DEPENDENCE ORDERING FOR QUEUEING NETWORKS WITH BREAKDOWN AND REPAIR

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Abstract

In this paper we introduce isotone differences stochastic ordering of Markov processes on lattice ordered state spaces as a device to compare internal dependencies of two such processes. We derive a characterization in terms of intensity matrices. This enables us to compare the internal dependency structure of different degradable Jackson networks in which the nodes are subject to random breakdowns and repairs. We show that the performance behavior and the availability of such networks can be compared.

Key words: dependence ordering, supermodular functions, Markov processes, Jackson networks, degradable networks with breakdowns and repairs

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1 Introduction

Let $Y = (Y(t), t \geq 0)$, $\widetilde{Y} = (\widetilde{Y}(t), t \geq 0)$ be two stationary homogeneous Markov processes with state space E, possessing identical invariant (stationary) distribution π. We assume that E is a lattice ordered Polish space equipped with a partial ordering $\prec$. Especially, we consider $E = \mathbb{R}$ with standard $\leq$ ordering, $E = \mathbb{N}^J$ with coordinate wise ordering $\leq^J$, and $E = \mathcal{P}(\mathrm{J})$, the power set of $\mathrm{J} = \{1, \ldots, J\}$, with inclusion ordering $\subseteq$.

The main difference among these examples is that $\mathbb{R}$ is totally (linearly) ordered, while $\mathbb{N}^J$ and $\mathcal{P}(\mathrm{J})$ are only lattice ordered. The dependence ordering for network processes that we have in mind requires methods and theory for non linearly ordered spaces, but up to now in the literature only processes with totally ordered state space have been investigated.

For describing iterated observations of stochastic processes we need product spaces and use the following notation. If $E^{(n)} = \times_{i=1}^{n} E_i$ is a product of partially lattice ordered spaces $(E_i, \prec_i)$, $i = 1, \ldots, n$, then the partial ordering $\prec^{(n)}$ on $E^{(n)}$ is the coordinate wise ordering, i.e. for $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in E^{(n)}$, $\mathbf{x} \prec^{(n)} \mathbf{y}$ if and only if $x_i \prec_i y_i$ for all $i = 1, \ldots, n$. If $(E_i, \prec_i) = (E, \prec)$ for $i = 1, \ldots, n$, then we write E^n instead of $E^{(n)}$ and $\prec^n$ instead of $\prec^{(n)}$. Clearly, if E_i are lattice ordered then $(E^{(n)}, \prec^{(n)})$ is lattice ordered as well.

We consider integral stochastic orders on a lattice ordered space $(E^{(n)}, \prec^{(n)})$. A function $f : E^{(n)} \to \mathbb{R}$ is *supermodular* if

$$f(x \wedge y) + f(x \vee y) \geq f(x) + f(y)\,. \tag{1.1}$$

Note that the definition of supermodular functions does not require that they are defined on a product space $E^{(n)}$. A function $f : E^{(2)} \to \mathbb{R}$ has *isotone differences* if for $x_1 \prec_1 x_1'$, $x_2 \prec_2 x_2'$ we have

$$f(x_1', x_2') - f(x_1, x_2') \geq f(x_1', x_2) - f(x_1, x_2). \tag{1.2}$$

A function $f : E^{(n)} \to \mathbb{R}$ has *isotone differences* if Eq. (1.2) is satisfied for any pair i, j of coordinates, whereas the remaining variables are fixed. If the E_i are totally ordered both definitions are equivalent. However, for not totally ordered spaces conditions (1.1) and (1.2) are not equivalent as mentioned by Heyman and Sobel [8]. We discuss both definitions in Section 2.1 and show that the class of functions with isotone differences on a product of general partially ordered spaces possesses similar closure properties as on products of totally ordered spaces, whereas for supermodular functions some of those properties may fail.

We say that two random elements X, Y of E are ordered in supermodular ordering ($X <_{\rm sm} Y$) if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all supermodular functions f on E, for which the expectations exist. Supermodular ordering on a totally ordered space is a dependence ordering in the sense of Joe [11]. In particular, if two real valued stationary processes $Y = (Y(t), t \geq 0)$ and $\widetilde{Y} = (\widetilde{Y}(t), t \geq 0)$ are comparable w.r.t $<_{\rm sm}$ then for each $t \geq 0$, $\mathbf{Cov}[Y(0), Y(t)] \leq \mathbf{Cov}[\widetilde{Y}(0), \widetilde{Y}(t)]$. Moreover one dimensional marginal distributions for both processes coincide. In case of uniform Markov processes in continuous (discrete) time on countable subsets of $\mathbb{R}$ supermodular ordering is characterized in terms of generators (transition probabilities) by Hu and Pan [9], see also Bäuerle [1], Bäuerle and Rolski [2] for previous results in this area. In order to obtain comparison results it is needed to assume stochastic monotonicity of both Y and $\widetilde{Y}$. We are not aware of any results concerning supermodular ordering of Markov processes without this restriction, except of the recent papers by Kulik [13] and Daduna and Szekli [5]. We refer to Müller and Stoyan [15] for a review of recent work and applications of the supermodular ordering in case of totally ordered state spaces. For partially ordered state spaces, some results extending Hu and Pan [9] concerning dependence orderings for Markov processes are given in the paper by Daduna and Szekli [5].

Motivating example for this paper is the Markov process defined by the joint queue lengths (in other words, network process) in a non-standard Jackson network of exponential queues. Standard Jackson networks are described in Jackson [10]. The non-standard feature incorporated here is that the nodes of the network are subject to breakdowns and repairs as studied by Sauer and Daduna [16]. Such networks are often called degradable (Jackson) networks.

It is well known that two (standard) network processes may have the same stationary distribution of the joint queue length process but their time behavior is completely different otherwise. Moreover, in equilibrium, for a fixed time instant the correlation in space (between nodes) vanishes. However, it is known that correlation in time does not vanish, see for instance Disney and Kiessler [7]. An investigation of dependence structure is treated systematically in Daduna and Szekli [4], see also Kanter [12]. Comparison of dependence in different network processes with the same equilibrium (stationary) distribution is a natural problem. Due to the multidimensional nonlinear state space the results of Hu and Pan [9] and others are not applicable. Therefore in Section 2.1 we study properties of supermodular-type functions on partially ordered spaces and collect definitions of the relevant dependence orderings. The results concerning orderings of Markov processes in continuous-time are given in the rest of Section 2. In Sections 3 and 4 we apply our results to the comparison of various networks of queues with unreliable nodes. Our results concerning comparison of Markov processes allow us to compare two (degradable) Jackson networks with the same stationary distribution but with different routing or different breakdown and repair structure. We are able to compare in this integrated (queueing/reliability) model the performance of the joint queue length process as well as the networks' availabilities. The main point to emphasize is that from the very construction the breakdown and repair process is Markovian by itself, and can (and will) be investigated directly, but the queue length process in the integrated model is not Markovian. Therefore results on dependence ordering of Markov processes can not be applied directly. As corollary we obtain new results on comparison of standard (completely reliable) networks. Related results for standard Jackson type networks are given by Daduna and Szekli [6]. In Kulik and Sauer [14] correlation inequalities concerning dependence in lag for unreliable networks are studied.

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2 Isotone differences ordering for Markov processes on lattice ordered state spaces

2.1 Functions with isotone differences and related orderings

Assume that E_i, $i = 1, \ldots, n$, are lattice ordered Polish spaces with Borel σ-algebra $\mathcal{E}_i$ and a closed partial ordering $\prec_i$, respectively. Then $(E^{(n)}, \prec^{(n)})$ is a lattice and hence for any $x, y \in E^{(n)}$ there exists a unique largest lower bound $x \wedge y \in E^{(n)}$ and a smallest upper bound $x \vee y \in E^{(n)}$. In this paper we use the following particular cases.

(i) Let $(E_i, \prec_i) = (E, \prec) = (\mathbb{R}, \leq)$ for $i = 1, \ldots, n$, with $\leq$ the standard linear order in $\mathbb{R}$. Then for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in E^n$, we have $x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$ and $x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n)$ with $x_i \wedge y_i = \min\{x_i, y_i\}$, $i = 1, \ldots, n$ and $x_i \vee y_i = \max\{x_i, y_i\}$, $i = 1, \ldots, n$.

(ii) Let $\mathrm{J} = \{1, \ldots, J\}$ and $\mathcal{P}(\mathrm{J})$ its power set. Taking $(E_i, \prec_i) = (E, \prec) = (\mathcal{P}(\mathrm{J}), \subseteq)$, $i = 1, \ldots, n$, with $\subseteq$ being inclusion ordering, we have for $\mathrm{A}_i, \mathrm{B}_i \in E_i$ that $\mathrm{A}_i \wedge \mathrm{B}_i = \mathrm{A}_i \cap \mathrm{B}_i$, $\mathrm{A}_i \vee \mathrm{B}_i = \mathrm{A}_i \cup \mathrm{B}_i$, $i = 1, \ldots, n$. It is clear that $(E, \subseteq)$ and $(E^n, \subseteq^n)$ are lattices.

We denote by $\mathcal{I}^*(E, \prec)$ ($\mathcal{D}^*(E, \prec)$) the set of all real valued $\prec$-increasing (decreasing) functions on $(E, \mathcal{E})$. Moreover, by $\mathcal{L}_{\mathrm{sm}}(E^{(n)}, \prec^{(n)})$ we denote the class of all real valued supermodular functions on $E^{(n)}$ as defined by Eq. (1.1). Related to this class is the class of functions with isotone differences, defined by Eq. (1.2). We denote this class by $\mathcal{L}_{\mathrm{idif}}(E^{(n)}, \prec^{(n)})$. Note that the definition of a function with isotone differences does not require that E_i are lattices. If, additionally, f is taken to be increasing we write $f \in \mathcal{L}_{\mathrm{i-idif}}(E^{(n)}, \prec^{(n)})$. If it is clear which space and which ordering we mean we omit $(E^{(n)}, \prec^{(n)})$ in these notations. The following lemma (for a proof see Heyman and Sobel [8, Theorem 8-1, p. 375]) implies that in case of totally ordered spaces both notions coincide. This is not the case when E_i, $i = 1 \ldots, n$, are only partially ordered lattices.

Lemma 2.1 (i) *Let $E_1, E_2, \ldots, E_n$ be lattices. If f on $(E^{(n)}, \prec^{(n)})$ is supermodular then it has also isotone differences.*

(ii) *Let $E_1, \ldots, E_n$ be totally ordered. If f has isotone differences on $(E^{(n)}, \prec^{(n)})$ then it is also supermodular.*

For $\mathcal{L}_{\mathrm{i-idif}}$ and $\mathcal{L}_{\mathrm{idif}}$ classes it is easy to prove preservation results which are parallel to the totally ordered case (cf. Theorem 3.9.3 in Müller and Stoyan [15]), see [3] for a list of such properties. In particular, we have the following lemma.

Lemma 2.2 *Let $f, g : (E^{(n)}, \prec^{(n)}) \to \mathbb{R}_+$ and assume that E_i, $i = 1, \ldots, n$ are partially ordered.*

(i) *If $f \in \mathcal{L}_{\mathrm{idif}}(E^{(n)}, \prec^{(n)})$ and $g_i : E_i \to E_i$ are $\prec_i$-increasing then $f(g_1(\cdot), \ldots, g_n(\cdot)) \in \mathcal{L}_{\mathrm{idif}}(E^{(n)}, \prec^{(n)})$.*

(ii) *If $f, g \in \mathcal{L}_{\mathrm{i-idif}}(E^{(n)}, \prec^{(n)})$ are nonnegative then $fg \in \mathcal{L}_{\mathrm{i-idif}}(E^{(n)}, \prec^{(n)})$. In particular, if $f \in \mathcal{I}^*(E_1, \prec_1)$, $g \in \mathcal{I}^*(E_2, \prec_2)$ with nonnegative values then for fg defined on $E^{(2)} = E_1 \times E_2$ by $(fg)(x, y) = f(x)g(y)$ for $x \in E_1, y \in E_2$ holds $fg \in \mathcal{L}_{\mathrm{i-idif}}(E^{(2)}, \prec^{(2)})$. Further for nonnegative $f \in \mathcal{D}^*(E_1, \prec_1)$ and $g \in \mathcal{D}^*(E_2, \prec_2)$ holds $fg \in \mathcal{L}_{\mathrm{idif}}(E^{(2)}, \prec^{(2)})$.*

The following example shows however that increasing transformations do not preserve supermodularity and a product of increasing functions need not to be supermodular on lattice ordered spaces which are not totally ordered.

Example 2.1 Let $E_1 = E_2 = \{\{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$ with standard ordering $\prec := \subseteq$. Define $g(\{0\}) = \{0,1\}$, $g(\{0,1\}) = g(\{0,2\}) = g(\{0,1,2\}) = \{0,1,2\}$ and $h(\mathrm{B}) = \mathrm{B}$ for each $\mathrm{B} \in E_1$. Both functions are increasing. The function $f(\mathrm{A}, \mathrm{B}) = \mathbb{I}_{(\mathrm{A} \cap \mathrm{B} = \mathrm{C})}(\mathrm{A}, \mathrm{B})$ for $\mathrm{C} = \{0,1,2\}$ is supermodular on $(E^{(2)}, \prec^{(2)})$. Define $\tilde{f}(\mathrm{A}, \mathrm{B}) = f(g(\mathrm{A}), h(\mathrm{B}))$. The function $\tilde{f}$ is not supermodular. Indeed, for $\mathrm{A}_1 = \{0,1\}$, $\mathrm{A}_2 = \{0,2\}$ and $\mathrm{B}_1 = \mathrm{B}_2 = \{0,1,2\}$ we have

$$
\begin{aligned}
& \tilde{f}(\mathrm{A}_1 \cap \mathrm{A}_2, \mathrm{B}_1 \cap \mathrm{B}_2) + \tilde{f}(\mathrm{A}_1 \cup \mathrm{A}_2, \mathrm{B}_1 \cup \mathrm{B}_2) - \tilde{f}(\mathrm{A}_1, \mathrm{B}_1) - \tilde{f}(\mathrm{A}_2, \mathrm{B}_2) \\
= \quad & f(g(\mathrm{A}_1 \cap \mathrm{A}_2), h(\mathrm{B}_1 \cap \mathrm{B}_2)) + f(g(\mathrm{A}_1 \cup \mathrm{A}_2), h(\mathrm{B}_1 \cup \mathrm{B}_2))
\end{aligned}
$$

$$
\begin{aligned}
& -f(g(\mathrm{A}_1),h(\mathrm{B}_1)) - f(g(\mathrm{A}_2),h(\mathrm{B}_2)) \\
= \;& \mathbb{I}_{(g(\mathrm{A}_1\cap \mathrm{A}_2)\cap h(\mathrm{B}_1\cap \mathrm{B}_2)=\mathrm{C})} + \mathbb{I}_{(g(\mathrm{A}_1\cup \mathrm{A}_2)\cap h(\mathrm{B}_1\cup \mathrm{B}_2)=\mathrm{C})} \\
& -\mathbb{I}_{(g(\mathrm{A}_1)\cap h(\mathrm{B}_1)=\mathrm{C})} - \mathbb{I}_{(g(\mathrm{A}_2)\cap h(\mathrm{B}_2)=\mathrm{C})} \\
= \;& \mathbb{I}_{(\{0,1\}\cap\{0,1,2\}=\{0,1,2\})} + \mathbb{I}_{(\{0,1,2\}\cap\{0,1,2\}=\{0,1,2\})} \\
& -\mathbb{I}_{(\{0,1,2\}\cap\{0,1,2\}=\{0,1,2\})} - \mathbb{I}_{(\{0,1,2\}\cap\{0,1,2\}=\{0,1,2\})} = -1
\end{aligned}
$$

In the above setting, a sufficient condition for $\tilde{f}$ to be supermodular is $k(A_1) \cap k(A_2) = k(A_1 \cap A_2)$ for both $k = h$ and $k = g$. Here, this is not the case. Moreover, since $\mathbb{I}_{(g(\mathrm{A})\cap\mathrm{B}=\mathrm{C})}(\mathrm{A},\mathrm{B}) = \mathbb{I}_{(g(\mathrm{A})=\mathrm{C})}(\mathrm{A})\mathbb{I}_{(\mathrm{B}=\mathrm{C})}(\mathrm{B})$, $\tilde{f}$ is a product of increasing functions not being supermodular. However $\tilde{f}$, using Lemma 2.2 (ii), has isotone differences. Therefore, we also showed that there exists a function which has isotone differences but is not supermodular.

From the above example it follows that supermodular ordering on lattice ordered state spaces in general is not preserved under increasing transformations. Therefore, the class of supermodular functions is restrictive in the study of dependence orderings of random processes on general partially ordered spaces. Hence, we limit our investigations to functions with isotone differences and we call the corresponding ordering the *isotone differences ordering.*

Definition 2.1 *Let $Y = (Y_1, \dots, Y_n)$, $\widetilde{Y} = (\widetilde{Y}_1, \dots, \widetilde{Y}_n)$ be random vectors with values in $(E^{(n)}, \mathcal{E}^{(n)}, \prec^{(n)})$.*

(i) *Y is smaller than $\widetilde{Y}$ in the isotone differences ordering ($Y <_{\mathrm{idif}(\prec^{(n)})} \widetilde{Y}$) if*

$$\mathbb{E}\left[f(Y_1 \dots, Y_n)\right] \leq \mathbb{E}\left[f(\widetilde{Y}_1, \dots, \widetilde{Y}_n)\right]$$

for all $f \in \mathcal{L}_{\mathrm{idif}}(E^{(n)}, \prec^{(n)})$ such that both expectations exist.

(ii) *Let $Y = (Y(t), t \in T)$, $\widetilde{Y} = (\widetilde{Y}(t), t \in T)$, $T \subseteq \mathbb{R}$ be stochastic processes. Then we write $Y <_{\mathrm{idif}(\prec^{(\infty)})} \widetilde{Y}$ if for all $n \geq 2$, all $t_1 < \dots < t_n \in T$ we have $(Y(t_1), \dots, Y(t_n)) <_{\mathrm{idif}(\prec^{(n)})} (\widetilde{Y}(t_1), \dots, \widetilde{Y}(t_n))$.*

For simplicity we skip $(\prec^{(n)})$ in the above notations if it will be clear which orderings are used. We also skip the necessary assumption that *the expectations exist* in statements similar to the above.

2.2 Continuous time Markov chains

Let $Y = (Y(t), t \geq 0)$, $\widetilde{Y} = (\widetilde{Y}(t), t \geq 0)$ be stationary homogeneous continuous time Markov processes with a countable lattice ordered state space $(E, \mathcal{E}, \prec)$ and transition kernel families $\mathbf{P}^Y := (P_t^Y : E \times \mathcal{E} \to [0,1], t \geq 0)$ and $\mathbf{P}^{\widetilde{Y}} := (P_t^{\widetilde{Y}} : E \times \mathcal{E} \to [0,1], t \geq 0)$, respectively. $\mathbf{P}^Y_{\leftarrow}$, $\mathbf{P}^{\widetilde{Y}}_{\leftarrow}$ are transition families for the time reversed processes. $Q^Y = (q^Y(x,y),\ x, y \in E)$, $Q^{\widetilde{Y}} = (q^{\widetilde{Y}}(x,y),\ x, y \in E)$, $Q^Y_{\leftarrow} = (q^Y_{\leftarrow}(x,y),\ x, y \in E)$, $Q^{\widetilde{Y}}_{\leftarrow} = (q^{\widetilde{Y}}_{\leftarrow}(x,y),\ x, y \in E)$ are the corresponding infinitesimal generators. We always assume that all processes are uniform (i.e. their generators are bounded and conservative).

A transition kernel family $\mathbf{P}^Y := (P_t^Y : E \times \mathcal{E} \to [0,1], t \geq 0)$ is $\prec$-monotone if all the P_t are $\prec$-monotone. Recall that a transition kernel P_t is $\prec$-monotone if the map $x \to \int_E P_t(x, dy) f(y)$ is $\prec$-increasing provided f is $\prec$-increasing. From now on we assume that either $\mathbf{P}^Y$ and $\mathbf{P}^{\widetilde{Y}}_{\leftarrow}$ or $\mathbf{P}^Y_{\leftarrow}$ and $\mathbf{P}^{\widetilde{Y}}$ are families of $\prec -$ monotone kernels (and refer to this property as to *the monotonicity assumption*).

The proof of the characterization theorem needs the following property of the isotone differences order, which is proved in [3, Lemma 2.7], see [15, Lemma 5.2.17] for supermodular order on totally ordered spaces.

Lemma 2.3 *Assume that kernel P_t is $\prec$-monotone and f has isotone differences on $(E^{n+1}, \prec^{n+1})$. Then a function $h : E^n \to \mathbb{R}$ defined by*

$$h(x_0, \ldots, x_{n-1}) = \int_E P_t(x_{n-1}, dx_n) f(x_0, \ldots, x_n)$$

has isotone differences on $(E^n, \prec^n)$.

Using Lemma 2.3 the proof of the characterization theorem follows the same arguments as Theorem 3.6 in Daduna and Szekli [5] for concordance ordering.

Theorem 2.1 *Under the above assumptions (recall especially the* the monotonicity assumption*) the following properties are equivalent:*

(i) $Y <_{\text{idif}} \widetilde{Y}$*;*

(ii) $(Y_0, Y_t) <_{\text{idif}} (\widetilde{Y}_0, \widetilde{Y}_t)$*, for all $t \geq 0$;*

(iii) *For any* $f \in \mathcal{L}_{\mathrm{idif}}(E^2, \prec^2)$ *we have*

$$\sum_{x\in E} \pi(x) \sum_{y\in E} f(x,y) q^{Y}(x,y) \le \sum_{x\in E} \pi(x) \sum_{y\in E} f(x,y) q^{\widetilde{Y}}(x,y)\,. \tag{2.3}$$

2.3 ε transformation

Consider a stationary Markov process Y with intensities $q^Y(x,y)$ and stationary distribution π. For $x_1 \prec x_2$, $y_1 \prec y_2$ and fixed $\varepsilon > 0$ such that the quantities in Eq. (2.4) define a new intensity matrix of a Markov process $\widetilde{Y^\varepsilon}$ (called ε transformation of Y) set

$$q^{\widetilde{Y^\varepsilon}}(x,y) = \begin{cases} q^Y(x,y) + \frac{\varepsilon}{\pi(x)}, & \text{if } (x = x_1, y = y_1) \text{ or } (x = x_2, y = y_2)\,, \\ q^Y(x,y) - \frac{\varepsilon}{\pi(x)}, & \text{if } (x = x_1, y = y_2) \text{ or } (x = x_2, y = y_1)\,, \\ q^Y(x,y), & \text{otherwise}\,. \end{cases} \tag{2.4}$$

Both Markov processes $\widetilde{Y^\varepsilon}$ and Y have the same stationary distribution (see the lemma below). Furthermore, $\widetilde{Y^\varepsilon}$ is more likely to stay in the same state compared to Y. Thus, $\widetilde{Y^\varepsilon}$ has longer memory and intuitively should be greater than Y according to suitably defined dependence orderings. This will be proved in Proposition 2.1.

Lemma 2.4 *The Markov processes* Y *and* $\widetilde{Y^\varepsilon}$ *with the transition intensities given by Eq. (2.4) have equal invariant distribution* π.

Proof . We have for the intensity matrix of Y: $\pi Q^Y = \mathbf{0}$, where $\mathbf{0}$ is the null-vector. It means that for any $y \in E$ we have

$$\sum_{x\in E} \pi(x) q^Y(x,y) = 0\,.$$

Now, for $y \neq y_1, y_2$ we have

$$\sum_{x\in E} \pi(x) q^{\widetilde{Y^\varepsilon}}(x,y) = \sum_{x\in E} \pi(x) q^Y(x,y) = 0\,.$$

Moreover, for $y = y_1$

$$\sum_{x\in E} \pi(x) q^{\widetilde{Y^\varepsilon}}(x,y_1) \quad = \quad \sum_{x\neq x_1,x_2} \pi(x) q^Y(x,y_1)$$

$$
\begin{aligned}
&+\pi(x_1)\left(q^Y(x_1,y_1)+\frac{\varepsilon}{\pi(x_1)}\right)+\pi(x_2)\left(q^Y(x_2,y_1)-\frac{\varepsilon}{\pi(x_2)}\right)\\
= &\sum_{x\in E}\pi(x)q^Y(x,y_1)=0\,.
\end{aligned}
$$

It follows analogously that $\sum_{x\in E}\pi(x)q^{\widetilde{Y^\varepsilon}}(x,y_2)=0$.

■

Proposition 2.1 *Consider Markov processes Y and $\widetilde{Y^\varepsilon}$ (with intensities given by Eq. (2.4)). Under the monotonicity assumption $Y <_{\text{idif}} \widetilde{Y^\varepsilon}$.*

Proof. From Theorem 2.1 it is sufficient to prove that for all $f \in \mathcal{L}_{\text{idif}}(E^2, \prec^2)$

$$
\sum_{x\in E}\pi(x)\sum_{y\in E}q^Y(x,y)f(x,y)\le\sum_{x\in E}\pi(x)\sum_{y\in E}q^{\widetilde{Y^\varepsilon}}(x,y)f(x,y)
$$

holds. The difference between left-hand-side and right-hand-side is

$$
\varepsilon\bigg(f(x_1,y_1)+f(x_2,y_2)-f(x_1,y_2)-f(x_2,y_1)\bigg)
$$

being nonnegative due to the assumptions on f.

■

3 Comparison of unreliable queueing networks

3.1 Degradable Jackson networks with unreliable nodes and repair

Consider a Jackson network of J numbered nodes, denoted by $\mathrm{J} = \{1,\dots,J\}$. Station $j \in \mathrm{J}$, is a single server queue with infinite waiting room under first-come-first-served (FCFS) regime. Indistinguishable customers arrive in a Poisson stream with intensity $\lambda > 0$ and are sent to node j with probability r_{0j}, $\sum_{j=1}^{J} r_{0j} = r \le 1$. $r_{00} := 1-r$ is the rejection probability for customers at their arrival. Customers arriving at node j from the outside or from other nodes request a service which is exponentially distributed with mean 1. Service at node j is provided with intensity $\mu_j > 0$. We denote by n_j the number of customers at node j including the one being served. All service times and arrival processes are assumed to be independent. A customer departing from node j immediately proceeds to node i with probability $r_{ji} \geq 0$ or departs from the network with probability r_{j0}. The routing is independent of the past of the system given the momentary node where the customer is. Let $\mathrm{J}_0 := \mathrm{J} \cup \{0\}$. We assume that the matrix $R := (r_{ij}, i, j \in \mathrm{J}_0)$ is aperiodic and irreducible. The servers at the nodes in the Jackson network are unreliable, i.e., the nodes may break down. Nodes may break down as an isolated event or in groups simultaneously, repair of nodes may end for each node individually or in groups as well. Nodes that broke down simultaneously need not return to service at the same time. We assume that the breakdown occurs independent of the load (queue lengths) of the nodes and that the repair is independent of the queue lengths as well. For more general behavior we refer to Sauer and Daduna [16].

Control of breakdowns and repairs is as follows: Let $\mathrm{I} \subset \mathrm{J}$ be the set of nodes in down status and $\mathrm{H} \subset \mathrm{J} \backslash \mathrm{I}, \mathrm{H} \neq \emptyset$, be some subset of nodes in up status. Then the nodes of H break down concurrently with intensity $\alpha(\mathrm{I}, \mathrm{I} \cup \mathrm{H})$. Nodes in down status neither accept new customers nor continue serving the old customers who will wait for the server's return. Therefore, the routing has to be changed so that customers attending to join a node in down status are rerouted to nodes in up status or to the outside. We will give three rerouting schemes below. These originate from mechanisms applied e.g. in production theory or in the theory of information blocking. Assume the nodes in I are under repair, $\mathrm{I} \subset \mathrm{J}, \mathrm{I} \neq \emptyset$. Then the nodes of $\mathrm{H} \subset \mathrm{I}, \mathrm{H} \neq \emptyset$, return from repair as a batch with intensity $\beta(\mathrm{I}, \mathrm{I} \setminus \mathrm{H})$ and immediately resume services. Routing then has to be updated anew as will be described below. The intensities for occurrence of breakdowns and repairs have to be set under constraints. A versatile class of intensities is defined as follows.

Definition 3.1 *Let* I *be the set of nodes in down status. The intensities for* *breakdowns, resp. repairs for* $\mathrm{H} \neq \emptyset$ *are defined by*

$$\alpha(\mathrm{I}, \mathrm{I} \cup \mathrm{H}) := \frac{a(\mathrm{I} \cup \mathrm{H})}{a(\mathrm{I})}, \quad \textit{resp.} \quad \beta(\mathrm{I}, \mathrm{I} \setminus \mathrm{H}) := \frac{b(\mathrm{I})}{b(\mathrm{I} \backslash \mathrm{H})}, \tag{3.5}$$

where a *and* b *are any functions,* $a, b : \mathcal{P}(\mathrm{J}) \to \mathbb{R}_+ = [0, \infty)$ *with* $a(\emptyset) = b(\emptyset) = 1$. *We set* $\frac{0}{0} := 0$.

The above intensities are assumed to be finite.

Rerouting matrices of interest are as follows, for details see Sauer and Daduna [16].

Definition 3.2 (BLOCKING) *Assume that the routing matrix of the original process is reversible. Assume the nodes in* I *are presently under repair. Then the routing probabilities are redefined on* $\mathrm{J}_0 \backslash \mathrm{I}$ *according to*

$$\tilde{r}^{\mathrm{I}}_{ij} = \begin{cases} r_{ij}, & i, j \in \mathrm{J}_0 \backslash \mathrm{I},\ i \neq j, \\ r_{ii} + \sum_{k \in \mathrm{I}} r_{ik}, & i \in \mathrm{J}_0 \backslash \mathrm{I},\ i = j. \end{cases} \tag{3.6}$$

Definition 3.3 (STALLING) *If there is any breakdown of some node, then the arrival stream to the network and all service processes are completely interrupted and resumed only when all nodes are repaired again.*

Definition 3.4 (SKIPPING) *Assume that the nodes in* I *are the nodes presently under repair. Then the routing matrix is redefined on* $\mathrm{J}_0 \setminus \mathrm{I}$ *according to:*

$$\hat{r}^{\mathrm{I}}_{jk} = r_{jk} + \textstyle\sum_{i \in \mathrm{I}} r_{ji} \hat{r}^{\mathrm{I}}_{ik}, \quad k, j \in \mathrm{J}_0 \backslash \mathrm{I},$$
$$\hat{r}^{\mathrm{I}}_{ik} = r_{ik} + \textstyle\sum_{l \in \mathrm{I}} r_{il} \hat{r}^{\mathrm{I}}_{lk}, \quad i \in \mathrm{I}, k \in \mathrm{J}_0 \backslash \mathrm{I}.$$

State space and Markov process describing the time evolution of the degradable Jackson network are constructed as follows: Let $X_j(t)$ be the number of customers present at node j at time $t \geq 0$. Then $X(t) = (X_1(t), \ldots, X_J(t))$ is the joint queue length vector at time $t \geq 0$ and $X := (X(t), t \geq 0)$ is the joint queue length process on state space $(E, \prec) := (\mathbb{N}^J, \leq^J)$ (where $\leq^J$ denotes the standard coordinate wise ordering). The availability status of the network at time $t \geq 0$ is indicated by $Y(t) = \mathrm{I}$, if $\mathrm{I} \subseteq \mathrm{J}$ is the set of nodes that are broken down (under repair). $Y := (Y(t), t \geq 0)$ is the availability (breakdown/repair) process. From this description it is easy to see that the joint process

$$Z := (Z(t), t \geq 0), \quad Z(t) := (X(t), Y(t)), \quad t \geq 0,$$

is a strong Markov process on state space $E = \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J$. We refer to $Z = (X, Y)$ as the queue length/availability or queueing/reliability process. States of Z are

$$(\mathrm{I};\ n_1, n_2, \ldots, n_J) \in \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J$$

with the meaning: I is the set of nodes under repair. The numbers $n_j \in \mathbb{N}$ indicate for nodes $j \in \mathrm{J}\backslash\mathrm{I}$, which work in up status, that there are n_j customers present; for nodes $j \in \mathrm{I}$ in down status the numbers $n_j \in \mathbb{N}$ indicate that there are n_j customers waiting for the return of the repaired server at node j. For these general models with breakdowns and repairs and with the above rerouting principles it was shown in Sauer and Daduna [16] that on the state space E the steady state distribution for Z is of product form.

Theorem 3.1 *For the degradable Jackson network with breakdown and repair intensities given by Eq. (3.5) and rerouting according to either* BLOCKING *or* STALLING*, or* SKIPPING *the traffic equation for the network in up status is*

$$\eta_j = r_{0j}\lambda + \sum_{i=1}^{J} \eta_i r_{ij} \quad , \ j \in \mathrm{J}, \tag{3.7}$$

with unique solution $\eta = (\eta_1, \ldots, \eta_J)$.
If for all $j \in \mathrm{J}$ we have $\eta_j < \mu_j$ then the joint queue length/availability process

$$Z = (X, Y) = (Z(t) = (X(t), Y(t)), t \geq 0)$$

is ergodic and has a stationary distribution of product form given by:

$$\pi(\mathrm{I};\ n_1, n_2, \ldots, n_J) = \pi^Y(\mathrm{I})\, \pi_0(n_1, n_2, \ldots, n_J)$$

for $(\mathrm{I};\ n_1, n_2, \ldots, n_J) \in E = \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J$ with

$$\pi_0(n_1, \ldots, n_J) = \prod_{j=1}^{J} \left(1 - \frac{\eta_j}{\mu_j}\right) \left(\frac{\eta_j}{\mu_j}\right)^{n_j}, \tag{3.8}$$

and with

$$\pi^Y(\mathrm{I}) = \left(\sum_{\mathrm{K}\subseteq\mathrm{J}} \frac{a(\mathrm{K})}{b(\mathrm{K})}\right)^{-1} \frac{a(\mathrm{I})}{b(\mathrm{I})} \quad \textit{for } \mathrm{I}\subseteq\mathrm{J}.$$

The marginal steady state $\pi_0(n_1, n_2, \ldots, n_J)$ of the non Markovian process X coincides with the equilibrium distribution in a standard (completely reliable) Jackson network, see (4.12) below. Nevertheless, the behavior of the queueing process X is completely different in all cases (standard Jackson, BLOCKING, STALLING, SKIPPING). Moreover, it is possible to vary the breakdown/repair intensities without changing the stationary distribution for the process Z. So the behavior over time presents itself for comparison in dependence orderings.

3.2 Comparison results for degradable Jackson networks with unreliable nodes and repair

The breakdown/repair process Y is Markov on the state space $\mathcal{P}(\mathrm{J})$ of all subsets of J, but the joint queue length process (network process) X is in this setting not a Markov process for itself. Applying the theory of dependence ordering for Markov processes we may directly investigate Y, but direct application to X is not possible. We first investigate both processes separately and then discuss the joint process Z.

3.2.1 Comparison of breakdown/repair processes

For a set K we denote the sets of its ancestors, descendants and relatives as follows:

$$\begin{aligned}
\{\mathrm{K}\}^{\uparrow} &:= \{\mathrm{I}\subseteq\mathrm{J} : \mathrm{K}\subseteq\mathrm{I}\} \quad , \quad \{\mathrm{K}\}^{\uparrow}_{+} := \{\mathrm{I}\subseteq\mathrm{J} : \mathrm{K}\subseteq\mathrm{I}, \mathrm{K}\neq\mathrm{I}\}\,,\\
\{\mathrm{K}\}^{\downarrow} &:= \{\mathrm{I}\subseteq\mathrm{J} : \mathrm{I}\subseteq\mathrm{K}\} \quad , \quad \{\mathrm{K}\}^{\downarrow}_{-} := \{\mathrm{I}\subseteq\mathrm{J} : \mathrm{I}\subseteq\mathrm{K}, \mathrm{K}\neq\mathrm{I}\}\,,\\
\{\mathrm{K}\}^{\prec} &:= \{\mathrm{K}\}^{\downarrow}_{-} \cup \{\mathrm{K}\}^{\uparrow}_{+}\,.
\end{aligned}$$

The Markov process $Y = (Y(t), t \geq 0)$ on the state space $\mathcal{P}(\mathrm{J})$ describes the availability of the network's nodes over time, i.e. $Y(t) = \mathrm{K}$ means that at time t

the set K consists of the nodes which are under repair. We have

$$q^Y(\mathrm{K},\mathrm{H}) = \begin{cases} \alpha(\mathrm{K},\mathrm{H}) = \frac{a(\mathrm{H})}{a(\mathrm{K})}, & \text{if } \mathrm{H} \in \{\mathrm{K}\}_+^{\uparrow}\,, \\ \beta(\mathrm{K},\mathrm{H}) = \frac{b(\mathrm{K})}{b(\mathrm{H})}, & \text{if } \mathrm{H} \in \{\mathrm{K}\}_-^{\downarrow}\,, \\ -\sum_{\mathrm{I}\in\{\mathrm{K}\}_+^{\uparrow}} \frac{a(\mathrm{I})}{a(\mathrm{K})} - \sum_{\mathrm{I}\in\{\mathrm{K}\}_-^{\downarrow}} \frac{b(\mathrm{K})}{b(\mathrm{I})}, & \text{if } \mathrm{H} = \mathrm{K}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

According to Eq. (2.4) we define for fixed $\mathrm{I}_1 \subset \mathrm{I}_2$, $\mathrm{J}_1 \subset \mathrm{J}_2$ new intensities by

$$q^{\widetilde{Y}^\varepsilon}(\mathrm{K},\mathrm{H}) = \begin{cases} q^Y(\mathrm{K},\mathrm{H}) + \frac{\varepsilon}{\pi(\mathrm{K})}, & \text{if } (\mathrm{K}=\mathrm{I}_1, \mathrm{H}=\mathrm{J}_1) \text{ or } (\mathrm{K}=\mathrm{I}_2, \mathrm{H}=\mathrm{J}_2)\,, \\ q^Y(\mathrm{K},\mathrm{H}) - \frac{\varepsilon}{\pi(\mathrm{K})}, & \text{if } (\mathrm{K}=\mathrm{I}_1, \mathrm{H}=\mathrm{J}_2) \text{ or } (\mathrm{K}=\mathrm{I}_2, \mathrm{H}=\mathrm{J}_1)\,, \\ q^Y(\mathrm{K},\mathrm{H}), & \text{otherwise}\,. \end{cases} \tag{3.10}$$

Applying Proposition 2.1 for Y, $\widetilde{Y}^\varepsilon$ we have the following comparison.

Proposition 3.1 *Assume that the breakdown/repair intensity process Y and the time inverse of $\widetilde{Y}^\varepsilon$ with intensity matrices $Q^Y = (q^Y(\mathrm{K},\mathrm{H}),\ \mathrm{K},\mathrm{H} \subseteq \mathrm{J})$ and $Q^{\widetilde{Y}^\varepsilon}_{\leftarrow}$ are stochastically monotone. Then $Y <_{\mathrm{idif}((\subseteq)^{(\infty)})} \widetilde{Y}^\varepsilon$.*

Recall that a family $\mathcal{A} \subseteq \mathcal{P}(\mathrm{J})$ is $\subseteq$-increasing if $K \in \mathcal{A}$ and $K \subseteq H$ implies $H \in \mathcal{A}$, and for $C \in \mathcal{P}(\mathrm{J}) : \{\mathrm{C}\}^{\uparrow} = \{\mathrm{A} : \mathrm{A} \supseteq \mathrm{C}\}$, $\{\mathrm{C}\}^{\downarrow} = \{\mathrm{A} : \mathrm{A} \subseteq \mathrm{C}\}$.

Example 3.1

1. Let $f(\mathrm{K},\mathrm{H}) = \mathrm{I\!I}_{(\mathrm{K}\in\mathcal{B},\mathrm{H}\in\mathcal{C})}(\mathrm{K},\mathrm{H})$, where $\mathcal{B},\mathcal{C} \subseteq \mathcal{P}(\mathrm{J})$ are $\subseteq$-increasing. Then f has isotone differences on $((\mathcal{P}(\mathrm{J}))^2, \subseteq^2)$. By Proposition 3.1 we obtain

$$\mathbb{P}(Y(t_1) \in \mathcal{B}, Y(t_2) \in \mathcal{C}) \leq \mathbb{P}(\widetilde{Y}^\varepsilon(t_1) \in \mathcal{B}, \widetilde{Y}^\varepsilon(t_2) \in \mathcal{C})\,.$$

2. Let $g(\mathrm{K},\mathrm{H}) = f(|\mathrm{K}|,|\mathrm{H}|)$ with $f \in \mathcal{L}_{\mathrm{idif}}(\mathrm{J}^2, \leq^2)$. Then $g \in \mathcal{L}_{\mathrm{idif}}((\mathcal{P}(\mathrm{J}))^2, \subseteq^2)$. Especially, by Proposition 3.1 we obtain by taking $f(|\mathrm{K}|,|\mathrm{H}|) = |\mathrm{K}||\mathrm{H}|$

$$\mathbf{Cov}\,[|Y(t_1)|, |Y(t_2)|] \leq \mathbf{Cov}\left[|\widetilde{Y}^\varepsilon(t_1)|, |\widetilde{Y}^\varepsilon(t_2)|\right]\,.$$

3. Let $f(\mathrm{A}_1,\ldots,\mathrm{A}_k) = \mathrm{I\!I}_{\{\mathrm{A}_1\cap\cdots\cap\mathrm{A}_k\in\{\mathrm{C}\}^{\uparrow}\}}$ for some $\mathrm{C} \in \mathcal{P}(\mathrm{J})$. Then $f \in \mathcal{L}_{\mathrm{idif}}((\mathcal{P}(\mathrm{J}))^k, \subseteq^k)$ and hence

$$\mathbb{P}(Y(t_1) \cap \cdots \cap Y(t_k) \in \{\mathrm{C}\}^{\uparrow}) \leq \mathbb{P}(\widetilde{Y}^\varepsilon(t_1) \cap \cdots \cap \widetilde{Y}^\varepsilon(t_k) \in \{\mathrm{C}\}^{\uparrow})\,,$$

i.e. the set of nodes in the first network which are under repair at time instants $t_1, \dots, t_k$ is smaller (in the stochastic sense) than the corresponding set for the second network. On the other hand, because $f(\mathrm{A}_1, \dots, \mathrm{A}_k) = \mathbb{I}_{\{\mathrm{A}_1 \cup \cdots \cup \mathrm{A}_k \in \{\mathrm{C}\}^{\downarrow}\}}$ has isotone differences as well, we obtain

$$\mathbb{P}(Y(t_1) \cup \cdots \cup Y(t_k) \in \{\mathrm{C}\}^{\downarrow}) \leq \mathbb{P}(\widetilde{Y}^{\varepsilon}(t_1) \cup \cdots \cup \widetilde{Y}^{\varepsilon}(t_k) \in \{\mathrm{C}\}^{\downarrow}),$$

i.e. the set of nodes in the first network which are under repair at least in one of the time instants $t_1, \dots, t_k$ is larger (in the stochastic sense) than the corresponding set for the second network.

Take now the transition intensities in the special form:

$$q^Y(\mathrm{K}, \mathrm{H}) = \begin{cases} \lambda(\mathrm{K})\delta(\mathrm{K}, \mathrm{H}), & \text{if } \mathrm{H} \in \{\mathrm{K}\}^{\prec}, \\ 0, & \text{otherwise}. \end{cases}$$

Here, the nodes in K are under repair and $\lambda(\mathrm{K})$ is the intensity of having a jump in Y and $\delta(\mathrm{K}, \mathrm{H})$ is the conditional probability that after the jump the set H is under repair. Especially, we put

$$\delta(\mathrm{K}, \mathrm{H}) = \frac{\mathbb{I}_{(\mathrm{H} \in \{\mathrm{K}\}^{\prec})}}{|\{\mathrm{K}\}^{\prec}|}$$

and $\lambda(\mathrm{K}) = |\{\mathrm{K}\}^{\prec}|$. If $\{T_n\}$ is a sequence of jump points of Y, then it means that $Y(T_n)$ is uniformly distributed on the set $\{\mathrm{K}\}^{\prec}$ given that $Y(T_n-) = \mathrm{K}$. Then $q^Y(\mathrm{K}, \mathrm{H}) = 1$ if and only if $\mathrm{H} \in \{\mathrm{K}\}^{\prec}$. Note that in this case π^Y is uniformly distributed on $\mathcal{P}(\mathrm{J})$. Note that we can also represent our intensities as

$$q^Y(\mathrm{K}, \mathrm{H}) = \frac{\mathbb{I}_{(\mathrm{H} \in \{\mathrm{K}\}^{\downarrow}_{-})}}{|\{\mathrm{K}\}^{\downarrow}_{-}|} \cdot |\{\mathrm{K}\}^{\downarrow}_{-}| + \frac{\mathbb{I}_{(\mathrm{H} \in \{\mathrm{K}\}^{\uparrow}_{+})}}{|\{\mathrm{K}\}^{\uparrow}_{+}|} \cdot |\{\mathrm{K}\}^{\uparrow}_{+}|.$$

Here, $|\{\mathrm{K}\}^{\downarrow}_{-}|$, $|\{\mathrm{K}\}^{\uparrow}_{+}|$ play the role of the repair and breakdown intensity, respectively. Then, for instance $\frac{\mathbf{I}_{(\mathrm{H} \in \{\mathrm{K}\}^{\downarrow}_{-})}}{|\{\mathrm{K}\}^{\downarrow}_{-}|}$ serves as repair probability for the set of nodes $\mathrm{K} \backslash \mathrm{H}$ given that we have repair. Now, we take another Markov process $\widetilde{Y}^{\varepsilon}$ with the same jump intensity $\lambda(\mathrm{K})$ and modified transition probabilities. Let $\mathrm{I}_1 \subset \mathrm{I}_2$, $\mathrm{J}_1 \subset \mathrm{J}_2$. Then we put

$$\widetilde{\delta}^{\varepsilon}(\mathrm{K}, \mathrm{H}) = \begin{cases} \frac{\mathbf{I}_{(\mathrm{H} \in \{\mathrm{K}\}^{\prec})} + \varepsilon}{|\{\mathrm{K}\}^{\prec}|}, & \text{if } (\mathrm{K} = \mathrm{I}_1, \mathrm{H} = \mathrm{J}_1) \text{ or } (\mathrm{K} = \mathrm{I}_2, \mathrm{H} = \mathrm{J}_2), \\ \frac{\mathbf{I}_{(\mathrm{H} \in \{\mathrm{K}\}^{\prec})} - \varepsilon}{|\{\mathrm{K}\}^{\prec}|}, & \text{if } (\mathrm{K} = \mathrm{I}_1, \mathrm{H} = \mathrm{J}_2) \text{ or } (\mathrm{K} = \mathrm{I}_2, \mathrm{H} = \mathrm{J}_1), \\ \delta(\mathrm{K}, \mathrm{H}), & \text{otherwise}. \end{cases}$$

From Proposition 3.1 both of these models are $<_{\mathrm{idif}}$-comparable under the monotonicity assumption.

3.2.2 Comparison of queue length processes

Now, we turn our attention to the (joint) queue length process X, which is not directly amenable to the theory of the previous sections, since it is non-Markovian. We have to operate on the joint queueing/reliability process $Z = (X, Y)$. We study in the sequel unreliable networks with the same routing matrices but different breakdown/repair processes Y and $\widetilde{Y}^{\varepsilon}$ as in Proposition 3.1. We denote the resulting processes by $Z = (X, Y)$ and $\widetilde{Z}^{\varepsilon} = (\widetilde{X}^{\varepsilon}, \widetilde{Y}^{\varepsilon})$. For the queueing processes X and $\widetilde{X}^{\varepsilon}$, of main interest in this section, we may consider the reliability processes Y and $\widetilde{Y}^{\varepsilon}$ as environment processes that strongly influence the behavior of X and $\widetilde{X}^{\varepsilon}$. As we shall see in Subsection 3.2.3 there occur severe problems with the full monotonicity of Z. But for comparison of X and $\widetilde{X}^{\varepsilon}$ only, we may overcome the problems by suitably selecting the order structure for Y, and $\widetilde{Y}^{\varepsilon}$ to apply Theorem 2.1. This will eventually result in the following theorem.

Theorem 3.2 *Assume that two Jackson networks have the same arrival intensities and the same rerouting matrices according to either* BLOCKING *or* STALLING *or* SKIPPING. *The breakdown/repair intensity matrices are given by Eq. (3.9) and Eq. (3.10). Assume also that breakdown/repair process and its time-reversal are stochastically monotone. Then we have*

$$X <_{\mathrm{idif}((\leq^{\mathrm{J}})^{(\infty)})} \widetilde{X}^{\varepsilon} ,$$

i.e. for all $n \geq 2$, $t_1 \leq \cdots \leq t_n$, and all functions f with isotone differences on $((\mathbb{N}^J)^n, (\leq^J)^n)$ holds

$$\mathbb{E}\left[f\left(X_{t_1}, \ldots, X_{t_n}\right)\right] \leq \mathbb{E}\left[f\left(\widetilde{X}^{\varepsilon}_{t_1}, \ldots, \widetilde{X}^{\varepsilon}_{t_n}\right)\right] .$$

The rest of this subsection is dedicated to the proof of the theorem, which is mainly by a lemma and a proposition which is of some independent interest. The first part is direct, the lengthy proof of the lemma is postponed to the appendix.

Lemma 3.1 *Assume that the inequality (2.3) of Theorem 2.1 for the intensities of Y and $\widetilde{Y}^{\varepsilon}$ on $(E, \prec) = (\mathcal{P}(\mathrm{J}), \subseteq)$ holds. Then this intensity inequality (2.3) holds for Z and $\widetilde{Z}^{\varepsilon}$ on $(E, \prec) = (\mathcal{P}(\mathrm{J}) \times \mathbb{N}^{\mathbb{J}}, \subseteq \times \leq^{\mathbb{J}})$ as well.*

Now we consider the following order: $(\mathrm{I}_1, \mathbf{n}_1) \prec (\mathrm{I}_2, \mathbf{n}_2)$ if and only if $\mathrm{I}_1 \subseteq_e \mathrm{I}_2$ and $\mathbf{n}_1 \leq^J \mathbf{n}_2$. Here, $\mathrm{I}_1 \subseteq_e \mathrm{I}_2$ if and only if $\mathrm{I}_1 = \mathrm{I}_2$. Z is clearly monotone w.r.t. to this ordering because standard Jackson networks with reliable nodes are monotone under the usual order on $\mathbb{N}^J$. The same holds for its time-reversal. Further, a function f which has isotone differences on $(E^n, (\subseteq \times \leq^J)^n)$ has isotone differences on $(E^n, (\subseteq_e \times \leq^J)^n)$. Therefore, from Lemma 3.1 the inequality (2.3) holds for all functions with isotone differences on $(E^n, (\subseteq_e \times \leq^J)^n)$. Because of the above mentioned monotonicity we have the following result.

Proposition 3.2 *Under the assumptions of Theorem 3.2 and with the ordering introduced above for the queueing/availability process holds*

$$Z <_{\mathrm{idif}((\subseteq_e \times \leq^{\mathrm{J}})^{(\infty)})} \widetilde{Z}^{\varepsilon} ,$$

i.e. for all $n \geq 2$, $t_1 \leq \cdots \leq t_n$, and all functions f with isotone differences on $(E^n, (\subseteq_e \times \leq^J)^n)$ holds

$$\mathbb{E}\left[f\left(Z_{t_1}, \ldots, Z_{t_n}\right)\right] \leq \mathbb{E}\left[f\left(\widetilde{Z}^{\varepsilon}_{t_1}, \ldots, \widetilde{Z}^{\varepsilon}_{t_n}\right)\right] .$$

In particular, because a function g with isotone differences on $(E^n, (\subseteq_e \times \leq^J)^n)$ has clearly isotone differences on $((\mathbb{N}^J)^n, (\leq^J)^n)$ we obtain now the statement of the Theorem 3.2 for the queue length process in a network with unreliable nodes.

Example 3.2 As for standard Jackson networks (Section 4) we can prove inequalities like (4.13) for the queue length process in networks with breakdowns and repairs.

3.2.3 Monotonicity problems for queueing/reliability processes

Since we are able to compare queueing processes in the standard Jackson networks (see Section 4 below) and breakdown/repair processes as demonstrated in Section

3.2.1, we could expect to be able to compare two networks with the same routing matrices and different breakdown/repair processes. Lemma 3.1 indicates in this direction. However, the joint network process $Z = (Y, X)$ is not monotone with respect to the standard orderings.

Example 3.3 Consider $E = \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J$, inclusion ordering on $\mathcal{P}(\mathrm{J})$, coordinate wise ordering on $\mathbb{N}^J$, and $\prec$ - coordinate wise ordering on E. The Markov processes Z and $\tilde{Z}^\varepsilon$ have the following intensities

$$q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) = \begin{cases} q^Y(\mathrm{I}_1, \mathrm{I}_2), & \text{if } \mathrm{I}_1 \neq \mathrm{I}_2 \text{ and } \mathbf{n}_1 = \mathbf{n}_2, \\ q_{\mathrm{I}_1}(\mathbf{n}_1, \mathbf{n}_2), & \text{if } \mathrm{I}_1 = \mathrm{I}_2 \text{ and } \mathbf{n}_1 \neq \mathbf{n}_2 \\ -q^Y(\mathrm{I}_1, \mathrm{I}_1) - q_{\mathrm{I}_1}(\mathbf{n}_1, \mathbf{n}_1), & \text{if } \mathrm{I}_1 = \mathrm{I}_2 \text{ and } \mathbf{n}_1 = \mathbf{n}_2, \\ 0, & \text{otherwise} \end{cases},$$

$$q^{\tilde{Z}^\varepsilon}(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) = \begin{cases} q^{\tilde{Y}^\varepsilon}(\mathrm{I}_1, \mathrm{I}_2), & \text{if } \mathrm{I}_1 \neq \mathrm{I}_2 \text{ and } \mathbf{n}_1 = \mathbf{n}_2, \\ q_{\mathrm{I}_1}(\mathbf{n}_1, \mathbf{n}_2), & \text{if } \mathrm{I}_1 = \mathrm{I}_2 \text{ and } \mathbf{n}_1 \neq \mathbf{n}_2, \\ -q^{\tilde{Y}^\varepsilon}(\mathrm{I}_1, \mathrm{I}_1) - q_{\mathrm{I}_1}(\mathbf{n}_1, \mathbf{n}_1), & \text{if } \mathrm{I}_1 = \mathrm{I}_2 \text{ and } \mathbf{n}_1 = \mathbf{n}_2, \\ 0, & \text{otherwise} \end{cases}.$$

Here $q_{\mathrm{I}}(\mathbf{n}_1, \mathbf{n}_2)$ are intensities of a standard Jackson networks according to Theorem 4.1 in Section 4 with valid node set $\mathrm{J} \setminus \mathrm{I}$, in which the routing matrix depends on the current state I of Y $(\tilde{Y}^\varepsilon)$ and is restricted to the set $\mathrm{J} \setminus \mathrm{I}$ according to BLOCKING, STALLING, or SKIPPING. In order to prove $\prec$-monotonicity we have to check (among others) that for $\mathrm{I}_1 \subseteq \mathrm{I}_2$, $\mathbf{n}_1 \leq^J \mathbf{n}_2$ the infinitesimal generator of the network fulfills

$$q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathcal{A}_1 \times \mathcal{A}_2) \leq q^Z(\mathrm{I}_2, \mathbf{n}_2; \mathcal{A}_1 \times \mathcal{A}_2)$$

for any increasing sets $\mathcal{A}_1 \in \mathcal{P}(\mathrm{J})$, $\mathcal{A}_2 \in \mathbb{N}^J$, whereas

$$q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathcal{A}_1 \times \mathcal{A}_2) := \sum_{(\mathrm{I}_3, \mathbf{n}_3) \in \mathcal{A}_1 \times \mathcal{A}_2} q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_3, \mathbf{n}_3).$$

Assume that $\mathrm{I}_1 \in \mathcal{A}_1$, $\mathbf{n}_1 \in \mathcal{A}_2$. We have

$$q^Z(\mathrm{I}, \mathbf{n}; \mathcal{A}_1 \times \mathcal{A}_2) = \sum_{\mathrm{K} \in \mathcal{A}_1} q^Y(\mathrm{I}, \mathrm{K}) + \sum_{\mathbf{m} \in \mathcal{A}_2} q_{\mathrm{I}}(\mathbf{n}, \mathbf{m}) \,. \tag{3.11}$$

(3.11) is clearly monotone w.r.t $\mathbf{n}$ since Q_{I} is a generator for a standard Jackson network with nodes $\mathrm{J} \setminus \mathrm{I}$ without breakdowns . Take $\mathcal{A}_1 = \mathcal{P}(\mathrm{J})$. Then the first part in the above expression is 0. Assume that STALLING is applied. Then Q_{I} is a zero matrix unless $\mathrm{I} = \emptyset$. For $\mathrm{I} = \emptyset$ the latter part in the above expression can be either positive or negative according to the choice of $\mathcal{A}_2$. Therefore (3.11) is not monotone w.r.t. I and the network process Z is not $\prec$-monotone. With BLOCKING or SKIPPING as rerouting rule the situation is more involved but does not lead to monotonicity either.

4 Standard queueing networks and their processes

In Section 3.1 we have described Jackson networks with unreliable nodes. To obtain standard networks is just a matter of trivializing the description by setting all breakdown rates to 0. Then we may neglect the reliability component Y of Z and obtain X as a Markov process. We use for simplicity the notion X for the joint queue length process as well. We present this network and its process here for its own because we need the reference for the proof in the appendix and in Example 3.3 and because even for these standard models the results thereafter are new. The following theorem is classical.

Theorem 4.1 (Jackson [10]) *Assume that for a standard Jackson network the assumptions on the network behavior as described in Section 3.1 before Theorem 3.1 hold. Then the queueing process X is a Markov process with infinitesimal operator $Q^X = (q(x,y) : x, y \in E)$ given by*

$$\begin{aligned}
q^X(n_1,\ldots,n_i,\ldots,n_J; n_1,\ldots,n_i+1,\ldots,n_J) &= \lambda r_{0i} \\
q^X(n_1,\ldots,n_i,\ldots,n_J; n_1,\ldots,n_i-1,\ldots,n_J) &= \mu_i r_{i0}, n_i > 0, \\
q^X(n_1,\ldots,n_i,\ldots,n_j,\ldots,n_J; n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_J) &= \mu_i r_{ij}, n_i > 0, \\
q^X(x,x) = -\sum_{y \in E \setminus \{x\}} q^X(x,y) \quad \textit{and} \quad q^X(x,y) &= 0 \textit{ otherwise.}
\end{aligned}$$

Assume that for the unique solution $\eta = (\eta_1, \ldots, \eta_J)$ of the traffic equation 3.7 holds $\eta_i < \mu_i$ for all $i \in \mathrm{J}$. Then X is ergodic and the unique invariant and *limiting distribution π of X is*

$$\pi_0(n_1, \ldots, n_J) = \prod_{j=1}^{J} \left(1 - \frac{\eta_j}{\mu_j}\right) \left(\frac{\eta_j}{\mu_j}\right)^{n_j}, \quad (n_1, \ldots, n_J) \in \mathbb{N}^J. \tag{4.12}$$

Consider network processes X, and its ε transformation $\widetilde{X^\varepsilon}$. It is well-known that both the network process and its time-reversal are stochastically monotone w.r.t. standard coordinate wise ordering $\leq^J$ (see Daduna and Szekli [4]). Therefore from Lemma 2.4 and Proposition 2.1 we have the following result.

Theorem 4.2 *For a Jackson network's state process X and its ε transformation $\widetilde{X^\varepsilon}$ holds*

$$X <_{\mathrm{idif}((\leq^J)^{(\infty)})} \widetilde{X^\varepsilon}.$$

In particular, if $f \in \mathcal{L}_{\mathrm{i-idif}}(E^k, (\leq^J)^k)$, $g \in \mathcal{L}_{\mathrm{i-idif}}(E^{n-k}, (\leq^J)^{n-k})$, then by Lemma 2.2, $fg \in \mathcal{L}_{\mathrm{i-idif}}(E^n, (\leq^J)^n)$, hence

$$\begin{aligned} &\mathbb{E}\left[f\left(X(t_1), \ldots, X(t_k)\right) g\left(X(t_{k+1}), \ldots, X(t_n)\right)\right] \leq \\ &\qquad \mathbb{E}\left[f\left(\widetilde{X^\varepsilon}(t_1), \ldots, \widetilde{X^\varepsilon}(t_k)\right) g\left(\widetilde{X^\varepsilon}(t_{k+1}), \ldots, \widetilde{X^\varepsilon}(t_n)\right)\right] \end{aligned} \tag{4.13}$$

which is related to Corollary 5.1 in Daduna and Szekli [4].

Example 4.1 Let $\mathbf{e}_i$ be an n-dimensional vector with 1 on ith coordinate and 0 otherwise. Consider the ε transformation specified in the following way: Fix $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}^J$ with $n_i > 0$ and i, j such that $1 \leq i < j \leq J$. For simplicity assume that λ and service intensities are equal to 1. Take in Eq. (2.4) $x_1 = \mathbf{n} - \mathbf{e}_i, x_2 = \mathbf{n}, y_1 = \mathbf{n} - \mathbf{e}_i, y_2 = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$ to obtain

$$\begin{aligned} q^{\widetilde{X^\varepsilon}}(\mathbf{n} - \mathbf{e}_i, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) &= q^X(\mathbf{n} - \mathbf{e}_i, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) - \varepsilon/\pi(\mathbf{n} - \mathbf{e}_i) \\ &= r_{0j} - \varepsilon/\pi(\mathbf{n} - \mathbf{e}_i) \\ q^{\widetilde{X^\varepsilon}}(\mathbf{n} - \mathbf{e}_i, \mathbf{n} - \mathbf{e}_i) &= q^X(\mathbf{n} - \mathbf{e}_i, \mathbf{n} - \mathbf{e}_i) + \varepsilon/\pi(\mathbf{n} - \mathbf{e}_i) \\ &= (1 - r) + \sum_{i=1}^{J} (1 - \delta_{0n_i})(1 - r_{ii}) \\ q^{\widetilde{X^\varepsilon}}(\mathbf{n}, \mathbf{n} - \mathbf{e}_i) &= q^X(\mathbf{n}, \mathbf{n} - \mathbf{e}_i) - \varepsilon/\pi(\mathbf{n}) = r_{i0} - \varepsilon/\pi(\mathbf{n}) \\ q^{\widetilde{X^\varepsilon}}(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) &= q^X(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) + \varepsilon/\pi(\mathbf{n}) = r_{ij} + \varepsilon/\pi(\mathbf{n}) \end{aligned}$$

We observe that the changes in transition intensities according to Eq. (2.4) correspond here to changes in routing probabilities in a similar way with the following meaning. We increase the rejection probability $r_{00} = 1 - r$ and the transition probability r_{ij} and at the same time we decrease the arrival probability r_{0j} and the departure probability r_{i0}. From Theorem 4.2 it follows that then the stationary joint queue length distribution increases in isotone differences ordering.

5 Appendix

Proof of **Lemma 3.1**. Let $\mathrm{I}, \mathrm{I}_1, \mathrm{I}_2 \in \mathcal{P}(\mathrm{J})$, $\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{N}^J$. Recall that Z and $\widetilde{Z}^\varepsilon$ are Markov processes with respective intensities given in Example 3.3. The $q_{\mathrm{I}}(\mathbf{n}_1, \mathbf{n}_2)$ there are intensities of a standard Jackson network with valid node set $\mathrm{J} \setminus \mathrm{I}$, in which the routing matrix depends on the current state I of Y $(\widetilde{Y}^\varepsilon)$ and is restricted to the set $\mathrm{J}_0 \backslash \mathrm{I}$ consistently according to BLOCKING, STALLING or SKIPPING. Therefore components belonging to node set I are frozen and must coincide in $\mathbf{n}_1$ and $\mathbf{n}_2$ to obtain a feasible transition. Due to our assumptions, the intensities $q_{\mathrm{I}}(\mathbf{n}_1, \mathbf{n}_2)$ are the same for the networks Z and $\widetilde{Z}^\varepsilon$ for each fixed I. Furthermore we have from Theorem 3.1 and Lemma 2.4:

$$\pi^Z(\mathrm{I}, \mathbf{n}) = \pi^Y(\mathrm{I})\, \pi_0(\mathbf{n}) = \pi^{\widetilde{Y}^\varepsilon}(\mathrm{I})\, \pi_0(\mathbf{n}) = \pi^{\widetilde{Z}^\varepsilon}(\mathrm{I}, \mathbf{n})\,.$$

Assume now that $f : \left(\mathcal{P}(\mathrm{J}) \times \mathbb{N}^J\right)^2 \to \mathbb{R}$ has isotone differences. Then we have

$$\begin{aligned}
\sum_{(\mathrm{I}_1,\mathbf{n}_1)\in\widetilde{E}} \pi(\mathrm{I}_1, \mathbf{n}_1) \sum_{(\mathrm{I}_2,\mathbf{n}_2)\in\widetilde{E}} q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) f(\mathrm{I}_1, \mathbf{n}_1, \mathrm{I}_2, \mathbf{n}_2) \quad = \\
= \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1, \mathbf{n}_1) \sum_{(\mathrm{I}_2\neq\mathrm{I}_1,\mathbf{n}_2=\mathbf{n}_1)} q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) f(\mathrm{I}_1, \mathbf{n}_1, \mathrm{I}_2, \mathbf{n}_2) \\
+ \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1, \mathbf{n}_1) \sum_{(\mathrm{I}_2=\mathrm{I}_1,\mathbf{n}_2\neq\mathbf{n}_1)} q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) f(\mathrm{I}_1, \mathbf{n}_1, \mathrm{I}_2, \mathbf{n}_2) \\
+ \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1, \mathbf{n}_1) \sum_{(\mathrm{I}_2=\mathrm{I}_1,\mathbf{n}_2=\mathbf{n}_1)} q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) f(\mathrm{I}_1, \mathbf{n}_1, \mathrm{I}_2, \mathbf{n}_2) \\
= \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1, \mathbf{n}_1) \sum_{\mathrm{I}_2\neq\mathrm{I}_1} q^Y(\mathrm{I}_1, \mathrm{I}_2) f(\mathrm{I}_1, \mathbf{n}_1, \mathrm{I}_2, \mathbf{n}_1) \\
+ \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1, \mathbf{n}_1) \sum_{\mathbf{n}_2\neq\mathbf{n}_1} q_{\mathrm{I}_1}(\mathbf{n}_1, \mathbf{n}_2) f(\mathrm{I}_1, \mathbf{n}_1, \mathrm{I}_1, \mathbf{n}_2)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) q^Z(\mathrm{I}_1,\mathbf{n}_1;\mathrm{I}_1,\mathbf{n}_1) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_1)\\
&= \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathrm{I}_2\neq\mathrm{I}_1} q^Y(\mathrm{I}_1,\mathrm{I}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathbf{n}_2\neq\mathbf{n}_1} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_2)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \left(- \sum_{(\mathrm{I}_2,\mathbf{n}_2)\neq(\mathrm{I}_1,\mathbf{n}_1)} q^Z(\mathrm{I}_1,\mathbf{n}_1;\mathrm{I}_2,\mathbf{n}_2) \right) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_1)\\
&= \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathrm{I}_2\neq\mathrm{I}_1} q^Y(\mathrm{I}_1,\mathrm{I}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathbf{n}_2\neq\mathbf{n}_1} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_2)\\
&\quad - \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \left(\sum_{\mathrm{I}_2\neq\mathrm{I}_1} q^Y(\mathrm{I}_1,\mathrm{I}_2) \right) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_1)\\
&\quad - \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \left(\sum_{\mathbf{n}_2\neq\mathbf{n}_1} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) \right) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_1)\\
&= \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathrm{I}_2} q^Y(\mathrm{I}_1,\mathrm{I}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathbf{n}_2} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_2)\\
&= \sum_{\mathbf{n}_1} \pi_0(\mathbf{n}_1) \left(\sum_{\mathrm{I}_1} \pi^Y(\mathrm{I}_1) \sum_{\mathrm{I}_2} q^Y(\mathrm{I}_1,\mathrm{I}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1) \right)\\
&\quad + \sum_{\mathrm{I}_1} \pi^Y(\mathrm{I}_1) \left(\sum_{\mathbf{n}_1} \pi_0(\mathbf{n}_1) \sum_{\mathbf{n}_2} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_2) \right) .
\end{aligned}
$$

Now, for given $\mathbf{n}_1$ the function $(\mathrm{I}_1,\mathrm{I}_2) \to f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1)$ has isotone differences. According to our assumption that Eq. (2.3) holds for Y and $\widetilde{Y}^\varepsilon$ the first term in the latter expression is not bigger than the corresponding one for $\widetilde{Y}^\varepsilon$. Moreover, the second term is the same for both Z and $\widetilde{Z}^\varepsilon$. Therefore, the sufficient condition Eq. (2.3) for ordering between Z and $\widetilde{Z}^\varepsilon$ is fulfilled.

■

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