hsing [12]

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1 Introduction

Let $\{\epsilon_i, i \geq 1\}$ be a centered sequence of i.i.d. random variables. Consider the class of stationary linear processes

$$X_i = \sum_{k=0}^{\infty} c_k \epsilon_{i-k}, \quad i \geq 1. \tag{1}$$

We assume that the sequence c_k, $k \geq 0$, is regularly varying with index $-\beta$, $\beta \in (1/2, 1)$. This means that $c_k \sim k^{-\beta} L_0(k)$ as $k \to \infty$, where L_0 is a slowly varying function at infinity. We shall refer to all such models as long range dependent (LRD) linear processes. In particular, if the variance exists (which is assumed throughout the whole paper), then the covariances $\rho_k := \mathrm{E} X_0 X_k$ decay at the hyperbolic rate, $\rho_k = k^{-(2\beta-1)} L(k)$,

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where $\lim_{k\to\infty} L(k)/L_0^2(k) = B(2\beta-1, 1-\beta)$ and $B(\cdot,\cdot)$ is the beta-function. Consequently, the covariances are not summable (cf. [9]).

Assume that X_1 has a continuous distribution function F. For $y \in (0,1)$ define $Q(y) = \inf\{x : F(x) \geq y\} = \inf\{x : F(x) = y\}$, the corresponding (continuous) quantile function. Given the ordered sample $X_{1:n} \leq \cdots \leq X_{n:n}$ of $X_1, \ldots, X_n$, let $F_n(x) = n^{-1}\sum_{i=1}^n 1_{\{X_i \leq x\}}$ be the empirical distribution function and $Q_n(\cdot)$ be the corresponding left-continuous sample quantile function, i.e. $Q_n(y) = X_{k:n}$ for $\frac{k-1}{n} < y \leq \frac{k}{n}$. Define $U_i = F(X_i)$ and $E_n(x) = n^{-1}\sum_{i=1}^n 1_{\{U_i \leq x\}}$, the associated uniform empirical distribution function. Denote by $U_n(\cdot)$ the corresponding uniform sample quantile function.

Let r be an integer and define

$$Y_{n,r} = \sum_{i=1}^n \sum_{1\leq j_1 < \cdots \leq j_r} \prod_{s=1}^r c_{j_s}\epsilon_{i-j_s}, \qquad n \geq 1,$$

so that $Y_{n,0} = n$, and $Y_{n,1} = \sum_{i=1}^n X_i$. If $p < (2\beta-1)^{-1}$, then

$$\sigma_{n,p}^2 := \mathrm{Var}(Y_{n,p}) \sim n^{2-p(2\beta-1)} L_0^{2p}(n). \tag{2}$$

Define now the general empirical, the uniform empirical, the general quantile and the uniform quantile processes respectively as follows:

$$\beta_n(x) = \sigma_{n,1}^{-1} n(F_n(x) - F(x)), \qquad x \in \mathbb{R},$$

$$\alpha_n(y) = \sigma_{n,1}^{-1} n(E_n(y) - y), \qquad y \in (0,1),$$

$$q_n(y) = \sigma_{n,1}^{-1} n(Q(y) - Q_n(y)), \qquad y \in (0,1),$$

$$u_n(y) = \sigma_{n,1}^{-1} n(y - U_n(y)), \qquad y \in (0,1).$$

The aim of this paper is to study the asymptotic behavior of trimmed sums based on the ordered sample $X_{1:n} \leq \cdots \leq X_{n:n}$ coming from the long range dependent sequence defined by (1).

Let $T_n(m,k) = \sum_{i=m+1}^{n-k} X_{i:n}$ and note that (see below for a convention concerning integrals)

$$T_n(m,k) = n\int_{m/n}^{1-k/n} Q_n(y)\,dy. \tag{3}$$

Ho and Hsing observed in [12] that, under appropriate conditions on F,

$$\sup_{y\in[y_0,y_1]}\left|q_n(y)+\sigma_{n,1}^{-1}\sum_{i=1}^{n}X_i\right|=o_P(1), \tag{4}$$

where $0<y_0<y_1<1$. Equation (4) means in principle that *the quantile process can be approximated by the partial sum, independently of* y. This observation, together with (3), yield the asymptotic normality of the trimmed means in case of heavy trimming $m=m_n=[\delta_1 n]$, $k=k_n=[\delta_2 n]$, where $0<\delta_1<\delta_2<1$ and $[\cdot]$ is the integer part, see [12, Corollary 5.2]. This agrees with the i.i.d. situation, see [18].

However, the representation (3) requires some additional assumptions on F. In order to avoid them, we may study asymptotics for the trimmed sums via the integrals of the form $\int\alpha_n(y)dQ(y)$. This approach was initiated in two beautiful papers by M. Csörgő, S. Csörgő, Horváth and Mason, [2], [3]. Then, S. Csörgő, Haeusler, Horváth and Mason took this route to provide the full description of the weak asymptotic behavior of the trimmed sums in the i.i.d. case. We refer to [4] for an extensive up-to-date discussion and the survey of results.

In the LRD case, instead of using the Brownian bridge approximation, we can use the reduction principle for the general empirical processes as studied in [9], [12], [13] or [19], see Lemma 7 below. We can use then the similar approach as the above mentioned authors to establish the asymptotic normality in case of moderate trimming, complementing Ho and Hsing results in the heavy trimming case. The results are similar to the i.i.d case. In case os sums of extreme values, however, we can have some interesting phenomena, for which we refer to [14].

We will use the following convention concerning integrals. If $-\infty<a<b<\infty$ and h, g are left-continuous and right-continuous functions, respectively, then

$$\int_a^b gdh=\int_{[a,b)}gdh \qquad \text{and} \qquad \int_a^b hdg=\int_{(a,b]}hdg,$$

whenever these integrals make sense as Lebesgue-Stjeltjes integrals. The integration by parts formula yields

$$\int_a^b gdh+\int_a^b hdg=h(b)h(b)-f(a)g(a).$$

We shall write $g\in RV_\alpha$ ($g\in SV$) if g is regularly varying at infinity with index α (slowly varying at infinity).

In what follows C will denote a generic constant which may be different at each of its appearances. Also, for any sequences a_n and b_n, we write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$. Further, let $\ell(n)$ be a slowly varying function, possibly different at each place it appears. On the other hand, $L(\cdot)$, $L_0(\cdot)$, $L_1(\cdot)$, etc., are slowly varying functions, fixed form the time they appear. Moreover, $g^{(k)}$ denotes the kth order derivative of a function g and Z is a standard normal random variable. For any stationary sequence $\{V_i, i \geq 1\}$, we will denote by V the random variable with the same distribution as V_1.

2 Statement of results and discussion

Let F_ϵ be the marginal distribution function of the centered i.i.d. sequence $\{\epsilon_i, i \geq 1\}$. Also, for a given integer p, the derivatives $F_\epsilon^{(1)}, \ldots, F_\epsilon^{(p+3)}$ of F_ϵ are assumed to be bounded and integrable. Note that these properties are inherited by the distribution F as well (cf. [12] or [19]). Furthermore, assume that $\mathrm{E}\epsilon_1^4 < \infty$. These conditions are needed to establish the reduction principle for the empirical process and will be assumed throughout the paper.

We need to impose some conditions on F. The first assumption is that the right tail of the distribution F satisfies the following Von-Mises condition:

$$\lim_{x\to\infty} \frac{xf(x)}{1-F(x)} = \alpha > 0. \tag{5}$$

The condition (5) together with its counterpart for the left tail will be referred to as $X \in MDA(\Phi_\alpha)$, since, in particular, (5) implies that X belongs to the maximal domain of attraction of the Fréchet distribution with index α. Then

$$Q(1-y) = y^{-1/\alpha} L_1(y^{-1}) \quad \text{as} \quad y \to 0 \tag{6}$$

and the density-quantile function $fQ(y) = f(Q(y))$ satisfies

$$fQ(1-y) = y^{1+1/\alpha} L_2(y^{-1}) \quad \text{as} \quad y \to 0, \tag{7}$$

where $L_2(u) = \alpha(L_1(u))^{-1}$.

The second type of assumption is that F belongs to the maximal domain of attraction of the double exponential Gumbel distribution. Then the corresponding Von-Mises condition implies

$$\lim_{y\to 0} \frac{fQ(1-y)\int_{1-y}^{1}(1-u)/fQ(u)du}{y^2} = 1. \tag{8}$$

Thus, with $L_3(y^{-1}) = \left(y^{-1}\int_{1-y}^1 (1-u)/fQ(u)du\right)^{-1}$ one has

$$fQ(1-y) = yL_3(y^{-1})$$

and L_3 is slowly varying at infinity. The above assumption, together with its left-tailed counterpart will be referred to as $X \in MDA(\Lambda)$.

Recall that $Q_n(y) = \inf\{x : F_n(x) \geq y\} = X_{k:n}$ if $\frac{k-1}{n} < y \leq \frac{k}{n}$. Let $T_n(m,k) = \sum_{i=m+1}^{n-k} X_{i:n}$ and

$$\mu_n(m,k) = n\int_{m/n}^{1-k/n} Q(y)dy.$$

The main result of this paper is the following theorem.

Theorem 1 (Moderate trimming) *Let p be the smallest positive integer such that $(p+1)(2\beta-1) > 1$ and assume that for $r = 1,\ldots,p$, $\int_0^1 F^{(r)}(Q(y))dQ(y) < \infty$. Let $k_n = n^{\xi}$, $\xi \in (0,1)$. Assume that either $F \in MDA(\Lambda)$ or $F \in MDA(\Phi_\alpha)$ for some $\alpha < \infty$ such that*

$$\begin{cases} \alpha > \frac{1-\xi}{1-\beta} & \text{if } \beta \geq \frac{3}{4}, \\ \alpha \geq 4 & \text{if } \beta \in (\frac{1}{2},\frac{3}{4}). \end{cases} \qquad (9)$$

Then

$$\sigma_{n,1}^{-1}(T_n(k_n,k_n) - \mu_n(k_n,k_n)) \xrightarrow{\mathrm{d}} Z.$$

2.1 Remarks

Remark 2 Let us discuss condition (9) of Theorem 1. Recall that the main assumption is $\mathrm{E}\epsilon_1^4 < \infty$ which implies $\mathrm{E}X_1^4 < \infty$ and thus $\alpha \geq 4$, in particular. If β is close to $\frac{3}{4}$ then there is no additional restriction on moments. However, the restriction on α is very sensitive for β close to $\frac{1}{2}$ or 1.

The condition $\mathrm{E}\epsilon_1^4 < \infty$ appeared since we used the reduction principle from [19]. This moment restriction can be weakened to $\mathrm{E}|\epsilon_1|^{2+\delta}$, $\delta > 0$, as indicated in [9]. However, in this case the rates in the reduction principle are not as good as in [19], which is crucial in our method. If one can improve those rates, one can have less restrictive conditions on α.

Remark 3 The conditions $\int_{1/2}^1 F^{(r)}(Q(y))dQ(y) < \infty$, $r = 1,\ldots,p$, are not restrictive at all, since they are fulfilled for most distributions with a regularly varying density-quantile function $f(Q(1-y))$, for which we refer to [16]. Consider for example the Pareto case, $Q(y) = C(1-y)^{-1/\alpha}$ for all y such that $0 < y_1 < y < 1$, where y_1 is fixed. Then for $r \geq 1$ and all x exceeding some value x_0, $F^{(r)}(x) = Cx^{-(\alpha+r)}$. Thus, we have

$$\int_{1/2\vee x_0}^{1} F^{(r)}(Q(y))dQ(y) = \int_{1/2\vee x_0}^{1} (1-y)^{(r-1)/\alpha} dy < \infty.$$

If, additionally, we impose the following *Csörgő-Révész-type conditions* (cf. also [1, Theorem 3.2.1]):

(CsR1) f exists on (a,b), where $a = \sup\{x : F(x) = 0\}$, $b = \inf\{x : F(x), x = 1\}$, $-\infty \leq a < b \leq \infty$,

(CsR2) $\inf_{x\in(a,b)} f(x) > 0$,

then in view of (CsR2) and the assumed boundness of derivatives $F^{(r)}(\cdot)$, the integral $\int_{1/2}^{1} F^{(r)}(Q(y))dQ(y)$ is finite.

The condition is trivially fulfilled for the exponential right-tail. Consider the standard normal distribution. Then $F^{(r)}(x) = f(x)W_{(r-1)}(x)$, where $W_{(r-1)}(\cdot)$ is a polynomial of order $r-1$. Consequently, the finiteness of the integral is equivalent to $\int_{1/2}^{1} Q^{r-1}(y)dy < \infty$, which is the case.

Remark 4 It is well-known that $\sigma_{n,1}^{-1}(T_n(0,0) - \mu_n(0,0)) = \sigma_{n,1}^{-1}T_n(0,0) \xrightarrow{\text{d}} Z$. Combining this with Theorem 1 we see that the extreme sum $\sum_{i=n-k_n+1}^{n} X_{i:n}$ is $o_P(\sigma_{n,1})$-negligible, namely

$$\sigma_{n,1}^{-1}\left(\sum_{i=n-k_n+1}^{n} X_{i:n} - n\int_{1-k_n/n}^{1} Q(y)dy\right) = o_P(1).$$

Moreover, we have $\sum_{i=n-k_n+1}^{n} X_{i:n} = o_P(\sigma_{n,1})$ as long as

$$\sigma_{n,1}^{-1} n \int_{1-k_n/n}^{1} Q(y)dy = o(1). \tag{10}$$

In other words

$$\frac{\sum_{i=n-k_n+1}^{n} X_{i:n}}{\sum_{i=1}^{n} X_i} = o_P(1), \tag{11}$$

the cumulated extremes have negligible contribution to the whole partial sum.

Remark 5 In the similar vein as in Theorem 1, we may consider partial sums $\sum_{i=1}^{n} G(X_i)$, where G is a measurable function. Assuming for example that $X_i, i \geq 1$ is a Gaussian sequence and that G has the Hermite rank τ, using the strong reduction principle for the empirical process based on subordinated random variables $Y_i = G(X_i)$ (see [7]) one can state the corresponding results, replacing $\sigma_{n,1}$ and Z by $\sigma_{n,\tau}$ and Z_τ respectively, where Z_τ is a (possibly non-Gaussian) random variable defined as the τ-fold integral with respect to a Brownian motion, see [11]. In the context of Theorem 1, however, one needs to assume that $G(X_1)$ is in the appropriate domain of attraction. For example, if $G(x) = \log(x^+)^\alpha$ and $X \in MDA(\Phi_\alpha)$, then $G(X) \in MDA(\Lambda)$. The result of Theorem 1 is still valid, since the Hermite rank of G is 1. On the other hand, if $G(x) = x^2 - 1$ or $G(x) = |x|$, then the Hermite is 2 and the maximal domains of attraction for $G(X)$ can be easily characterized by those of X.

Remark 6 Note that L-statistics can be written as integrals of the form $\int_0^1 J(y)Q_n(y)dy$ with some function J. Therefore, our approach can be also applied to establish asymptotic results for L-statistics.

3 Proofs

Let p be a positive integer and let

$$\begin{aligned} S_{n,p}(x) &= \sum_{i=1}^{n}(1_{\{X_i \leq x\}} - F(x)) + \sum_{r=1}^{p}(-1)^{r-1}F^{(r)}(x)Y_{n,r} \\ &=: \sum_{i=1}^{n}(1_{\{X_i \leq x\}} - F(x)) + V_{n,p}(x), \end{aligned}$$

where $F^{(r)}$ is the rth order derivative of F. Setting $U_i = F(X_i)$ and $x = Q(y)$ in the definition of $S_n(\cdot)$ we arrive at its uniform version,

$$\begin{aligned} \tilde{S}_{n,p}(y) &= \sum_{i=1}^{n}(1_{\{U_i \leq y\}} - y) + \sum_{r=1}^{p}(-1)^{r-1}F^{(r)}(Q(y))Y_{n,r} \\ &=: \sum_{i=1}^{n}(1_{\{U_i \leq y\}} - y) + \tilde{V}_{n,p}(y). \end{aligned}$$

Denote

$$d_{n,p} = \begin{cases} n^{-(1-\beta)}L_0^{-1}(n)(\log n)^{5/2}(\log\log n)^{3/4}, & (p+1)(2\beta-1) \geq 1 \\ n^{-p(\beta-\frac{1}{2})}L_0^{p}(n)(\log n)^{1/2}(\log\log n)^{3/4}, & (p+1)(2\beta-1) < 1 \end{cases}.$$

We shall need the following lemma.

Lemma 7 ([19]) *Let p be a positive integer. Then, as $n \to \infty$,*

$$\mathrm{E}\sup_{x\in\mathbf{R}}\left|\sum_{i=1}^{n}(1_{\{X_i\le x\}}-F(x))+\sum_{r=1}^{p}(-1)^{r-1}F^{(r)}(x)Y_{n,r}\right|^2=O(\Xi_n+n(\log n)^2),$$

where

$$\Xi_n=\begin{cases} O(n), & (p+1)(2\beta-1)>1 \\ O(n^{2-(p+1)(2\beta-1)}L_0^{2(p+1)}(n)), & (p+1)(2\beta-1)<1 \end{cases}.$$

Using Lemma 7 we obtain (cf. [6])

$$\begin{aligned}&\sigma_{n,p}^{-1}\sup_{x\in\mathbf{R}}|S_n(x)|\\ &=\begin{cases} O_{a.s}(n^{-(\frac{1}{2}-p(\beta-\frac{1}{2}))}L_0^{-p}(n)(\log n)^{5/2}(\log\log n)^{3/4}), & (p+1)(2\beta-1)>1 \\ O_{a.s}(n^{-(\beta-\frac{1}{2})}L_0(n)(\log n)^{1/2}(\log\log n)^{3/4}), & (p+1)(2\beta-1)<1 \end{cases}.\end{aligned}$$

Since (see (2))

$$\frac{\sigma_{n,p}}{\sigma_{n,1}}\sim n^{-(\beta-\frac{1}{2})(p-1)}L_0^{p-1}(n)$$

we obtain

$$\begin{aligned}&\sup_{x\in\mathbf{R}}|\beta_n(x)+\sigma_{n,1}^{-1}V_{n,p}(x)|=\\ &=\frac{\sigma_{n,p}}{\sigma_{n,1}}\sup_{x\in\mathbf{R}}\left|\sigma_{n,p}^{-1}\sum_{i=1}^{n}(1_{\{X_i\le x\}}-F(x))+\sigma_{n,p}^{-1}V_{n,p}(x)\right|=o_{a.s}(d_{n,p}).\end{aligned}$$

Consequently, via $\{\alpha_n(y),y\in(0,1)\}=\{\beta_n(Q(y)),y\in(0,1)\}$,

$$\sup_{y\in(0,1)}|\alpha_n(y)+\sigma_{n,1}^{-1}\tilde{V}_{n,p}(y)|=O_{a.s}(d_{n,p}). \tag{12}$$

By (12), the next lemma is obvious.

Lemma 8 *Let p be a positive integer. Assume that for $r=1,\ldots,p$, $\int_0^1 F^{(r)}(Q(y))dQ(y)<\infty$. Then for any $0<a_n$ such that $a_n\to\infty$, $a_n=o(n)$,*

$$\sigma_{n,1}^{-1}\int_{a_n/n}^{1-a_n/n}\tilde{V}_{n,p}(y)dQ(y)\xrightarrow{\mathrm{d}}Z.$$

3.1 Integral functionals of the empirical process

Let $\psi_\mu(y) = (y(1-y))^\mu$, $y \in [0,1]$, $\mu > 0$.

We start with following results, which may be of independent interest. Let $\varepsilon_n = n^{-\kappa}$, $\kappa > 0$ and consider the following class of functions:

$$\mathcal{L}_0 = \mathcal{L}_0(\mu) = \left\{ K : K : [0,1] \to \mathbb{R}, \text{nondecreasing}, \int_0^1 \psi_\mu(y) dK(y) < \infty \right\}.$$

Lemma 9 *Fix $\kappa > 0$, $p \in \mathbb{N}$. Let $\mu > 0$ be such that*

$$\begin{cases} \mu < \frac{1-\beta}{\kappa}, & \text{if } (p+1)(2\beta-1) \geq 1, \\ \mu < \frac{p(\beta-\frac{1}{2})}{\kappa}, & \text{if } (p+1)(2\beta-1) < 1. \end{cases} \tag{13}$$

Then

$$\sup_{K \in \mathcal{L}_0} \sup_{\varepsilon_n < s < t < 1-\varepsilon_n} \left| \int_s^t \left(\alpha_n(y) + \sigma_{n,1}^{-1} \tilde{V}_{n,p}(y) \right) dK(y) \right| = o_{a.s}(1). \tag{14}$$

The most interesting case is $K = Q$.

Corollary 10 *Assume that either* $\mathrm{E}|X|^\alpha < \infty$ *for all* $\alpha > 0$ *or (6) holds with some* $\alpha < \infty$ *such that*

$$\begin{cases} \alpha > \frac{\kappa}{1-\beta} & \text{if } \beta \geq \frac{3}{4}, \\ \alpha \geq 4 & \text{if } \beta \in (\frac{1}{2}, \frac{3}{4}). \end{cases}$$

Let p be the smallest positive integer such that $(p+1)(2\beta-1) > 1$. Then

$$\sup_{1/n^\kappa < s < t < 1-1/n^\kappa} \left| \int_s^t \left(\alpha_n(y) + \sigma_{n,1}^{-1} \tilde{V}_{n,p}(y) \right) dQ(y) \right| = o_{a.s}(1). \tag{15}$$

Proof of Lemma 9. From (12) and the choice of μ,

$$\sup_{y \in (\epsilon_n, 1-\epsilon_n)} \frac{|\alpha_n(y) + \sigma_{n,1}^{-1} \tilde{V}_{n,p}(y)|}{\psi_\mu(y)} = O_{a.s}(1). \tag{16}$$

We have for all $\varepsilon_n < s < t < 1-\varepsilon_n$ and all $K \in \mathcal{L}_0$,

$$\begin{aligned} & \left| \int_s^t \left(\alpha_n(y) - \sigma_{n,1}^{-1} \tilde{V}_{n,p}(y) \right) dK(y) \right| \\ \leq & \int_s^t \sup_{y \in (\varepsilon_n, 1-\varepsilon_n)} \frac{|\alpha_n(y) - \sigma_{n,1}^{-1} \tilde{V}_{n,p}(y)|}{\psi_\mu(y)} \psi_\mu(y) dK(y) \\ \leq & \; o_{a.s}(1) \int_0^1 \psi_\mu(y) dK(y) = o_{a.s}(1). \end{aligned}$$

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Proof of Corollary 10. If $\beta \geq \frac{3}{4}$, set $p = 1$ in (13). If $F \in MDA(\Phi_\alpha)$, then $\mathrm{E}|X|^{\alpha-\delta} < \infty$ for all $\delta \in (0,\alpha)$. Since we assumed $\alpha > \kappa/(1-\beta)$, choose $\delta \in (0,\alpha)$ such that $\alpha - \delta > \kappa/(1-\beta)$ still holds. Set $\mu = (\alpha-\delta)^{-1} < (1-\beta)/\kappa$. Then we have $\mathrm{E}|X|^{1/\mu+\delta/2} < \infty$. The latter condition is sufficient for the finiteness of $\int_0^1 \psi_\mu(y)dQ(y)$ (see [17, Remark 2.4]). Consequently, $Q \in \mathcal{L}_0$ and (14) applies.

If $\beta < \frac{3}{4}$ and $F \in MDA(\Phi_\alpha)$, then take in (13) the smallest integer p such that $(p+1)(2\beta-1) > 1$. Now, $(1-\beta)^{-1} < 4$. Further, since $\mathrm{E}\epsilon_1^4 < \infty$, we have $\mathrm{E}X_1^4 < \infty$ and thus $\alpha \geq 4$. Consequently, $\alpha > (1-\beta)^{-1}$. We may choose $\delta \in (0,\alpha)$ such that $\alpha - \delta > (1-\beta)^{-1}$ and continue as in the previous case.

Likewise, if $\mathrm{E}|X|^\alpha < \infty$ for all $\alpha > 0$ then choose α such big so that $\alpha > (1-\beta)^{-1}$. Consequently, we may continue as in the case of Q being regularly varying. Therefore, (15) has been proved.

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3.2 Proof of Theorem 1

Integration by parts yields

$$\sigma_{n,1}^{-1}\left(\sum_{i=k_n+1}^{n-k_n} X_i - \mu_n(k_n,k_n)\right) = -\int_{k_n/n}^{1-k_n/n} \alpha_n(y)dQ(y)$$
$$+\sigma_{n,1}^{-1} n \int_{k_n/n}^{U_{k_n:n}} (E_n(y) - k_n/n)dQ(y) +$$
$$\sigma_{n,1}^{-1} n \int_{U_{n-k_n:n}}^{1-k_n/n} (E_n(y) - (1-k_n/n))dQ(y) =: B_1 + B_2 + B_3.$$

Note that $F \in MDA(\Lambda)$ implies that $\mathrm{E}|X|^\alpha < \infty$ for all $\alpha > 0$ (see [8, p. 148]). Corollary 10 applied with $\kappa = 1-\xi$, together with Lemma 8 yields the asymptotic normality for B_1. It suffices to show that $B_3 = o_P(1)$. The term B_2 is treated in the same way.

Lemma 11 *For any $k_n \to \infty$, $k_n = o(n)$*

$$\frac{U_{n-k_n:n}}{1-k_n/n} \xrightarrow{p} 1.$$

Proof. In view of (12) one obtains

$$\sup_{y\in(0,1)} |u_n(y)| = \sup_{y\in(0,1)} |\alpha_n(y)| = O_P(1).$$

Consequently,

$$\begin{aligned}\sup_{y\in(0,1)} |y - U_n(y)| &= \sup_{y\in(0,1)} \sigma_{n,1} n^{-1} |u_n(y)| = \sup_{y\in(0,1)} \sigma_{n,1} n^{-1} |\alpha_n(y)| \\ &= O_P(\sigma_{n,1} n^{-1}).\end{aligned}$$

Thus, the result follows by noting that $U_n(1 - k_n/n) = U_{n-k_n:n}$.

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An easy consequence of (12) is the following result.

Lemma 12 *For any* $k_n \to 0$,

$$\sup_{y\in(1-k_n/n,1)} |\alpha_n(y)| = O_{a.s.}(d_{n,p}) + O_P(f(Q(1 - k_n/n))).$$

To prove that $B_3 = o_P(1)$, let y be in the interval with the endpoints $U_{n-k_n:n}$ and $1 - k_n/n$. Then

$$\left|1 - E_n(y) - \frac{k_n}{n}\right| \le |E_n(1 - k_n/n) - (1 - k_n/n)|.$$

Case 1, $Y \in MDA(\Phi_\alpha)$: By Lemma 11 we have

$$Q(1 - k_n/n)/Q(U_{n-k_n:n}) \xrightarrow{p} 1. \tag{17}$$

Thus, by (17) and Lemma 12

$$\begin{aligned}B_3 &\le \sigma_{n,1}^{-1} Q(1 - k_n/n)|\alpha_n(1 - k_n/n)| \frac{|Q(1 - k_n/n) - Q(U_{n-k_n:n})|}{Q(1 - k_n/n)} \\ &= \sigma_{n,1}^{-1} Q(1 - k_n/n)\alpha_n(1 - k_n/n) o_p(1) \\ &= o_p\left(\sigma_{n,1}^{-1} Q(1 - k_n/n) f Q(1 - k_n/n)\right) + o_p\left(\sigma_{n,1}^{-1} Q(1 - k_n/n) d_{n,p}\right) = o_P(1).\end{aligned}$$

Case 2, $X \in MDA(\Lambda)$: Let

$$T_n(\lambda) = \sigma_{n,1}^{-1} |\alpha_n(1 - k_n/n)| \left|Q(r_n^+(\lambda)) - Q(r_n^-(\lambda))\right|,$$

where $r_n^+(\lambda) = 1 - \frac{k_n}{\lambda n}$, $r_n^-(\lambda) = 1 - \frac{k_n}{\lambda n}$ and $1 < \lambda < \infty$ is arbitrary. Applying the argument as in the proof of Theorem 1 in [5] we have

$$\liminf_{n\to\infty} P(|B_3| < |T_n(\lambda)|) \ge \lim \inf_{n\to\infty} P(r_n^-(\lambda) \le U_{n-k_n:n} \le r_n^+(\lambda)).$$

In view of Lemma 11, the lower bound is 1. Thus, $\lim_{n\to\infty} P(|B_3| < |T_n(\lambda)|) = 1$. Further, by Lemma 4 in [15]

$$\lim_{n\to\infty}(Q(r_n^+(\lambda)) - Q(r_n^-(\lambda)))L_3(n/k_n) = -\log\lambda.$$

Thus, for large n

$$\begin{aligned} T_n(\lambda) &= \sigma_{n,1}^{-1}|\alpha_n(1-k_n/n)|(L_3(n/k_n))^{-1}|Q(r_n^+(\lambda)) - Q(r_n^-(\lambda))|L_3(n/k_n) \\ &\le C_1\frac{\sigma_{n,1}^{-1}}{L_3(n/k_n)}fQ(1-k_n/n)(\log\lambda) + C_2\frac{\sigma_{n,1}^{-1}}{L_3(n/k_n)}d_{n,p}\log\lambda \end{aligned}$$

almost surely with some constants C_1, C_2. The both terms, for arbitrary λ, converges to 0. Thus, we have for sufficiently large n, $T_n(\lambda) \le C_1 \log\lambda$ almost surely. Thus, $\lim_{n\to\infty} P(|T_n(\lambda)| \le C_1\log\lambda) = 1$. Consequently,

$$\begin{aligned} &\lim_{n\to\infty} P(|B_3| > C_1\log\lambda) = \\ &= \lim_{n\to\infty} P(|B_3| > C_1\log\lambda, |T_n(\lambda)| \le C_1\log\lambda) + \lim_{n\to\infty} P(|T_n(\lambda)| > C_1\log\lambda) \\ &\le \lim_{n\to\infty} P(|B_3| > |T_n(\lambda)|) + 0 = 0 \end{aligned}$$

and thus $B_3 = o_P(1)$.

$\odot$

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References

[1] Csörgő, M. (1983). *Quantile Processes with Statistical Applications.* CBMS-NSF Regional Conference Series in Applied Mathematics.

[2] Csörgő, M., Csörgő, S, Horváth, L., Mason, D. M. (1986). Weighted empirical and quantile processes. *Ann. Probab.* **14**, 31–85.

[3] Csörgő, M., Csörgő, S, Horváth, L., Mason, D. M. (1986). Normal and stable convergence of integral functions of the empirical distribution function. *Ann. Probab.* **14**, 86–118.

[4] Csörgő, S., Haeusler, E. and Mason, D. M. (1991). The quantile-transform–empirical-process approach to limit theorems for sums of order statistics. *Sums, trimmed sums and extremes*, 215–267, Progr. Probab., **23**, Birkhäuser, Boston, MA.

[5] Csörgő, S, Horváth, L., Mason, D. M.(1986). What portion of the sample makes a partial sum asymptotically stable or normal? *Probab. Theory Relat. Fields* **72**, 1–16.

[6] Csörgő, M. and Kulik, R. (2006). Reduction principles for quantile and Bahadur-Kiefer processes of long-range dependent linear sequences. *Preprint, submitted.*

[7] Csörgő, M., Szyszkowicz, B. and Wang, L. (2006). Strong Invariance Principles for Sequential Bahadur-Kiefer and Vervaat Error Processes of Long-Range Dependence Sequences. *Ann. Statist.*

[8] Embrechts, P., Kluppelberg, C. and Mikosch, T. (1997). *Modelling extremal events. For insurance and finance.* Applications of Mathematics **33**. Springer-Verlag, Berlin.

[9] Giraitis, L. and Surgailis, D. (2002). The reduction principle for the empirical process of a long memory linear process. *Empirical process techniques for dependent data*, 241–255, Birkhäuser Boston, Boston, MA.

[10] de Haan, L. (1975). On Regular Variation and Its Applications to the Weak Convergence of Sample Extremes. Math. Centre Tracts 32, Amsterdam.

[11] Dobrushin, R.L.

[12] Ho, H.-C. and Hsing, T. (1996). On the asymptotic expansion of the empirical process of long-memory moving averages. *Ann. Statist.* **24**, 992–1024.

[13] Koul, H.L. and Surgailis, D. (2002). Asymptotic expansion of the empirical process of long memory moving averages. *Empirical process techniques for dependent data*, 213–239, Birkhäuser Boston, Boston, MA.

[14] Kulik, R. (2006). Sums of extreme values of subordinated long-range dependent sequences: moving averages with finite variance. *Preprint, submitted.*

[15] Lo, G. S. (1989). A note on the asymptotic normality of sums of extreme values. *J. Statist. Plann. Inference* **22**, 127–136.

[16] Parzen, E. (1979). Nonparametric statistical data modeling. With comments by John W. Tukey, Roy E. Welsch, William F. Eddy, D. V. Lindley, Michael E. Tarter and Edwin L. Crow, and a rejoinder by the author. *J. Amer. Statist. Assoc.* **74**, 105–131.

[17] Shao, Q.-M. and Yu, H. (1996). Weak convergence for weighted empirical processes of dependent sequences. *Ann. Probab.* **24**, 2098–2127.

[18] Stigler, S. M. (1973). The asymptotic distribution of the trimmed mean. *Ann. Statist.* **1**, 472–477.

[19] Wu, W.B. (2003). Empirical processes of long-memory sequences. Bernoulli **9**, no. 5, 809–831.