

Weak convergence of Vervaat and Vervaat Error processes of long-range dependent sequences

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Abstract

Following Csörgő, Szyszkowicz and Wang (Ann. Statist. **34**, (2006), 1013–1044) we consider a long range dependent linear sequence. We prove weak convergence of the uniform Vervaat and the uniform Vervaat error processes, extending their results to distributions with unbounded support and removing normality assumption.

1 Introduction

Let $\{\epsilon_i, i \geq 1\}$ be a centered sequence of i.i.d. random variables. Consider the class of stationary linear processes

$$X_i = \sum_{k=0}^{\infty} c_k \epsilon_{i-k}, \quad i \geq 1. \quad (1)$$

We assume that the sequence $c_k, k \geq 0$, is regularly varying with index $-\beta$, $\beta \in (1/2, 1)$ (written as $c_k \in RV_{-\beta}$). This means that $c_k \sim k^{-\beta} L(k)$ as $k \rightarrow \infty$, where L is slowly varying at infinity. We shall refer to all such models as long range dependent (LRD) linear processes. In particular, if the variance exists, then the covariances $\rho_k := EX_0 X_k$ decay at the hyperbolic rate,

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$\rho_k = c_\beta L^2(k) k^{-(2\beta-1)}$, where $c_\beta = \mathbb{E}(\epsilon_1^2) \int_0^\infty x^{-\beta} (1+x)^{-\beta} dx$. Consequently, the covariances are not summable.

Assume that X_1 has a continuous distribution function F . For $y \in (0, 1)$ define $Q(y) = \inf\{x : F(x) \geq y\} = \inf\{x : F(x) = y\}$, the corresponding (continuous) quantile function. Given the ordered sample $X_{1:n} \leq \dots \leq X_{n:n}$ of X_1, \dots, X_n , let $F_n(x) = n^{-1} \sum_{i=1}^n 1_{\{X_i \leq x\}}$ be the empirical distribution function and $Q_n(\cdot)$ be the corresponding left-continuous sample quantile function. Define $U_i = F(X_i)$ and $E_n(x) = n^{-1} \sum_{i=1}^n 1_{\{U_i \leq x\}}$, the associated uniform empirical distribution. Denote by $U_n(\cdot)$ the corresponding uniform sample quantile function.

Assume that $E\epsilon_1^2 < \infty$. Let r be an integer and define

$$Y_{n,r} = \sum_{i=1}^n \sum_{1 \leq j_1 < \dots < j_r < \infty} \prod_{s=1}^r c_{j_s} \epsilon_{i-j_s}, \quad n \geq 1,$$

so that $Y_{n,0} = n$, and $Y_{n,1} = \sum_{i=1}^n X_i$. If $p < (2\beta - 1)^{-1}$, then

$$\sigma_{n,p}^2 := \text{Var}(Y_{n,p}) \sim \text{const.} n^{2-p(2\beta-1)} L^{2p}(n). \quad (2)$$

In particular

$$\sigma_{n,1}^2 \sim \frac{c_\beta}{(1-\beta)(3-2\beta)} n^{3-2\beta} L^2(n) =: n^{3-2\beta} L_0^2(n).$$

Define now the general empirical, the uniform empirical, the general quantile and the uniform quantile processes respectively as follows:

$$\beta_n(x) = \sigma_{n,1}^{-1} n(F_n(x) - F(x)), \quad x \in \mathbb{R}, \quad (3)$$

$$\alpha_n(y) = \sigma_{n,1}^{-1} n(E_n(y) - y), \quad y \in [0, 1], \quad (4)$$

$$q_n(y) = \sigma_{n,1}^{-1} n(Q(y) - Q_n(y)), \quad y \in (0, 1), \quad (5)$$

$$u_n(y) = \sigma_{n,1}^{-1} n(y - U_n(y)), \quad y \in [0, 1]. \quad (6)$$

Let

$$\tilde{R}_n(y) = \alpha_n(y) - u_n(y), \quad y \in [0, 1], \quad (7)$$

be the uniform Bahadur-Kiefer process. This process was introduced by Kiefer in [11], though not explicitly, in order to study the behavior of quantile processes via that of empirical, as initiated by Bahadur [1] for $y \in (0, 1)$ fixed.

Let

$$\tilde{V}_n(t) = 2\sigma_{n,1}^{-1}n \int_0^t \tilde{R}_n(y)dy, \quad t \in [0, 1],$$

the uniform Vervaat process and

$$\tilde{W}_n(t) = 2\sigma_{n,1}^{-1}n \int_0^t \tilde{R}_n(y)dy - \alpha_n^2(y), \quad t \in [0, 1],$$

be the uniform Vervaat error process as in [6].

Assume for a while that $\{\eta_n\}_{n \geq 1}$ is a stationary and standardized long-range dependent Gaussian sequence with a covariance structure

$$\gamma(k) := \mathbb{E}(\eta_1 \eta_{k+1}) = k^{-D} \tilde{L}(k), \quad 0 < D < 1,$$

where \tilde{L} is slowly varying at infinity. Let G be an arbitrary real-valued measurable function and define $Y_n = G(\eta_n)$, $n \geq 1$. Let F_Y be the continuous distribution function of Y_1 and $Q_Y(\cdot)$ the corresponding continuous quantile function. Define $V_i = F_Y(Y_i)$. As in Dehling and Taqqu [10], expand $1_{\{X_n \leq x\}} - F(x)$ in terms of Hermite polynomials,

$$1_{\{Y_n \leq x\}} - F_Y(x) = \sum_{l=\tau_x}^{\infty} c_l(x) H_l(\eta_n) / l!,$$

where $H_l(x)$ is the l th Hermite polynomial,

$$c_l(x) = \mathbb{E} \left[\left(1_{\{G(\eta_1) \leq x\}} - F_Y(x) \right) H_l(\eta_1) \right],$$

and for any $x \in \mathbb{R}$, τ_x (the Hermite rank) is the index of the first non-zero coefficient of the expansion. The uniform version is obtained as

$$1_{\{V_n \leq y\}} - y = \sum_{l=\tau_y}^{\infty} J_l(y) H_l(\eta_n) / l!,$$

where now $J_l(y) = c_l(Q_Y(y))$ for any $y \in (0, 1)$.

Let $\tilde{\sigma}_{n,\tau}^2 = n^{2-\tau D} \tilde{L}^\tau(n)$. Replace the constants $\sigma_{n,1}$ with $\tilde{\sigma}_{n,\tau}$ in the definitions of $\tilde{R}_n(\cdot)$, $\tilde{V}_n(\cdot)$ and $\tilde{W}_n(\cdot)$. In [6] Csörgő, Szyszkowicz and Wang (CsSzW) proved that the uniform Bahadur-Kiefer process $\tilde{R}_n(\cdot)$ converges weakly in $D([0, 1])$. This phenomenon is exclusive for long range dependent sequences, since in the i.i.d. case the (uniform) Bahadur-Kiefer process cannot converge weakly. However, as it was first shown by Vervaat [16], in

the i.i.d case the uniform Vervaat process does converge weakly. Obviously, in the LRD case, weak convergence of the uniform Vervaat process is implied by that of $\tilde{R}_n(\cdot)$, namely (see [6, Theorem 3.1]):

$$\tilde{V}_n(t) \Rightarrow \frac{2}{(2 - \tau D)(1 - \tau D)} J_\tau^2(t) Z_\tau^2, \quad n \rightarrow \infty, \quad (8)$$

where \Rightarrow denotes weak convergence in $D([0, 1])$ equipped with the sup-norm, and Z_τ is a random variable defined by an appropriate integral with respect to Brownian motion (see [10]). In particular, if $\tau = 1$, then Z_1 is standard normal. Further, CsSzW [6] observed that, similarly to the i.i.d case, the limiting process associated with $\tilde{V}_n(\cdot)$ agrees with that of $\alpha_n^2(\cdot)$. Therefore, it makes sense to consider the uniform Vervaat error process $\tilde{W}_n(\cdot)$. They showed that this process converges weakly as well, via concluding

$$n\sigma_{n,1}^{-1} \tilde{W}_n(t) \Rightarrow \frac{2^{5/2}}{(2 - \tau D)^{3/2}(1 - \tau D)^{3/2}} J_\tau^2(t) J_\tau'(t) Z_\tau^3, \quad n \rightarrow \infty. \quad (9)$$

This property is also exclusive for the LRD case. We refer to [2], [7], [20], [5] as well as the Introduction in [6] for motivations, probabilistic properties and applications of Bahadur-Kiefer, Vervaat and Vervaat error processes.

We note in passing that, though the results in CsSzW [6] for the uniform Bahadur-Kiefer process and, consequently, for the uniform Vervaat and Vervaat error processes, are true, their proofs have gaps, unless F is assumed to have finite support. Moreover, even then, the limiting process in (9) should be corrected via multiplying it by $\frac{1}{2}$.

In case of the Bahadur-Kiefer process, the problem of F possibly having an infinite support was solved in [3] in a more general setting in the case of LRD linear sequences by using weighted approximations. However, in general, this is still not suitable for establishing the weak convergence of the Vervaat process $\tilde{V}_n(\cdot)$, unless some specific conditions are imposed on the model. The reason for the problems arising in [6], and faced up to in [3], is that, unlike in the i.i.d. case, the uniform quantile process contains information about the quantile function associated with the random variables X_n .

Therefore, coming back to LRD linear sequences, the aim of this paper is to present an appropriate approximation result for the uniform Bahadur-Kiefer process, which will be suitable to treat the uniform Vervaat process to obtain (8), when F is assumed to have infinite support. Further, we will obtain the correct version of the weak convergence of the uniform Vervaat error

process. The approach is via weighted approximation of the Bahadur-Kiefer process like in [3]. Thus, first we get the correct limiting behaviour of the Vervaat error process, second, we remove assumptions on bounded support of F , third, we remove the normality assumption on ϵ_i . This approach in fact requires very precise knowledge on the behavior of the density-quantile function $f(Q(y))$.

To state our results, Let F_ϵ be the distribution function of the centered i.i.d. sequence $\{\epsilon_i, i \geq 1\}$. Assume that for a given integer p , the derivatives $F_\epsilon^{(1)}, \dots, F_\epsilon^{(p+3)}$ of F_ϵ are bounded and integrable. Note that these properties are inherited by the distribution F as well (cf. [18]). These conditions will be assumed throughout the paper with $p = 2$.

We shall need the following conditions on $fQ(\cdot) = f(Q(\cdot))$ and $f'Q(\cdot) = f'(Q(\cdot))$:

- (A) $\sup_{y \in (0,1)} |gQ(y)|/(y(1-y))^{1-\mu} = O(1)$ for some $1/2 > \mu > 0$ and $g = f, f'$;
- (B) $\sup_{y \in (0,1)} |(gQ(y))'(y)|(y(1-y))^\mu = O(1)$ for any $\mu > 0$ and $g = f, f'$.
Note that $(fQ(y))'(y)fQ(y) = f'Q(y)$;
- (C) $\sup_{y \in (0,1)} |(gQ(y))''(y)|(y(1-y))^{1+\mu} = O(1)$ for any $\mu > 0$ and $g = f, f'$.

We shall prove the following results.

Theorem 1.1 *Assume that conditions (A)-(C) are fulfilled. Then, as $n \rightarrow \infty$,*

$$\sup_{y \in [\delta_n, 1-\delta_n]} \left| n\sigma_{n,1}^{-1} \tilde{R}_n(y) - \sigma_{n,1}^{-2} f'(Q(y)) \left(\sum_{i=1}^n X_i \right)^2 \right| = o_{\text{a.s.}}(1),$$

where $\delta_n = Cn^{-(2\beta-1)}L_0^2(n)(\log \log n)$.

Corollary 1.2 *Under the conditions of Theorem 1.1, as $n \rightarrow \infty$,*

$$n\sigma_{n,1}^{-1} \tilde{R}_n(y) 1_{\{y \in [\delta_n, 1-\delta_n]\}} \Rightarrow f'Q(y) Z_1^2.$$

Theorem 1.3 *Under the conditions of Theorem 1.1, as $n \rightarrow \infty$,*

$$\tilde{V}_n(t) \Rightarrow f^2Q(t) Z_1^2.$$

Theorem 1.4 *Under the conditions of Theorem 1.1, as $n \rightarrow \infty$,*

$$\sigma_{n,1}^{-1} n \tilde{W}_n(t) \Rightarrow \frac{1}{((3-2\beta)(1-\beta))^{3/2}} f^2 Q(t) (fQ)'(t) Z_1^3. \quad (10)$$

Remark 1.5 A few words on the conditions (A)-(C). Assume that $F = \Phi$ (the standard normal distribution). It follows from [13] that (A) is fulfilled. Further, $(\phi(\Phi^{-1}(y)))' = -\Phi^{-1}(y)$ is unbounded (this is actually the reason, why the proofs in [6] do not work), but (B) holds. Furthermore, $(\phi(\Phi^{-1}(y)))' = -\frac{1}{\phi(\Phi^{-1}(y))}$, and it follows from [13] that (C) is fulfilled.

Furthermore, one can check that the conditions (A)-(B) are fulfilled for distributions with exponential or Pareto tails.

Remark 1.6 In Theorem 1.1 we are not able to obtain the a.s. approximation on $(0, 1)$. From this theorem, weak convergence of $\tilde{R}_n(y) 1_{\{y \in [\delta_n, 1-\delta_n]\}}$ follows, as in Corollary 1.2. We are not able to obtain weak convergence on $(0, 1)$ either. However, this was not our concern in this paper. It can be done via weight functions (see [3] for more details). Nevertheless, this convergence is good enough to obtain weak convergence of both the uniform Vervaat and the uniform Vervaat error processes. The weak convergence limit in Theorem 1.4 differs from that of Proposition 3.2 in [6] by the already mentioned factor of $\frac{1}{2}$. To see this, assume that $E(\epsilon_1^2) = 1$ and note that parametrization of the Gaussian and the linear model yields $\tilde{L}(n) = c_\beta L^2(n)$, $D = 2\beta - 1$. Plugging this into (10) we see, that the result (9) should be corrected by replacing $2^{5/2}$ with $2^{3/2}$.

The problem in the proof of Proposition 3.2 in [6] comes from an inappropriate use of their Proposition 2.5.

In what follows C will denote a generic constant which may be different at each time it appears. Further, $\ell(n)$ is a slowly varying function at infinity, possibly different at each time it appears.

2 Proofs

Recall that

$$\delta_n = n^{-(2\beta-1)} L_0^2(n) (\log \log n)$$

and let

$$a_n = n^{-(\beta-1/2)} L_0(n) (\log \log n)^{1/2},$$

$$d_{n,p} = \begin{cases} n^{-(1-\beta)} L_0^{-1}(n) (\log n)^{5/2} (\log \log n)^{3/4}, & (p+1)(2\beta-1) > 1 \\ n^{-p(\beta-\frac{1}{2})} L_0^p(n) (\log n)^{1/2} (\log \log n)^{3/4}, & (p+1)(2\beta-1) < 1 \end{cases},$$

Note that $d_{n,2} = o(a_n)$ if $\beta < 3/4$ and $\sigma_{n,1}^{-1} = o(d_{n,2})$.

2.1 Preliminary results

We recall the following law of the iterated logarithm for partial sums $\sum_{i=1}^n X_i$ (see, e.g., [17]):

$$\limsup_{n \rightarrow \infty} \sigma_{n,1}^{-1} (\log \log n)^{-1/2} \left| \sum_{i=1}^n X_i \right| \stackrel{\text{a.s.}}{=} c(\beta, 1), \quad (11)$$

where where $c^2(\beta, p) = \left(\int_0^\infty x^{-\beta} (1+x)^{-\beta} dx \right) (1-\beta)^{-1} (3-2\beta)^{-1}$. Also, if $p < (2\beta-1)^{-1}$, then

$$Y_{n,p} = O(\sigma_{n,p}). \quad (12)$$

Lemma 2.1 *Let $p \geq 1$ be an arbitrary integer such that $p < (2\beta-1)^{-1}$. Then, as $n \rightarrow \infty$,*

$$Y_{n,p} = O_{\text{a.s.}}(\sigma_{n,p} (\log n)^{1/2} \log \log n). \quad (13)$$

Proof. Let $B_n^2 = \sigma_{n,p}^2 \log n (\log \log n)^2$. By (2), [19, Lemma 4] and Karata's Theorem we have for $2^{d-1} < n \leq 2^d$,

$$\begin{aligned} \left\| \frac{Y_{n,p}}{B_{2^d}} \right\|_2^2 &\leq \frac{1}{B_{2^d}^2} \left(\sum_{j=0}^d 2^{(d-j)/2} \sigma_{2^j,p} \right)^2 \leq \frac{1}{B_{2^d}^2} \left(\sum_{j=0}^d 2^{j(1-p(2\beta-1))/2} L_0^p(2^j) \right)^2 \\ &\sim \frac{1}{B_{2^d}^2} 2^{2d-dp(2\beta-1)} L_0^{2p}(2^d) \sim d^{-1} (\log d)^{-2}. \end{aligned}$$

Therefore, the result follows by the Borel-Cantelli lemma. \odot

Let $\tilde{V}_{n,p}(y) = \sum_{r=1}^p F^{(r-1)}(Q(y)) Y_{n,r}$. As an easy consequence of (11) and (13) we obtain the next result.

Lemma 2.2 *Let $p \geq 1$ be an arbitrary integer such that $p < (2\beta-1)^{-1}$. We have*

$$\limsup_{n \rightarrow \infty} \sigma_{n,1}^{-1} (\log \log n)^{-1/2} \sup_{y \in (0,1)} |\tilde{V}_{n,p}(y)| \stackrel{\text{a.s.}}{=} c(\beta, 1). \quad (14)$$

The next result gives the reduction principle for the empirical processes.

Theorem 2.3 ([18]) *Let p be a positive integer. Then, as $n \rightarrow \infty$,*

$$\mathbb{E} \sup_{x \in \mathbb{R}} \left| \sum_{i=1}^n (1_{\{X_i \leq x\}} - F(x)) + \sum_{r=1}^p (-1)^{r-1} F^{(r)}(x) Y_{n,r} \right|^2 = O(\Xi_n + n(\log n)^2),$$

where

$$\Xi_n = \begin{cases} O(n), & (p+1)(2\beta-1) > 1 \\ O(n^{2-(p+1)(2\beta-1)} L_0^{2(p+1)}(n)), & (p+1)(2\beta-1) < 1 \end{cases}.$$

Using Theorem 2.3 and the same argument as in the proof of Lemma 2.1, we obtain

$$\begin{aligned} & \sigma_{n,p}^{-1} \sup_{x \in \mathbb{R}} |S_{n,p}(x)| \\ &= \begin{cases} O_{\text{a.s.}}(n^{-(\frac{1}{2}-p(\beta-\frac{1}{2}))} L_0^{-p}(n) (\log n)^{5/2} (\log \log n)^{3/4}), & (p+1)(2\beta-1) > 1 \\ O_{\text{a.s.}}(n^{-(\beta-\frac{1}{2})} L_0(n) (\log n)^{1/2} (\log \log n)^{3/4}), & (p+1)(2\beta-1) < 1 \end{cases}. \end{aligned}$$

Since (see (2))

$$\frac{\sigma_{n,p}}{\sigma_{n,1}} \sim n^{-(\beta-\frac{1}{2})(p-1)} L_0^{p-1}(n), \quad (15)$$

we obtain

$$\begin{aligned} & \sup_{x \in \mathbb{R}} |\beta_n(x) + \sigma_{n,1}^{-1} V_{n,p}(x)| = \quad (16) \\ &= \frac{\sigma_{n,p}}{\sigma_{n,1}} \sup_{x \in \mathbb{R}} \left| \sigma_{n,p}^{-1} \sum_{i=1}^n (1_{\{X_i \leq x\}} - F(x)) + \sigma_{n,p}^{-1} V_{n,p}(x) \right| = o_{\text{a.s.}}(d_{n,p}). \end{aligned}$$

Consequently, via $\{\alpha_n(y), y \in (0, 1)\} = \{\beta_n(Q(y)), y \in (0, 1)\}$,

$$\sup_{y \in (0,1)} |\alpha_n(y) + \sigma_{n,1}^{-1} \tilde{V}_{n,p}(y)| = O_{\text{a.s.}}(d_{n,p}). \quad (17)$$

We shall use this result with $p = 2$. Then, as mentioned before, $d_{n,2} = o(a_n)$ if $\beta < 3/4$.

2.2 Results on the uniform empirical and quantile processes

We have

$$\frac{\tilde{V}_{n,2}(y)}{(y(1-y))^{1/2}} = \frac{f(Q(y))}{(y(1-y))^{1/2}} \sum_{i=1}^n X_i + \frac{f^{(1)}(Q(y))}{(y(1-y))^{1/2}} Y_{n,2}.$$

Write

$$\frac{f^{(1)}(Q(y))}{(y(1-y))^{1/2}} = \frac{f^{(1)}(Q(y))}{f(Q(y))} (y(1-y))^\mu \frac{f(Q(y))}{(y(1-y))^{1/2+\mu}}$$

with $\mu < 1/2$. Using (A), (B) and (11) we have $\frac{\tilde{V}_{n,2}(y)}{(y(1-y))^{1/2}} = O_{\text{a.s.}}((\log \log n)^{1/2})$ uniformly on $(0, 1)$.

Lemma 2.4 *Under the conditions of Theorem 1.1, as $n \rightarrow \infty$,*

$$\sup_{y \in [\delta_n, 1-\delta_n]} \frac{|\alpha_n(y)|}{\sqrt{y(1-y)}} = O_{\text{a.s.}}((\log \log n)^{1/2}).$$

Proof. We have

$$\begin{aligned} & \sup_{y \in [\delta_n, 1-\delta_n]} \frac{|\alpha_n(y)|}{\sqrt{y(1-y)}} \\ & \leq \sup_{y \in [\delta_n, 1-\delta_n]} \frac{|\alpha_n(y) + \sigma_{n,1}^{-1} \tilde{V}_{n,2}(y)|}{\sqrt{y(1-y)}} + O_{\text{a.s.}}((\log \log n)^{1/2}) \\ & = O_{\text{a.s.}}[\delta_n^{-1/2} d_{n,2}] + O_{\text{a.s.}}((\log \log n)^{1/2}) = O_{\text{a.s.}}((\log \log n)^{1/2}), \end{aligned}$$

using (17). ◻

Using the method of [4, Theorem 2], we obtain the same result for uniform quantile process.

Lemma 2.5 *Under the conditions of Theorem 1.1, with some $C_0 \in (0, \infty)$, as $n \rightarrow \infty$,*

$$\sup_{y \in [C_0 \delta_n, 1-C_0 \delta_n]} \frac{|u_n(y)|}{\sqrt{y(1-y)}} = O_{\text{a.s.}}((\log \log n)^{1/2}).$$

Next, we study the distance between the empirical and quantile processes.

Lemma 2.6 *Under the conditions of Theorem 1.1, as $n \rightarrow \infty$,*

$$\sup_{y \in (0,1)} |u_n(y) - \alpha_n(y)| = O_{\text{a.s.}}(a_n (\log \log n)^{1/2}).$$

Proof. From (17),

$$\sup_{y \in [C_0 \delta_n, 1-C_0 \delta_n]} |u_n(y) - \alpha_n(y)|$$

$$\begin{aligned}
&\leq \sigma_{n,1}^{-1} \sup_{y \in [C_0 \delta_n, 1 - C_0 \delta_n]} |\tilde{V}_{n,2}(y) - \tilde{V}_{n,2}(U_n(y))| + O_{\text{a.s.}}(\sigma_{n,1}^{-1}) \\
&\leq \sigma_{n,1}^{-1} \left| \sum_{i=1}^n X_i \right| \sup_{y \in [C_0 \delta_n, 1 - C_0 \delta_n]} (fQ)'(\theta) |y - U_n(y)| \\
&\quad + \sigma_{n,1}^{-1} (f'Q)'(\theta) |y - U_n(y)| Y_{n,2} + O_{\text{a.s.}}(\sigma_{n,1}^{-1}),
\end{aligned}$$

where $\theta = \theta(y, n)$ is such that $|\theta - y| \leq \sigma_{n,1} n^{-1} |u_n(y)| = O_{\text{a.s.}}((y(1-y)\sigma_{n,1} n^{-1} \log \log n)^{1/2})$ by Lemma 2.5.

Now, via Lemma 2.5,

$$\sup_{y \in [\delta_n, 1 - \delta_n]} (fQ)'(\theta) |y - U_n(y)| \leq \sup_{y \in [\delta_n, 1 - \delta_n]} |(fQ)'(\theta)| \sqrt{y(1-y)} O_{\text{a.s.}}((\log \log n)^{1/2})$$

and the bound is $O(1) O_{\text{a.s.}}((\log \log n)^{1/2})$. Indeed, by (B),

$$\sup_{y \in [\delta_n, 1 - \delta_n]} |(fQ)'(\theta)| \sqrt{y(1-y)} = \sup_{y \in [\delta_n, 1 - \delta_n]} |(fQ)'(\theta)| (\theta(1-\theta))^{1/2} \left(\frac{y(1-y)}{\theta(1-\theta)} \right)^{1/2}.$$

The second order terms, in view of (A), is treated in the similar way.

By the same argument as in [4, Theorem 3],

$$\frac{y(1-y)}{\theta(1-\theta)} = O(1). \tag{18}$$

This, together with the condition (B) implies the bound.

Also,

$$\sup_{y \in (0, \delta_n]} |u_n(y)| = O_{\text{a.s.}}(a_n (\log \log n)^{1/2}) \tag{19}$$

by the same argument as in [4, Theorem 3]. Also,

$$\sup_{y \in (0, \delta_n]} |\alpha_n(y)| = O_{\text{a.s.}}(fQ[\delta_n] (\log \log n)^{1/2}) + O_{\text{a.s.}}(d_{n,2}) = O_{\text{a.s.}}[\delta_n^{1-\mu} \ell(n)] + O_{\text{a.s.}}(d_{n,2}) \tag{20}$$

via the reduction principle, (A) and (B). Indeed,

$$\tilde{V}_{n,2}[\delta_n] = fQ[\delta_n] \sigma_{n,1}^{-1} \sum_{i=1}^n X_i + f'(Q[\delta_n]) \sigma_{n,1}^{-1} Y_{n,2}.$$

The first part is $O_{\text{a.s.}}[\delta_n^{1-\mu} \ell(n)]$ by (A). For the second part, write

$$f'(Q[\delta_n]) \frac{\sigma_{n,2}}{\sigma_{n,1}} = \frac{f'(Q[\delta_n])}{f(Q[\delta_n])} [\delta_n(1-\delta_n)]^\mu \frac{f(Q[\delta_n])}{[\delta_n(1-\delta_n)]^\mu} \frac{\sigma_{n,2}}{\sigma_{n,1}} = O(1) \frac{\delta_n^{1-\mu}}{\delta_n^\mu} \frac{\sigma_{n,2}}{\sigma_{n,1}}$$

by (A) and (B). The above bound is $O(1)$ since $\mu < 1/2$. Consequently, (20) follows.

Therefore, the result of lemma follows. \odot

From (12) with $p = 2$, Lemma 2.6 together with the reduction principle we conclude:

Corollary 2.7 *Under the conditions of Theorem 1.1, as $n \rightarrow \infty$,*

$$\sup_{y \in (0,1)} |u_n(y) + \sigma_{n,1}^{-1} \tilde{V}_{n,2}(y)| = O_{\text{a.s.}}(a_n(\log \log n)^{1/2}),$$

$$\sup_{y \in (0,1)} |u_n(y) + \sigma_{n,1}^{-1} f(Q(y)) \sum_{i=1}^n X_i| = O_{\text{a.s.}}(a_n(\log \log n)^{1/2}(\log n)^{1/2})$$

and

$$\sup_{y \in (0,1)} |u_n(y)| = O_{\text{a.s.}}((\log \log n)^{1/2}).$$

2.3 Proof of Theorem 1.1

We have via the reduction principle,

$$\begin{aligned} & \sup_{y \in (0,1)} |\alpha_n(y) - u_n(y) - \sigma_{n,1}^{-1}(\tilde{V}_{n,2}(y) - \tilde{V}_{n,2}(U_n(y)))| \\ & \leq \sup_{y \in (0,1)} |\alpha_n(y) - \alpha_n(U_n(y)) - \sigma_{n,1}^{-1}(\tilde{V}_{n,2}(y) - \tilde{V}_{n,2}(U_n(y)))| + O_{\text{a.s.}}(\sigma_{n,1}^{-1}) = O_{\text{a.s.}}(d_{n,2}). \end{aligned}$$

Next, let $\psi(y) = (y(1-y))^\mu$, $\mu > 0$. Then

$$\begin{aligned} & \sup_{y \in [\delta_n, 1-\delta_n]} \psi(y) |(\tilde{V}_{n,2}(y) + \tilde{V}_{n,2}(U_n(y))) - n^{-1} \tilde{V}_{n,1}(y) \tilde{V}_{n,2}^{(1)}(y)| \\ & = \sigma_{n,1} n^{-1} \sup_{y \in [\delta_n, 1-\delta_n]} \psi(y) |(fQ)'(y)| \left(u_n(y) + \sigma_{n,1}^{-1} \tilde{V}_{n,2}(y) \right) + \\ & \quad \frac{1}{2} \sup_{y \in [\delta_n, 1-\delta_n]} \psi(y) (y(1-y)) |(fQ)''(\theta)| \frac{(y - U_n(y))^2}{y(1-y)} \sum_{i=1}^n X_i + \\ & \quad \sup_{y \in [\delta_n, 1-\delta_n]} \psi(y) (y(1-y))^{1/2} |(f'Q)'(\theta)| \frac{y - U_n(y)}{(y(1-y))^{1/2}} Y_{n,2} \end{aligned}$$

with the very same θ as in Lemma 2.6. From the condition (B), (14) and Corollary 2.7, the first term is

$$O_{\text{a.s.}}(\sigma_{n,1} n^{-1} a_n (\log \log n)^{1/2} \sigma_{n,1} (\log \log n)^{1/2}) = O_{\text{a.s.}}(n^{5/2-3\beta} \ell(n)).$$

As to the second term, by the condition (C) and (18) we have

$$\begin{aligned} & \sup_{y \in [\delta_n, 1-\delta_n]} (y(1-y))^{1+\mu} |(fQ)''(\theta)| \\ &= \sup_{y \in [\delta_n, 1-\delta_n]} (\theta(1-\theta))^{1+\mu} |(fQ)''(\theta)| \left(\frac{y(1-y)}{\theta(1-\theta)} \right)^{1+\mu} = O(1). \end{aligned}$$

Thus, via Lemma 2.5 and (14), the order of the second term is no greater than $O_{\text{a.s.}}(\sigma_{n,1}^2 n^{-2} \sigma_{n,1} (\log \log n)^{1/2}) = O_{\text{a.s.}}(n^{5/2-3\beta} \ell(n))$.

For the third term, via condition (A) and (18)

$$(f'Q)'(\theta)(y(1-y))^{1/2+\mu} = (f'Q)'(\theta)(\theta(1-\theta))^{1/2+\mu} \left(\frac{y(1-y)}{\theta(1-\theta)} \right)^{1/2+\mu} = O(1).$$

Consequently, the third term is $O_{\text{a.s.}}(\sigma_{n,1} n^{-1} \sigma_{n,2} \ell(n)) = O_{\text{a.s.}}(n^{5/2-3\beta} \ell(n))$.

Note further that

$$\begin{aligned} & (\sigma_{n,1} n)^{-1} \tilde{V}_{n,1}(y) \tilde{V}'_{n,2} \\ &= (\sigma_{n,1} n)^{-1} (fQ)'(y) fQ(y) \left(\sum_{i=1}^n X_i \right)^2 + (f'Q)'(y) fQ(y) (\sigma_{n,1} n)^{-1} \sum_{i=1}^n X_i Y_{n,2}. \end{aligned}$$

Since $\psi(y)(f'Q)'(y) fQ(y) = O(1)$ we conclude that the second term is $O_{\text{a.s.}}(\sigma_{n,2} n^{-1} \ell(n)) = O_{\text{a.s.}}(d_{n,2})$ uniformly on $(0, 1)$.

Thus, since $(fQ)'(y) fQ(y) = f'Q(y)$,

$$\begin{aligned} & \sup_{y \in [\delta_n, 1-\delta_n]} \psi(y) \left| \tilde{R}_n(y) - \sigma_{n,1}^{-1} n^{-1} f'Q(y) \left(\sum_{i=1}^n X_i \right)^2 \right| \\ & \leq C \sup_{y \in (0,1)} \left| \alpha_n(y) - u_n(y) - \sigma_{n,1}^{-1} (\tilde{V}_{n,2}(y) - \tilde{V}_{n,2}(U_n(y))) \right| \\ & \quad + \sup_{y \in [\delta_n, 1-\delta_n]} \psi(y) \sigma_{n,1}^{-1} \left| (\tilde{V}_{n,2}(y) - \tilde{V}_{n,2}(U_n(y))) - f'Q(y) \left(\sum_{i=1}^n X_i \right)^2 \right| + O_{\text{a.s.}}(d_{n,2}) \\ & = O_{\text{a.s.}}(d_{n,2}) + O_{\text{a.s.}}(\sigma_{n,1}^{-1} n^{5/2-3\beta} \ell(n)) = O_{\text{a.s.}}(d_{n,2}). \end{aligned}$$

Therefore,

$$\sup_{y \in [\delta_n, 1-\delta_n]} \left| n \sigma_{n,1}^{-1} \tilde{R}_n(y) - \sigma_{n,1}^{-2} f'Q(y) \left(\sum_{i=1}^n X_i \right)^2 \right| = O_{\text{a.s.}}(n \sigma_{n,1}^{-1} d_{n,2} \delta_n^{-\mu}) = o_{\text{a.s.}}(1)$$

by choosing $0 < \mu < 1/2$.

◊

2.4 Proof of Theorem 1.3

We have for $y < 1/2$,

$$2\sigma_{n,1}^{-1}n \int_0^y \tilde{R}_n(y)dy = 2\sigma_{n,1}^{-1}n \int_{(0,y) \cap [\delta_n, 1-\delta_n]} \tilde{R}_n(y)dy \\ + O\left(\sigma_{n,1}^{-1}n \int_0^{\delta_n} |u_n(y)|\right) + O\left(\sigma_{n,1}^{-1}n \int_0^{\delta_n} |\alpha_n(y)|\right).$$

The second integral is at most of the order

$$O_{\text{a.s.}}\left(\sigma_{n,1}^{-1}n\delta_n \sup_{y \in (0, \delta_n]} |u_n(y)|\right) = o_{\text{a.s.}}(1)$$

by (19). The same holds for the third one. A similar reasoning applies for $y > 1/2$. Thus, the result follows from Corollary 1.2.

◊

2.5 Proof of Theorem 1.4

As in [6], let

$$A_n(t) = 2\sigma_{n,1}^{-1}n \int_{U_n(t)}^t (\alpha_n(y) - \alpha_n(t))dy.$$

Then,

$$\sup_{t \in (0,1)} |A_n(t) - \tilde{W}_n(t)| = O_{\text{a.s.}}(n^{-(2\beta-1)}\ell(n)).$$

Via the reduction principle and the second part of Corollary 2.7,

$$\sup_{t \in (0,1)} |A_n(t) - B_n(t)| =: \sup_{t \in (0,1)} \left| A_n(t) + 2\sigma_{n,1}^{-2}n \int_{U_n(t)}^t (\tilde{V}_{n,2}(y) - \tilde{V}_{n,2}(t))dy \right| \\ \leq 4\sigma_{n,1}^{-1}n \sup_{y \in (0,1)} |y - U_n(y)| \sup_{y \in (0,1)} |\alpha_n(y) + \sigma_{n,1}^{-1}\tilde{V}_{n,2}(y)| = O_{\text{a.s.}}(d_{n,2}(\log \log n)^{1/2}).$$

Let $C(t) = \int_0^t fQ(y)dy$, $D(t) = \int_0^t f'Q(y)dy$. Then

$$B_n(t) = 2\sigma_{n,1}^{-2}n \left(\sum_{i=1}^n X_i \int_{U_n(t)}^t (fQ(y) - fQ(t))dy - Y_{n,2} \int_{U_n(t)}^t (f'Q(y) - f'Q(t))dy \right)$$

$$\begin{aligned}
&= 2\sigma_{n,1}^{-2}n \sum_{i=1}^n X_i (C(t) - C(U_n(t)) - (t - U_n(t))fQ(t)) - \\
&\quad 2\sigma_{n,1}^{-2}n Y_{n,2} \left(D(t) - D(U_n(t)) - (t - U_n(t))f'Q(t) \right) \\
&= 2\sigma_{n,1}^{-2}n \frac{(fQ)'(t)}{2} (t - U_n(t))^2 \sum_{i=1}^n X_i + 2\sigma_{n,1}^{-2}n \frac{(fQ)''(\theta)}{6} (t - U_n(t))^3 \sum_{i=1}^n X_i - \\
&\quad 2\sigma_{n,1}^{-2}n \frac{(f'Q)'(t)}{2} (t - U_n(t))^2 Y_{n,2} - 2\sigma_{n,1}^{-2}n \frac{(f'Q)''(\theta)}{6} (t - U_n(t))^3 Y_{n,2}.
\end{aligned}$$

where θ is from Lemma 2.6. Consequently, by (14),

$$\begin{aligned}
&\sup_{t \in [\delta_n, 1-\delta_n]} \left| B_n(t) - \sigma_{n,1}^{-2}n \sum_{i=1}^n X_i (fQ)'(t) (t - U_n(t))^2 \right| \tag{21} \\
&= O_{\text{a.s.}}(\sigma_{n,1}^{-1}n\ell(n)) \times \sup_{t \in [\delta_n, 1-\delta_n]} \frac{|t - U_n(t)|^3}{(t(1-t))^{3/2}} (fQ)''(\theta) (t(1-t))^{3/2} + \\
&\quad O_{\text{a.s.}}(\sigma_{n,2}\sigma_{n,1}^{-2}n\ell(n)) \times \sup_{t \in [\delta_n, 1-\delta_n]} \frac{|t - U_n(t)|^2}{t(1-t)} (f'Q)'(\theta) (t(1-t)) + \\
&\quad O_{\text{a.s.}}(\sigma_{n,2}\sigma_{n,1}^{-2}n\ell(n)) \times \sup_{t \in [\delta_n, 1-\delta_n]} \frac{|t - U_n(t)|^3}{(t(1-t))^{3/2}} (f'Q)''(\theta) (t(1-t))^{3/2}.
\end{aligned}$$

Therefore, by (B), (C) and Lemma 2.5, the bound in (21) is of the order $O_{\text{a.s.}}(n^{-(2\beta-1)}\ell(n)) = O_{\text{a.s.}}(d_{n,2}\ell(n))$. Consequently,

$$\sup_{t \in [\delta_n, 1-\delta_n]} \left| \tilde{W}_n(t) + \sigma_{n,1}^{-2}n \sum_{i=1}^n X_i (fQ)'(t) (t - U_n(t))^2 \right| = O_{\text{a.s.}}(d_{n,2}\ell(n)).$$

Therefore, by (12), (14) and Corollary 2.7 one obtains weak convergence and strong asymptotics of $\tilde{W}_n(t)1_{\{t \in [\delta_n, 1-\delta_n]\}}$. Further, by (19),

$$\sigma_{n,1}^{-2}n^2 \int_0^{\delta_n} |u_n(y)| dy = O_{\text{a.s.}}(\sigma_{n,1}^{-2}n^2\delta_n) \sup_{y \in (0, \delta_n)} |u_n(y)| = o_{\text{a.s.}}(1)$$

and the same holds if one replaces $u_n(y)$ with $\alpha_n(y)$. Thus,

$$\sigma_{n,1}^{-2}n^2 \int_0^{\delta_n} |\tilde{R}_n(y)| dy = o_{\text{a.s.}}(1).$$

Finally, $\sigma_{n,1}^{-1}n \sup_{t \in (0, \delta_n]} \alpha_n^2(t) = O_{\text{a.s.}}(d_{n,2}\ell(n))$ and $\sigma_{n,1}^{-1}n d_{n,2}\ell(n) = o(1)$. \odot

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