Limit theorems for self-normalized linear processes

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Abstract

In this article we prove a self-normalized central limit theorem and an invariance principle in the case of strictly stationary linear processes $X_t = \sum_{k=1}^{\infty} c_k Z_{t-k}$ assuming that the i.i.d. random variables $\{Z_t\}$ are in the domain of attraction of the normal law.

Key words: self-normalized, weak invariance principle, linear processes

1 Introduction

Let $\{X_t\}_{t\geq 1}$ be a sequence of independent identically distributed random variables. Suppose that $\mu = EX_1 = 0$ and let $S_n = \sum_{t=1}^{n} X_t$, $V_n^2 = \sum_{t=1}^{n} X_t^2$. The standard t-statistics may be expressed as

$$t_n = \frac{S_n/V_n}{\sqrt{(n-(S_n/V_n)^2)/(n-1)}} \ .$$

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It is known (see Efron (1969)) that the limiting behaviour of t_n agrees with that of S_n/V_n. Giné et al. (1997) showed that $t_n \xrightarrow{d} N(0,1)$, or equivalently $S_n/V_n \xrightarrow{d} N(0,1)$, if and only if the following condition holds:

(L) $L_X(x) = EX_1^2 1_{\{|X|\leq x\}}$ is slowly varying at ∞.

This proves a long-standing conjecture (Logan et al. (1973)).

Other fluctuation results for the sequence of self-normalized observations have been proved by various authors: the law of iterated logarithm was obtained in Griffin and Kuelbs (1989), the Berry-Esseen theorem can be found in Bentkus and Götze (1996), the large deviation principle in Shao (1997), a Darling-Erdös type result in Csörgő et al. (2003a) and the functional central limit theorem in Račkauskas and Suquet (2001). Recently, Mason (2005) proved the Giné, Götze and Mason result using convergence of arrays.

A common feature of all these results is that the distributional assumptions under which a self-normalized limit theorem would hold are in general milder than the assumptions of the corresponding classical limit theorem; in particular, most of these results do not require that the variance be finite.

The recent paramount result of Csörgő et al. (2003b) shows that under (L), the behaviour *in probability* of the self-normalized process $\{S_{[nt]}/V_n\}_{t\in[0,1]}$ coincides with the behavior of a standard Brownian motion $W = \{W(t)\}_{t\in[0,1]}$. More precisely, on an appropriate probability space we have

$$\sup_{t\in(0,1]} \left| \frac{S_{[nt]}}{V_n} - \frac{W(nt)}{\sqrt{n}} \right| = o_P(1). \tag{1}$$

For an extensive discussion concerning self-normalized limit theorems we refer to the survey articles, Shao (1998) and Csörgő et al. (2004).

In the present paper we consider a class of linear processes

$$X_t = \sum_{k=1}^{\infty} c_k Z_{t-k} \ , \tag{2}$$

where $\{Z, Z_t\}_{t=-\infty,\dots,\infty}$ is an i.i.d. centered sequence and

$$\sum_{k=1}^{\infty} |c_k| < \infty \ . \tag{3}$$

We prove that if L_Z is slowly varying then $S_n/(\beta V_n) \xrightarrow{\mathrm{d}} N(0,1)$ for an explicit parameter β. This parameter is given by $\beta^2 = \left(\sum_{k=0}^{\infty} c_k\right)^2 / (\sum_{k=0}^{\infty} c_k^2)$. Compared to ϕ-mixing or ρ-mixing sequences β can be computed explicitly (see Balan and Kulik (2005) for the ϕ-mixing case).

A common feature of all limiting results is that self-normalized linear processes behave (up to a constant) the same as the i.i.d. innovations $\{Z_t\}_{t\geq 1}$. This fact comes from a particular decomposition of linear processes (see (5)) which was used in Wang et al. (2001) and provides essentially better results than the method used in Phillips and Solo (1992). Using this, we will prove an analog to the invariance principle (1) for linear processes.

The paper is organized as follows. In Section 2 we state our results. The proof of the central limit theorem is included in Section 3. The proof of the invariance principle is presented in Section 4.

2 Self-normalized limit laws

Assume that $\{X_t\}_{t\geq 1}$ is an infinite order linear process (2) such that (3) holds. Let $L_Z(x) = EZ_1^2 1_{\{|Z_1|<x\}}$ and assume that L_Z is slowly varying at infinity ($L_Z \in SV$). Let $\eta_n^2 \sim nL_Z(\eta_n)$ (for a precise definition of η_n we refer to Csörgő et al. (2003b)) and $\beta^2 := \left(\sum_{k=0}^{\infty} c_k\right)^2 / \sum_{k=0}^{\infty} c_k^2$.

The first result provides a self-normalized central limit theorem (CLT) for linear processes.

Theorem 2.1 *Assume (3). If $L_Z \in SV$ and $l_n \to \infty$, $l_n = o(\sqrt{n})$ then*

$$\frac{S_n}{V_n} = \frac{\sum_{t=1}^n X_t}{\sqrt{\sum_{t=1}^n X_t^2}} \xrightarrow{\mathrm{d}} N(0, \beta^2).$$

If the variance of Z is finite, then the self-normalized limit laws follows from the CLT for linear processes (see (Brockwell and Davis, 1987, Theorem 7.1.2)) together with a law of large numbers, (Brockwell and Davis, 1987, Proposition 7.3.5).

Theorem 2.2 *Under conditions of Theorem 2.1, without changing its distribution, we can redefine the sequence $\{Z_j\}_{j\geq 1}$ on a larger probability space together with a standard Brownian motion $W = \{W(t)\}_{t\geq 0}$ such that*

$$\sup_{t\in(0,1]} \left| \frac{S_{[nt]}}{\beta V_n} - \frac{W(nt)}{\sqrt{n}} \right| = o_P(1) \; . \tag{4}$$

Clearly, the weak convergence for self-normalized linear processes follows from Theorem 2.2.

3 Central Limit Theorem

The proof is based on an interplay between law of large numbers type results and the central limit theorem for linear processes. In the i.i.d. case we refer to Gut (2005) for a short and easy discussion concerning such connections.

Throughout the paper we shall use the following decomposition of the linear process. For arbitrary finite $m \geq 1$ we have

$$\sum_{t=1}^{n} X_t = \left(\sum_{k=0}^{m} c_k\right) \sum_{t=1}^{n} Z_t + \sum_{t=1}^{m} Z_{1-t} \sum_{k=t}^{m} c_k + \sum_{t=0}^{m-1} Z_{n-t} \sum_{k=t+1}^{m} c_k + \sum_{k=m+1}^{\infty} c_k \sum_{t=1}^{n} Z_{t-k} \ . \quad (5)$$

Consider an array

$$\hat{Z}_{j,n} = Z_j 1_{\{|Z_j| \leq \eta_n\}} \ , \quad j \geq 1 \ , n \geq 1$$

and define $\bar{Z}_{j,n} = Z_j 1_{\{|Z_j| > \eta_n\}}$. In view of (Csörgő et al., 2003b, Lemma 1(c)) one gets

$$n E \bar{Z}_{1,n} = o(\eta_n) \ . \quad (6)$$

Lemma 1 *Let $R_{t,n} = \sum_{k=0}^{\infty} c_k^2 \hat{Z}_{t-k,n}^2$. Under the conditions of Theorem 2.1,*

$$\frac{\sum_{t=1}^{n} R_{t,n}}{\eta_n^2} \xrightarrow{p} \sum_{k=0}^{\infty} c_k^2 \ .$$

Proof. By (Giné et al. (1997))

$$\lim_{n\to\infty} \sum_{t=1}^{n} Z_t^2/\eta_n^2 = \lim_{n\to\infty} \sum_{t=1}^{n} \hat{Z}_{t,n}^2/\eta_n^2 = 1 \ , \quad (7)$$

in probability.

We show first that for each $\varepsilon > 0$

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left(\eta_n^{-2}\left|\sum_{t=1}^n R_{t.n} - \left(\sum_{k=0}^m c_k^2\right)\sum_{t=1}^n \hat{Z}_{t,n}^2\right| > \varepsilon/3\right) =: \lim_{m\to\infty} \limsup_{n\to\infty} I_1 = 0 \, .$$

Since $\{R_{t,n}\}_{t\ge 1}$ is a linear process, we may use the decomposition (5) replacing c_k with c_k^2 and Z_{t-k} with $\hat{Z}_{t-k,n}^2$:

$$\begin{aligned}\sum_{t=1}^n R_{t,n} &= \left(\sum_{k=0}^m c_k^2\right)\sum_{t=1}^n \hat{Z}_{t,n}^2 + \sum_{t=1}^m \hat{Z}_{1-t,n}^2 \sum_{k=t}^m c_k^2 + \sum_{t=0}^{m-1} \hat{Z}_{n-t,n}^2 \sum_{k=t+1}^m c_k^2 + \sum_{k=m+1}^\infty c_k^2 \sum_{t=1}^n \hat{Z}_{t-k,n}^2 \\ &=: I_1 + I_2 + I_3 + I_4 \, .\end{aligned}$$

For the second and the third part one has

$$I_2 + I_3 \le \max_{-m\le i\le n} \hat{Z}_{i,n}^2 \sum_{t=0}^m \left(\sum_{k=t}^m c_k^2 + \sum_{k=t+1}^m c_k^2\right) \, .$$

Since $L_Z \in SV$ we have $\max_{1\le i\le n} Z_i^2 / \sum_{t=1}^n Z_t^2 \xrightarrow{p} 0$ (see O'Brien (1980)). This, together with (7) gives $\max_{-m\le i\le n} \hat{Z}_{t,n}^2/\eta_n^2 \xrightarrow{p} 0$. Thus, the second and the third part are asymptotically negligible. Moreover,

$$E\left(\sum_{k=m+1}^\infty c_k^2 \sum_{t=1}^n \hat{Z}_{t-k,n}^2\right)^2 \le \left(\sum_{k=m+1}^\infty c_k^2\right)^2 E\left(\sum_{t=1}^n \hat{Z}_{t-k,n}^2\right)^2 \, .$$

Dividing both sides by η_n^4 we have for the term I_4

$$\frac{EI_4^2}{\eta_n^4} \le \left(\sum_{k=m+1}^\infty c_k^2\right)^2 \left\{\frac{nE\hat{Z}_{1,n}^4}{\eta_n^4} + \frac{2n^2\left(E\hat{Z}_{0,n}^2\right)^2}{\eta_n^4}\right\} = \left(\sum_{k=m+1}^\infty c_k^2\right)^2 (o(1)+2)$$

as $n\to\infty$. Letting then $m\to\infty$ we obtain $\lim_{m\to\infty}\limsup_{n\to\infty}\frac{EI_4^2}{\eta_n^4} = 0$.

Now,

$$\begin{aligned}&\lim_{n\to\infty} P\left(\left|\eta_n^{-2}\sum_{t=1}^n R_{t,n} - \sum_{k=0}^\infty c_k^2\right| > \varepsilon\right) = \lim_{m\to\infty}\lim_{n\to\infty} P\left(\left|\eta_n^{-2}\sum_{t=1}^n R_{t,n} - \sum_{k=0}^\infty c_k^2\right| > \varepsilon\right) \\ &\le \lim_{m\to\infty}\limsup_{n\to\infty} I_1 + \lim_{m\to\infty}\limsup_{n\to\infty} P\left(\eta_n^{-2}\sum_{t=1}^n \hat{Z}_{t,n}^2 \left|\sum_{k=0}^m c_k^2 - \sum_{k=0}^\infty c_k^2\right| > \varepsilon/3\right) + \\ &\quad \limsup_{n\to\infty} P\left(\left|\eta_n^{-2}\sum_{t=1}^n \hat{Z}_{t,n}^2 - 1\right|\sum_{k=0}^\infty c_k^2 > \varepsilon/3\right) .\end{aligned}$$

The second part is 0 in view of (7) and sumability of $\sum_{k=0}^{\infty} c_k^2$, the third one is 0 by (7).

□

Lemma 2 *Under the conditions of Theorem 2.1,*

$$\frac{\sum_{t=1}^{n} X_t^2}{\eta_n^2} \xrightarrow{p} \sum_{k=0}^{\infty} c_k^2 \ .$$

Proof. In view of the remark following Theorem 2.1 we may assume that $\mathrm{Var} Z = \infty$. We have

$$\begin{aligned}\sum_{t=1}^{n} X_t^2 &= \sum_{t=1}^{n}\sum_{k=0}^{\infty} c_k^2 Z_{t-k}^2 + 2\sum_{t=1}^{n}\sum_{k<r} c_k c_r Z_{t-k} Z_{t-r} \\ &= \sum_{t=1}^{n}\sum_{k=0}^{\infty} c_k^2 \hat{Z}_{t-k,n}^2 + \sum_{t=1}^{n}\sum_{k=0}^{\infty} c_k^2 \bar{Z}_{t-k,n}^2 + 2\sum_{t=1}^{n}\sum_{k<r} c_k c_r Z_{t-k} Z_{t-r} =: I_1 + I_2 + I_3 \ .\end{aligned}$$

By Lemma 1 the term I_1 gives the required asymptotics. As for I_2 we have

$$\frac{1}{\eta_n^2}\sum_{t=1}^{n}\sum_{k=0}^{\infty} c_k^2 \bar{Z}_{t-k,n}^2 \le \left(\frac{1}{\eta_n}\sum_{t=1}^{n}\sum_{k=0}^{\infty} |c_k||\bar{Z}_{t-k,n}|\right)^2 \ .$$

Using (6) and the fact that for each n the sequence $\{\bar{Z}_{t,n}\}_{t=-\infty,\infty}$ is stationary, one obtains

$$E\sum_{t=1}^{n}\sum_{k=0}^{\infty} |c_k||\bar{Z}_{t-k,n}| \le n\sum_{k=0}^{\infty} |c_k| E|\bar{Z}_{t-k,n}| = CnE|\bar{Z}_{1,n}| = o(\eta_n).$$

Thus, $I_2 = o_P(\eta_n^2)$. In view od $\mathrm{Var} Z = \infty$ one has $L_Z(\eta_n) \to \infty$. Recalling that $\eta_n^2 \sim nL_Z(\eta_n)$ we have for the last term

$$E|I_3| \le \left(\sum_{k=0}^{\infty} |c_k|\right)^2 \sum_{t=1}^{n} E|Z_{t-1}|E|Z_{t-2}| \le n\left(\sum_{k=0}^{\infty} |c_k|\right)^2 (E|Z_1|)^2 = o(\eta_n^2) \ .$$

□

The next result shows that the maximal term in the linear process is asymptotically negligible compared to the self-normalizer V_n. This generalizes Theorem 1 in O'Brien (1980).

Lemma 3 *Under the conditions of Theorem 2.1,* $\max_{1\le t\le n}|X_t|/V_n \xrightarrow{p} 0$.

Proof. Let $\gamma^2 = \sum_{k=0}^{\infty} c_k^2$. By Lemma 2 we have for some $\delta \in (0,1)$

$$
\begin{aligned}
&P\left(\frac{\max_{1\le t\le n}|X_t|}{V_n} > \varepsilon\right)\\
&\le P\left(\frac{\max_{1\le t\le n}|X_t|}{V_n} > \varepsilon, V_n > (1-\delta)\eta_n\gamma\right) + o(1)\\
&\le P\left(\max_{1\le t\le n}|X_t| > \eta_n\varepsilon(1-\delta)\gamma\right) + o(1)\\
&\le P\left(\max_{1\le t\le n}\left|\sum_{k=0}^{m} c_k Z_{t-k}\right| > \eta_n\varepsilon(1-\delta)\gamma/3\right) + P\left(\max_{1\le t\le n}\left|\sum_{k=m+1}^{\infty} c_k \hat{Z}_{t-k,n}\right| > \eta_n\varepsilon(1-\delta)\gamma/3\right)\\
&\quad + P\left(\max_{1\le t\le n}\left|\sum_{k=m+1}^{\infty} c_k \bar{Z}_{t-k,n}\right| > \eta_n\varepsilon(1-\delta)\gamma/3\right) + o(1)\\
&= I_1 + I_2 + I_3 + o(1)\ .
\end{aligned}
$$

Since $\max_{1\le t\le n}|Z_t|/\eta_n \xrightarrow{p} 0$ (cf. the proof of Lemma 1) we have for the first part

$$
I_1 \le P\left(\sum_{k=0}^{m}|c_k| \max_{-m\le t\le n}|Z_t| > \eta_n\varepsilon(1-\delta)\gamma/3\right) \to 0
$$

as $n\to\infty$. For I_2 we have

$$
\begin{aligned}
I_2 &\le nP\left(\left|\sum_{k=m+1}^{\infty} c_k \hat{Z}_{t-k,n}\right| > \eta_n\varepsilon\gamma(1-\delta)/3\right) \le Cn\eta_n^{-2}E\left|\sum_{k=m+1}^{\infty} c_k \hat{Z}_{t-k,n}\right|^2\\
&\le Cn\eta_n^{-2}\left\{\sum_{k=m+1}^{\infty} c_k^2 E\hat{Z}_{1-k,n}^2 + \sum_{k=m+1}^{\infty}\sum_{j=m+1,j\ne k} c_k c_j \left(E|\hat{Z}_{1,n}|\right)^2\right\}\\
&\le \left(\sum_{k=m+1}^{\infty} c_k^2\right)\frac{nE\hat{Z}_{1,n}^2}{\eta_n^2} + \left(\sum_{k=m+1}^{\infty}|c_k|\right)^2 n\left(\frac{E|\hat{Z}_{1,n}|}{\eta_n}\right)^2 .
\end{aligned}
$$

By the definition, $\frac{nE\hat{Z}_{1,n}^2}{\eta_n^2} = \frac{nL_Z(\eta_n)}{\eta_n^2} = O(1)$. Since $E|\hat{Z}_{1,n}| < E|Z_1| < \infty$ we let first $n \to \infty$ to obtain that the second part is either $o(1)$ (if $EZ_1^2 = \infty$) or bounded by $(E|Z_1|)\sigma^{-2}\left(\sum_{k=m+1}^{\infty}|c_k|\right)^2$. In either case we let $m \to \infty$ to obtain $I_2 = o(1)$. Similarly, letting $n \to \infty$,

$$I_3 \le n\eta_n^{-1}E\left|\sum_{k=m+1}^{\infty} c_k \bar{Z}_{1-k,n}\right| \le n\eta_n^{-1}E|\bar{Z}_{1,n}|\sum_{k=m+1}^{\infty}|c_k| \to 0\ .$$

□

Proposition 4 *Under conditions of Theorem 2.1,* $\frac{S_n}{\eta_n} \xrightarrow{\mathrm{d}} N\left(0, (\sum_{k=0}^{\infty} c_k)^2\right)$.

Proof. The first component in the decomposition (5) gives the required asymptotics as the limit of an i.i.d. random variables. For the second term

$$P\left(\left|\sum_{t=1}^{m} Z_{1-t}\sum_{k=t}^{m} c_k\right| > \varepsilon\eta_n\right) \le \frac{1}{\varepsilon\eta_n}\sum_{t=1}^{m} E|Z_{1-t}|\sum_{k=t}^{m}|c_k| \to 0$$

as $n \to \infty$ since m is finite, and similarly for the third one. It remains to deal with the last term in (5). We bound it by two parts:

$$I_n := \left|\sum_{k=m+1}^{\infty} c_k \sum_{t=1}^{n} \hat{Z}_{t-k,n}\right| \ , \quad J_n := \left|\sum_{k=m+1}^{\infty} c_k \sum_{t=1}^{n} \bar{Z}_{t-k,n}\right| \ .$$

We have

$$\lim_{m\to\infty}\limsup_{n\to\infty} P(I_n > \varepsilon\eta_n) \le \lim_{m\to\infty}\limsup_{n\to\infty}\eta_n^{-2}E\left|\sum_{k=m+1}^{\infty} c_k\sum_{t=1}^{n}\hat{Z}_{t-k,n}\right|^2$$
$$\le \lim_{m\to\infty}\limsup_{n\to\infty}\eta_n^{-2}\left(\sum_{k=m+1}^{\infty}|c_k|\right)^2 E\left(\sum_{t=1}^{n}\hat{Z}_{t-k,n}\right)^2 \le \frac{nE\hat{Z}_{0,n}^2}{\eta_n^2}\left(\sum_{k=m+1}^{\infty}|c_k|\right)^2$$
$$\le C\lim_{m\to\infty}\left(\sum_{k=m+1}^{\infty}|c_k|\right)^2 = 0\ .$$

As for J_n we deal with it in the same way as with the term I_2 in the proof of Lemma 2. Thus, the proof is finished by (Billingsley, 1968, Theorem 4.2)

$\square$

Proof of Theorem 2.1. The proof follows from Proposition 4 and Lemma 2.

$\square$

4 Invariance principle

Proof of Theorem 2.2. Using decomposition (5) we obtain

$$\max_{j\le n}\left|\frac{S_j}{\beta V_n}-\frac{W(j)}{\sqrt{n}}\right|$$
$$\le \max_{j\le n}\left|\frac{\left(\sum_{k=0}^{m}c_k\right)\sum_{t=1}^{j}Z_t}{\beta V_n}-\frac{W(j)}{\sqrt{n}}\right|+\max_{j\le n}\left|\frac{\sum_{k=m+1}^{\infty}c_k\sum_{t=1}^{j}Z_{t-k}}{\beta V_n}\right|$$
$$+\left|\frac{\sum_{t=1}^{m}Z_{1-t}\sum_{k=t}^{m}c_k}{\beta V_n}\right|+\max_{j\le n}\left|\frac{\sum_{t=0}^{m-1}Z_{j-t}\sum_{k=t+1}^{m}c_k}{\beta V_n}\right| .$$

Since $V_n^2/\eta_n^2 \xrightarrow{p} \sum_{k=0}^{\infty}c_k^2$, the third and the fourth part are treated in the same way as I_2+I_3 in the proof of Lemma 1. Also, the second part is treated in the same way as the terms I_n, J_n in the proof of Proposition 4. As for the first part,

$$\max_{j\le n}\left|\frac{\left(\sum_{k=0}^{m}c_k\right)\sum_{t=1}^{j}Z_t}{\beta V_n}-\frac{W(j)}{\sqrt{n}}\right|$$
$$\le \max_{j\le n}\left|\frac{\sum_{t=1}^{j}Z_t}{\sqrt{\sum_{t=1}^{n}Z_t^2}}-\frac{W(j)}{\sqrt{n}}\right|+\max_{j\le n}\left|\frac{\left(\sum_{k=0}^{m}c_k\right)\sum_{t=1}^{j}Z_t}{\beta V_n}-\frac{\sum_{t=1}^{j}Z_t}{\sqrt{\sum_{t=1}^{n}Z_t^2}}\right| =: A_1+A_2 .$$

In view of Theorem 1 in Csörgő et al. (2003b), $A_1=o_P(1)$. The term A_2 is bounded by

$$A_2\le\left|\frac{\left(\sum_{k=0}^{m}c_k\right)\sqrt{\sum_{t=1}^{n}Z_t^2}}{\beta V_n}-1\right|\max_{j\le n}\left|\frac{\sum_{t=1}^{j}Z_t}{\sqrt{\sum_{t=1}^{n}Z_t^2}}\right| =: B_1\cdot B_2 .$$

We have $B_2 = O_P(1)$. Thus, it suffices to prove that for all $\varepsilon > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left(\left|(\beta V_n)^{-1}\left(\sum_{k=0}^{m} c_k\right)\sqrt{\sum_{t=1}^{n} Z_t^2} - 1\right| > \varepsilon\right) = 0\ . \tag{8}$$

We have

$$\left|\frac{\left(\sum_{k=0}^{m} c_k\right)\sqrt{\sum_{t=1}^{n} Z_t^2}}{\beta V_n} - 1\right| \le \left|\frac{\left(\sum_{k=0}^{\infty} c_k\right)\sqrt{\sum_{t=1}^{n} Z_t^2}}{\beta V_n} - 1\right| + \left|\sum_{k=m+1}^{\infty} c_k\right| \left|\frac{\sqrt{\sum_{t=1}^{n} Z_t^2}}{\beta V_n}\right|$$

Since $V_n^2/\eta_n^2 \xrightarrow{p} \sum_{k=0}^{\infty} c_k^2$ and $\sum_{t=1}^{n} Z_t^2/\eta_n^2 \xrightarrow{p} 1$, the first part in the above expression converges in probability to 0 as $n \to \infty$. Moreover, $\left|\sqrt{\sum_{t=1}^{n} Z_t^2}/V_n\right| = O_P(1)$ as $n \to \infty$. Thus, letting $m \to \infty$ we prove (8).

$\square$

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References

Balan, R., Kulik, R., 2005. Self-normalized weak invariance principle for mixing sequences. Tech. rep., Laboratory for Reearch in Statistics and Probability, Carleton University and University of Ottawa.

Bentkus, V., Götze, F., 1996. The berry-esseen bound for student's statistic. Ann. Probab. 24, 491–503.

Billingsley, P., 1968. Convergence of probability measures.

Brockwell, P. J., Davis, R. A., 1987. Time series: theory and methods. Springer Series in Statistics. Springer-Verlag, New York.

Csörgő, M., Szyszkowicz, B., Wang, Q., 2003a. Darling-erdös theorem for self-normalzied sums. Ann. Probab. 31, 676–692.

Csörgő, M., Szyszkowicz, B., Wang, Q., 2003b. Donsker's theorem for self-normalized partial sums processes. Ann. Probab. 31, 1228–1240.

Csörgő, M., Szyszkowicz, B., Wang, Q., 2004. On weighted approximations and strong limit theorems for self-normalized partial sums processes. In: Asymptotic methods in stochastics. Fields Institute, pp. 489–521.

Efron, B., 1969. Student's t-test under symmetry conditions. J. Amer. Statist. Assoc. 64, 1278–1302.

Giné, E., Götze, F., Mason, D., 1997. When is the student t-statistic asymptotically standard normal? Ann. Probab. 25 (1514–1531).

Griffin, P. S., Kuelbs, J. D., 1989. Self-normalized laws of the iterated logarithm. Ann. Probab. 17, 1571–1601.

Gut, A., 2005. Gnedenko-raikov's theorem, central limit theory and the weak law of large numbers, prerint.

Logan, B. F., Mallows, C. L., Rice, S. O., Shepp, L. A., 1973. Limit distributions of self-normalized sums. Ann. Probab. 1, 788–809.

Mason, D., 2005. The asymptotic distribution of self-normalized triangular arrays, to appear.

O'Brien, G., 1980. A limit theorem for sample maxima and heavy branches in galton-watson trees.

Phillips, P. C. B., Solo, V., 1992. Asymptotics for linear processes. Ann. Statist. 20 (2), 971–1001.

Račkauskas, A., Suquet, C., 2001. Invariance principles for adaptive self-normalized partial sums processes. Stochastic Process. Appl. 95, 63–81.

Shao, Q.-M., 1997. Self-normalized large deviations. Ann. Probab. 25, 285–328.

Shao, Q.-M., 1998. Recent developments on self-normalized limit theorems. In: Asymptotic methods in probability and statistics. North-Holland, pp. 467–480.

Wang, Q., Lin, Y.-X., Gulati, C. M., 2001. Asymptotics for moving average processes with dependent innovations. Statist. Probab. Lett. 54 (4), 347–356.

Wang, Q., Lin, Y.-X., Gulati, C. M., 2002. The invariance principle for linear processes with applications. Econometric Theory 18, 119–139.