Dependence orderings for some functionals of multivariate point processes

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Abstract

We study dependence orderings for functionals of k-variate point processes $\mathbf{\Phi}$ and $\mathbf{\Psi}$. We view the first process as a collection of counting measures, whereas the second as the sequences of interpoint distances. Subsequently, we establish regularity properties of stationary sequences which generalize known results for iid case. The theoretical results are illustrated by many special cases including comparison of multivariate sums and products, comparison of multivariate shock models and queueing systems.

Keywords: point process; supermodular ordering; directionally convex ordering

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1 Introduction

The aim of this paper is to study dependence orderings for functionals of stationary multivariate point processes. Especially, we consider the *supermodular ordering* which is positive dependence ordering in the sense of Joe [12]. It implies positive orthant orderings, concordance ordering and hence comparison of covariance functions, minima and maxima. Moreover, we consider the *directionally convex ordering.* Note that it is not a dependence ordering, being not closed under increasing transforms and weak convergence, but have some similar properties as supermodular ordering.

Point processes can be viewed in several ways. We can consider them as sequences of interpoint distances, as random measures or as piecewise deterministic step functions. Each notion requires its own definition of stochastic ordering. Therefore we introduce appropriate definitions motivated by Kwieciński and Szekli [15]. In contrary to strong stochastic ordering (Rolski and Szekli [29]) a little is known about dependence orderings of point processes. We refer to Müller and Stoyan [26] for a review.

In many stochastic models (queues, ruin theory, shock models) some characteristics can be represented as functionals on point processes. Such models require stationarity (and ergodicity) of input stream. Therefore we study sufficient conditions (in terms of the ordering of base point processes) for the comparison of general functionals on stationary multivariate point processes. These results allow to obtain bounds for stochastic models with stationary (not necessary renewal) input stream. They extend results for example of Li and Xu [17], [18]. As a byproduct we obtain regularity properties of sequences of stationary random variables which extend results for the iid case (Ross [30], Makowski and Philips [20]).

The paper is organized as follows. In section 2 we describe multivariate point processes, classes of functions and define stochastic orderings for point processes. In section 3 we present our main results which are illustrated by several special cases (section 4). The proofs are given in section 5. From the proofs we get some regularity properties in section 6. In section 7 we apply our results to some stochastic models, including workload in queues and multivariate shock models. We mention some possible extensions in section 8 and present properties of the classes of functions and stochastic orderings in the Appendix.

2 Preliminaries

2.1 Multivariate point processes

A simple description of a k-variate ($k \le \infty$) point process is the one given by a sequence $\mathbf{\Phi} \equiv \{(T_n^1, \ldots, T_n^k)\}_{n=-\infty}^{\infty}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $T_0^i \le 0 < T_1^i$, $T_n^i < T_{n+1}^i$, $i = 1, \ldots, k$, $n \in \mathbb{Z}$ and $\lim_{n\to\pm\infty} T_n^i = \pm\infty$ ($\mathbf{\Phi}$ is nonexplosive). Denote by $\{X_n^i\}_{n=-\infty}^{\infty}$ a sequence of interpoint distances, i.e. $X_n^i = T_n^i - T_{n-1}^i$ (the interval X_1^i contains 0). Then a k-variate point process $\mathbf{\Phi}$ can be seen as a random element assuming its values in $(\mathbb{R}_+^\infty)^k$.

Let $\mathcal{N}$ be a set of locally finite integer valued measures on $\mathbb{R}$. Equivalently, we view $\mathbf{\Phi}$ as a random measure $\mathbf{\Phi} : \Omega \to \mathcal{N}^k$ with the coordinate functions $\mathbf{\Phi} = (\Phi^1, \ldots, \Phi^k)$, $\Phi^i : \Omega \to \mathcal{N}$. Then for all Borel sets B, $N_{\mathbf{\Phi}}^i(B) := \Phi^i(B)$ is the corresponding counting variable. However, if it is clear which point process do we mean we shall write shortly N^i instead of $N^i_{\mathbf{\Phi}}$. The corresponding counting processes $(N^i(t), t \geq 0)$, $i = 1, \dots, k$ are given by $N^i(t) := N^i((0,t])$. We shall assume that $\mathbf{\Phi}$ is (time) stationary, i.e. the distribution of

$$(N^1(B^1_1 + t), \dots, N^1(B^1_{r_1} + t), \dots, \dots, N^k(B^k_1 + t), \dots, N^1(B^k_{r_k} + t))$$

is independent of $t \in \mathbb{R}$, for any natural numbers $r_i \geq 1$, $i = 1, \dots, k$ and all Borel sets B^i_j, $j = 1, \dots, r_i$. We denote by $\lambda^i := \mathbb{E}[N^i(1)]$ the intensity of Φ^i.

We assume that we have another point process $\mathbf{\Psi}$ with the corresponding points $\{(\mathcal{T}^1_n, \dots, \mathcal{T}^k_n)\}_{n\geq 1}$, $k \leq \infty$ and interpoint distances $U^i_n = \mathcal{T}^i_n - \mathcal{T}^i_{n-1}$, $i = 1, \dots, k$.

We shall denote realizations (in $\mathcal{N}^k$) of $\mathbf{\Phi}$ by ν and realizations of $\mathbf{\Psi}$ by μ. The corresponding realizations of counting measures (counting functions) of $\mathbf{\Phi}$ and sequences of interpoint distances of $\mathbf{\Psi}$ we denote by $n^i(\nu)(\cdot)$ $(n^i(\nu)(t))$ and $\{u^i_n(\mu)\}$, respectively.

In the case $k = 1$ we shall write T_n (X_n, N, λ) and $\mathcal{T}_n$ (U_n) instead of writing these quantities with the superscript 1.

We say that sequences $\{U^i_n\}_{n\geq 1}$, (or, shortly $\{U^i_n\}$), $i = 1, \dots, k$ are *jointly stationary* if for any $n_i \geq 1$, $i = 1, \dots, k$, $m \geq 1$,

$$((U^1_1, \dots, U^1_{n_1}), \dots, (U^k_1, \dots, U^k_{n_k})) \stackrel{\mathrm{d}}{=} ((U^1_{1+m}, \dots, U^1_{n_1+m}), \dots, (U^k_{1+m}, \dots, U^k_{n_k+m}))\,.$$

In the sequel we shall write $(Y^i_1, \dots, Y^i_{r_i}, i = 1, \dots, k)$ for a vector

$$(Y^1_1, \dots, Y^1_{r_1}, \dots, Y^k_1, \dots, Y^k_{r_k})\,.$$

We assume that all random elements with tilde (for instance $\tilde{\mathbf{\Psi}}$, $\tilde{\mathbf{\Phi}}$) are defined on a possibly different probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

2.2 Classes of functions

We denote by $\mathcal{L}_{\mathrm{i}}$ $(\mathcal{L}_{\mathrm{cx}}, \mathcal{L}_{\mathrm{icx}})$ the class of increasing (convex, increasing and convex) functions $f : \mathbb{R} \to \mathbb{R}$.

Define for $1 \leq l \leq m$, $\epsilon > 0$ and arbitrary function $\varphi : \mathbb{R}^m \to \mathbb{R}$ the difference operator Δ^ϵ_l by

$$\Delta^\epsilon_l \varphi(u_1, \dots, u_m) = \varphi(u_1, \dots, u_{l-1}, u_l + \epsilon, u_{l+1}, \dots, u_m) - \varphi(u_1, \dots, u_m)$$

for given $u_1, \dots, u_m$.

We denote arbitrary m-dimensional intervals by $\mathcal{J} \subseteq \mathbb{R}^m$, i.e. $\mathcal{J} = I^1 \times \cdots \times I^m$, where I^j is a (possibly infinite ended) interval on $\mathbb{R}$ for $j = 1, \dots, m$. A function $\varphi : \mathbb{R}^m \to \mathbb{R}$ is *supermodular* on $\mathcal{J}$ if for all $1 \leq l < j \leq m$, $\epsilon_l, \epsilon_j > 0$ and $\mathbf{u} = (u_1, \dots, u_m) \in \mathcal{J}$ such that $(u_1, \dots, u_{l-1}, u_l + \epsilon_l, u_{l+1}, \dots, u_m) \in \mathcal{J}$ we have

$$\Delta^{\epsilon_l}_l \Delta^{\epsilon_j}_j \varphi(\mathbf{u}) \geq 0\,.$$

A function $\varphi : \mathbb{R}^m \to \mathbb{R}$ is *directionally convex* on $\mathcal{J}$ if it is supermodular on $\mathcal{J}$ and convex w.r.t. each coordinate on I^j, $j = 1, \dots, m$ or, equivalently

$$\Delta^{\epsilon_l}_l \Delta^{\epsilon_j}_j \varphi(\mathbf{u}) \geq 0$$

for all $1 \leq l \leq j \leq m$. We denote by $\mathcal{L}_{\mathrm{sm}}(\mathcal{J})$ ($\mathcal{L}_{\mathrm{dcx}}(\mathcal{J})$) the class of all supermodular (directionally convex) functions on $\mathcal{J}$. Moreover, we denote the class of increasing directionally convex functions on $\mathcal{J}$ by $\mathcal{L}_{\mathrm{idcx}}(\mathcal{J})$ and symmetric supermodular functions on $\mathcal{J}$ by $\mathcal{L}_{\mathrm{ssm}}(\mathcal{J})$. We skip $\mathcal{J}$ in this notation if $\mathcal{J} = \mathbb{R}^m$. We collect needed closure and regularity properties of these classes in the Appendix.

Definition 2.1 For a fixed $c \in \mathbb{R}$, let $\bar{\mathcal{J}} = \{\mathcal{J}_m\}_{m\geq 1}$ be a sequence of intervals, $\mathcal{J}_m \subseteq \mathbb{R}^m$, such that for all $m \geq 1$ and $(u_1,\ldots,u_m) \in \mathcal{J}_m$, $(u_1,\ldots,u_m,c) \in \mathcal{J}_{m+1}$. We say that a sequence $\{f^{(m)}\}_{m\geq 1}$ of functions $f^{(m)} : \mathbb{R}^m \to \mathbb{R}$ is *extendable* on $\bar{\mathcal{J}}$ with parameter $c \in \mathbb{R}$ if

$$f^{(m+1)}(u_1,\ldots,u_m,c) = f^{(m)}(u_1,\ldots,u_m), \text{ for all } m \geq 1 \text{ and } (u_1,\ldots,u_m) \in \mathcal{J}_m .$$

We denote by $\mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$ the class of all sequences which are extendable on $\bar{\mathcal{J}}$ with parameter c.

Example 2.2 We give some examples of sequences of symmetric supermodular functions $\{f^{(m)}\}$ such that $f^{(m)} \in \mathcal{L}_{\mathrm{ssm}}(\mathcal{J}_m)$ and $\{f^{(m)}\}_{m\geq 1} \in \mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$ for some $\bar{\mathcal{J}}$, c.

1. $\{f^{(m)}(u_1,\ldots,u_m) = h(\min\{u_1,\ldots,u_m\})\}_{m\geq 1} \in \mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$, $\mathcal{J}_m = (-\infty, c]^m$, for all $c \in \mathbb{R}$ and increasing h;
2. $\{f^{(m)}(u_1,\ldots,u_m) = h(\max\{u_1,\ldots,u_m\})\}_{m\geq 1} \in \mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$, $\mathcal{J}_m = [c,\infty)^m$, for all $c \in \mathbb{R}$ and decreasing h;
3. $\{f^{(m)}(u_1,\ldots,u_m) = \varphi(\prod_{n=1}^m u_n^d)\}_{m\geq 1} \in \mathcal{E}_1(\bar{\mathcal{J}})$, $\mathcal{J}_m = [0,\infty)^m$, for $d \geq 0$ and all increasing convex φ;
4. $\{f^{(m)}(u_1,\ldots,u_m) = \prod_{n=1}^m \mathbb{I}_{(-\infty,t]}(u_n)\}_{m\geq 1} \in \mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$, $\mathcal{J}_m = \mathbb{R}^m$, for all $c \leq t$, $t \in \mathbb{R}$;
5. $\{f^{(m)}(u_1,\ldots,u_m) = \prod_{n=1}^m \mathbb{I}_{[t,\infty)}(u_n)\}_{m\geq 1} \in \mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$, $\mathcal{J}_m = \mathbb{R}^m$, for all $c \geq t$, $t \in \mathbb{R}$;
6. $\{f^{(m)}(u_1,\ldots,u_m) = \varphi(\sum_{n=1}^m u_n)\}_{m\geq 1} \in \mathcal{E}_0(\bar{\mathcal{J}})$, $\mathcal{J}_m = \mathbb{R}^m$, for all convex φ.

The functions defined in 3 and 6 are directionally convex for $d \geq 1$. Note, that for $c_n \geq 0$, $n \geq 1$, the sequence $\{f^{(m)}(u_1,\ldots,u_m) = \sum_{n=1}^m c_n u_n\}_{m\geq 1} \in \mathcal{E}_0(\bar{\mathcal{J}})$, $\mathcal{J}_m = \mathbb{R}^m$, consists of nonsymmetric functions if $\{c_n\}$ is not a constant sequence.

2.3 Stochastic ordering

For arbitrary random vectors $(Y_1,\ldots,Y_n)$, $(\tilde{Y}_1,\ldots,\tilde{Y}_n)$ defined on probability spaces $(\Omega,\mathcal{F},\mathbb{P})$ and $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ respectively, we write $(Y_1,\ldots,Y_n) <_{\mathrm{a}} (\tilde{Y}_1,\ldots,\tilde{Y}_n)$ if $\mathbb{E}[\varphi(Y_1,\ldots,Y_n)] \leq \tilde{\mathbb{E}}[\varphi(\tilde{Y}_1,\ldots,\tilde{Y}_n)]$ for all $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi \in \mathcal{L}_{\mathrm{a}}$, where $\mathcal{L}_{\mathrm{a}}$ denotes one of the classes $\mathcal{L}_{\mathrm{sm}}$, $\mathcal{L}_{\mathrm{ssm}}$, $\mathcal{L}_{\mathrm{dcx}}$, $\mathcal{L}_{\mathrm{idcx}}$. Similarly, for random sequences $\{Y_n\}_{n\geq 1}$ and $\{\tilde{Y}_n\}_{n\geq 1}$ we write $\{Y_n\} <_{\mathrm{a}} \{\tilde{Y}_n\}$ if for all $n \geq 1$, $(Y_1,\ldots,Y_n) <_{\mathrm{a}} (\tilde{Y}_1,\ldots,\tilde{Y}_n)$.

Let $\mathbf{\Psi}$ ($\tilde{\mathbf{\Psi}}$) be a k-variate stationary point process with the corresponding inter-point distances $\{U_n^i\}$ ($\{\tilde{U}_n^i\}$), $i = 1,\ldots,k$. We write

- $\mathbf{\Psi} <_{\mathrm{h-a-}\infty} \tilde{\mathbf{\Psi}}$ if $(U_1^i,\ldots,U_n^i) <_{\mathrm{a}} (\tilde{U}_1^i,\ldots,\tilde{U}_n^i)$, $i = 1,\ldots,k$, $n \in \mathbb{N}$,

- $\mathbf{\Psi} <_{\mathrm{v-a-}\infty} \tilde{\mathbf{\Psi}}$ if $(U_n^1, \ldots, U_n^k) <_{\mathrm{a}} (\tilde{U}_n^1, \ldots, \tilde{U}_n^k)$, $n \geq 1$,
- $\mathbf{\Psi} <_{\mathrm{m-a-}\infty} \tilde{\mathbf{\Psi}}$ if $(\{U_n^1\}, \ldots, \{U_n^k\}) <_{\mathrm{a}} (\{\tilde{U}_n^1\}, \ldots, \{\tilde{U}_n^k\})$, i.e. if for all $n \geq 1$, $k \geq 1$,

$$\Big((U_1^1, \ldots, U_n^1), \ldots, (U_1^k, \ldots, U_n^k)\Big) <_{\mathrm{a}} \Big(\Big(\tilde{U}_1^1, \ldots, \tilde{U}_n^1\Big), \ldots, \Big(\tilde{U}_1^k, \ldots, \tilde{U}_n^k\Big)\Big) .$$

Let $\mathbf{\Phi}$ ($\mathbf{\Phi}$) be a k-variate point process with the corresponding counting measures N^i ($\tilde{N}^i$), $i = 1, \ldots, k$. We write

- $\mathbf{\Phi} <_{\mathrm{v-a-D}} \tilde{\mathbf{\Phi}}$ if for all $t \geq 0$,

$$(N^1(t), \ldots, N^k(t)) <_{\mathrm{a}} (\tilde{N}^1(t), \ldots, \tilde{N}^k(t)) ,$$

- $\mathbf{\Phi} <_{\mathrm{h-a-D}} \tilde{\mathbf{\Phi}}$ if for all $0 \leq t_1 < t_2 < \cdots < t_r$, $r \geq 1$,

$$(N^i(t_1), \ldots, N^i(t_r)) <_{\mathrm{a}} (\tilde{N}^i(t_1), \ldots, \tilde{N}^i(t_r)) ,$$

  $i = 1, \ldots, k$,
- $\mathbf{\Phi} <_{\mathrm{m-a-D}} \tilde{\mathbf{\Phi}}$ if for all $0 \leq t_1 < t_2 < \cdots < t_r$, $r \geq 1$,

$$(N^i(t_1), \ldots, N^i(t_r), i = 1, \ldots, k) <_{\mathrm{a}} (\tilde{N}^i(t_1), \ldots, \tilde{N}^i(t_r), i = 1, \ldots, k) .$$

Let $\mathcal{I} = \{I_n\}_{n \geq 1}$ be a partition of $\mathbb{R}_+$ such that I_r, $r \geq 1$ have the same length. We write

- $\mathbf{\Phi} <_{\mathrm{v-a-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ if for all $r \geq 1$,

$$(N^1(I_r), \ldots, N^k(I_r)) <_{\mathrm{a}} (\tilde{N}^1(I_r), \ldots, \tilde{N}^k(I_r)) ,$$

- $\mathbf{\Phi} <_{\mathrm{h-a-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ if for all $(I_1, \ldots, I_r)$, $r \geq 1$,

$$(N^i(I_1), \ldots, N^i(I_r)) <_{\mathrm{a}} (\tilde{N}^i(I_1), \ldots, \tilde{N}^i(I_r)) ,$$

  $i = 1, \ldots, k$,
- $\mathbf{\Phi} <_{\mathrm{m-a-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ if for all $(I_1, \ldots, I_r)$, $r \geq 1$,

$$(N^i(I_1), \ldots, N^i(I_r), i = 1, \ldots, k) <_{\mathrm{a}} (\tilde{N}^i(I_1), \ldots, \tilde{N}^i(I_r), i = 1, \ldots, k) .$$

Here, $<_{\mathrm{v-a-\cdot}}$ ($<_{\mathrm{h-a-\cdot}}$, $<_{\mathrm{m-a-\cdot}}$) means "vertical" ("horizontal", "matrix") ordering. On the other hand, $<_{\cdot-\cdot-\infty}$ ($<_{\cdot-\cdot-\mathcal{N}}$, $<_{\cdot-\cdot-\mathrm{D}}$) stands for the comparison of point processes considered as random elements of $(\mathbb{R}_+^\infty)^k$, $(\mathcal{N}^k$, $(\mathrm{D}([0,\infty)))^k)$, where $\mathrm{D}([0,\infty))$ is the space of right-hand-side continuous functions with left-hand-side limits.

Of course, if $\{X_n^i\}_{n \geq 1}$ is independent of $\{X_n^j\}_{n \geq 1}$ and $\{\tilde{X}_n^i\}_{n \geq 1}$ is independent of $\{\tilde{X}_n^j\}_{n \geq 1}$, $1 \leq i < j \leq k$ then $\mathbf{\Phi} <_{\mathrm{m-a-}\infty} \tilde{\mathbf{\Phi}}$ is equivalent to $\mathbf{\Phi} <_{\mathrm{h-a-}\infty} \tilde{\mathbf{\Phi}}$ and if $\{(X_n^1, \ldots, X_n^k)\}_{n \geq 1}$, $\{(\tilde{X}_n^1, \ldots, \tilde{X}_n^k)\}_{n \geq 1}$ are sequences of independent random vectors then $\mathbf{\Phi} <_{\mathrm{m-a-}\infty} \tilde{\mathbf{\Phi}}$ is equivalent to $\mathbf{\Phi} <_{\mathrm{v-a-}\infty} \tilde{\mathbf{\Phi}}$. The same relationships hold for stochastic orderings of counting measures.

For 1-variate point processes ($k = 1$) we shall omit subscript 1. Then for stationary processes $<_{\mathrm{a-D}}$ ($<_{\mathrm{a-}\mathcal{N}}$, $<_{\mathrm{a-}\infty}$) is exactly $<_{\mathrm{h-a-D}}$ ($<_{\mathrm{h-a-}\mathcal{N}}$, $<_{\mathrm{1-a-}\infty}$), whereas $<_{\mathrm{v-a-}\mathcal{N}}$ is equivalent to $<_{\mathrm{v-a-D}}$ and means that for all $t \geq 0$, $N(t) <_{\mathrm{a}} \tilde{N}(t)$. Note that for 1-variate point processes definitions for $<_{\mathrm{h-st-}\mathcal{N}}$ and $<_{\mathrm{h-st-D}}$ coincide with $<_{\mathrm{st-}\mathcal{N}}$ and $<_{\mathrm{st-D}}$ orderings defined in Kwieciński and Szekli [15].

A number of examples of sequences comparable in a $:=<_{\mathrm{sm}}, <_{\mathrm{idcx}}$ orderings, including Markov-renewal sequences, stochastically monotone Markov chains can be found in Bäuerle [3, 4], Bäuerle and Rolski [5], Frostig [9], Hu and Pan [10], Li and Xu [17, 18, 19], Kulik and Szekli [13, 14], Meester and Shanthikumar [22], Müller [23], Müller and Scarsini [25], Müller and Pflug [24], Shaked and Shanthikumar [31], Szekli *et al.* [35].

3 Main results

For $i = 1, \ldots, k$ denote arbitrary sequences of functions $\{f_i^{(m)} : \mathbb{R}^m \to \mathbb{R}^{d_i}\}_{m\geq 1}$, $1 \leq d_i, m < \infty$ by $\mathbf{f} \equiv (\{f_1^{(m)}\}_{m\geq 1}, \ldots, \{f_k^{(m)}\}_{m\geq 1})$. We write shortly $f_i^{(m)}(\{u_n^i\})$ for $f_i^{(m)}(u_1^i, \ldots, u_m^i)$. For Borel sets $B_1, \ldots, B_r$ define a functional $H_{\mathbf{f}}(\cdot,\cdot)(B_1, \ldots, B_r) : \mathcal{N}^k \times \mathcal{N}^k \to \mathbb{R}^{r\cdot d_1} \times \cdots \times \mathbb{R}^{r\cdot d_k}$ in the following way:

$$H_{\mathbf{f}}(\mu, \nu)(B_1, \ldots, B_r) \equiv (f_i^{(n^i(\nu)(B_1))}(\{u_n^i(\mu)\}), \ldots, f_i^{(n^i(\nu)(B_r))}(\{u_n^i(\mu)\}), i = 1, \ldots, k)\,.$$

We write shortly $H_{\mathbf{f}}(\mu, \nu)(\mathbf{B})$ for the above expression, and in particular, we write $H_{\mathbf{f}}(\mu, \nu)(\mathbf{t})$ and $H_{\mathbf{f}}(\mu, \nu)(\mathbf{I})$ for $B_j = (0, t_j]$ and $B_j = I_j$, $j = 1, \ldots, r$, respectively. Here, $\mathbf{t} = (t_1, \ldots, t_r) \geq \mathbf{0}$, $\mathbf{I} = (I_1, \ldots, I_r)$ and $\mathcal{I} = \{I_n\}_{n\geq 1}$ is the previously defined partition of $\mathbb{R}_+$. For the case $r = 1$ we shall write $H_{\mathbf{f}}(\mu, \nu)(t)$ and $H_{\mathbf{f}}(\mu, \nu)(I)$ instead of writing $\mathbf{t} = (t_1)$, $\mathbf{I} = (I_1)$, respectively.

Our aim is to compare $H_{\mathbf{f}}(\Psi, \Phi)(B_1, \ldots, B_r)$, with $H_{\mathbf{f}}(\tilde{\Psi}, \tilde{\Phi})(B_1, \ldots, B_r)$, in the supermodular and increasing directionally convex order under suitable assumptions on $f_i^{(m)}$. In order to do this we need to formalize a notion of monotonicity of sequences $\{f_i^{(m)}\}_{m\geq 1}$, $i = 1, \ldots, k$. Denote by $\leq$ the coordinatewise ordering on $\mathbb{R}^d$, $1 \leq d \leq \infty$. We say that a sequence $\{f^{(m)} : \mathbb{R}^m \to \mathbb{R}^d\}_{m\geq 1}$ is increasing w.r.t. m if $m \leq m'$ implies $f^{(m)}(u_1, \ldots, u_m) \leq f^{(m')}(u_1, \ldots, u_{m'})$ for all sequences $\{u_n\}_{n=1}^{\infty}$. We say that a function $f^{(m)} : \mathbb{R}^m \to \mathbb{R}^d$ is increasing w.r.t. $\{u_n\}$ if $\{u_n\}_{n=1}^m \leq \{\tilde{u}_n\}_{n=1}^m$ implies $f^{(m)}(\{u_n\}) \leq f^{(m)}(\{\tilde{u}_n\})$. Analogously we define decreasingness. We say that a function is monotone if it is increasing or decreasing. Moreover, functions $g_1, \ldots, g_k$ are *monotone in the same direction* if all are either increasing or decreasing. Now, we state our main results. The proofs are given in Section 5.

Proposition 3.1 *Assume that*

(i) $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ *are stationary,* $\mathbf{\Phi}$ *is independent of* $\mathbf{\Psi}$ *and* $\tilde{\mathbf{\Phi}}$ *is independent of* $\tilde{\mathbf{\Psi}}$*,*

(ii) $\mathbf{\Phi} <_{\mathrm{m-sm-D}} \tilde{\mathbf{\Phi}}$ ($\mathbf{\Phi} <_{\mathrm{m-sm-}\mathcal{N}} \tilde{\mathbf{\Phi}}$),

(iii) $\{(U_n^1, \ldots, U_n^k)\}_{n\geq 1}$, $\{(\tilde{U}_n^1, \ldots, \tilde{U}_n^k)\}_{n\geq 1}$ *are sequences of independent random vectors,*

(iv) $\mathbf{\Psi} <_{\mathrm{v-sm-}\infty} \tilde{\mathbf{\Psi}}$,

(v) *The sequences* $\{f_i^{(m)} : \mathbb{R}^m \to \mathbb{R}^{d_i}\}_{m\geq 1}$, $i = 1, \ldots, k$, *are monotone in the same direction w.r.t.* m,

(vi) *The functions* $f_i^{(m)}$, $m \geq 1$, $i = 1, \ldots, k$ *are increasing w.r.t.* $\{u_n^i\}$.

Then for all $\mathbf{t} \geq \mathbf{0}$, $(\mathbf{I})$

$$H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(\mathbf{t}) <_{\mathrm{sm}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(\mathbf{t}) \,,$$

$$(H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(\mathbf{I}) <_{\mathrm{sm}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(\mathbf{I}))\,.$$

Let $\bar{\mathcal{J}} = \{\mathcal{J}_m\}_{m\geq 1}$ be a sequence of m-dimensional intervals and $\mathcal{K}_m^i := \mathrm{supp}(U_1^i, \ldots, U_m^i) \subseteq \mathcal{J}_m$ be the support of $(U_1^i, \ldots, U_m^i)$, $i = 1, \ldots, k$.

Proposition 3.2 *Assume that*

(i) $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ *are stationary,* $\mathbf{\Phi}$ *is independent of* $\mathbf{\Psi}$ *and* $\tilde{\mathbf{\Phi}}$ *is independent of* $\tilde{\mathbf{\Psi}}$,

(ii) $\mathbf{\Phi} <_{\mathrm{v-idcx-D}} \tilde{\mathbf{\Phi}}$ $(\mathbf{\Phi} <_{\mathrm{v-idcx-}\mathcal{N}} \tilde{\mathbf{\Phi}})$,

(iii) *For some* $c \in \mathbb{R}$ *and* $\bar{\mathcal{J}}$ *the sequences* $\{f_i^{(m)} : \mathbb{R}^m \to \mathbb{R}\}_{m\geq 1} \in \mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$, $i = 1, \ldots, k$ *and are increasing w.r.t.* m,

(iv) *For all* $i = 1, \ldots, k$, $\{U_n^i\}_{n\geq 1}$ $(\{\tilde{U}_n^i\}_{n\geq 1})$ *is a stationary sequence independent of* $\{U_n^j\}_{n\geq 1}$ $(\{\tilde{U}_n^j\}_{n\geq 1})$, $j \neq i$, *such that* $\sup_{n,i} U_n^i \leq c$ *or* $\inf_{n,i} U_n^i \geq c$,

(v) $\mathbf{\Psi} <_{\mathrm{h-idcx-}\infty} \tilde{\mathbf{\Psi}}$,

(vi) *The functions* $f_i^{(m)} \in \mathcal{L}_{\mathrm{idcx}}(\mathcal{J}_m)$, $i = 1, \ldots, k$, $m \geq 1$ *are symmetric on* $\mathcal{J}_m$ *and increasing w.r.t.* $\{u_n^i\}$.

Then for all $t \geq 0$, (I)

$$H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(t) <_{\mathrm{idcx}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(t) \,,$$

$$(H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(I) <_{\mathrm{idcx}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(I))\,.$$

Remark 3.3 (i) If in Proposition 3.1 we have $\mathbf{\Psi} = \tilde{\mathbf{\Psi}}$ then we can relax independence assumption and monotonicity of functions w.r.t. $\{u_n^i\}$. On the other hand, if $\mathbf{\Phi} = \tilde{\mathbf{\Phi}}$ then we can relax monotonicity of functions w.r.t. m.

(ii) If in Proposition 3.2 we have $\mathbf{\Psi} = \tilde{\mathbf{\Psi}}$ we can assume that functions are symmetric and supermodular instead of directionally convex and increasing w.r.t. $\{u_n^i\}$. On the other hand, if $\mathbf{\Phi} = \tilde{\mathbf{\Phi}}$ then we can relax independence assumptions, monotonicity w.r.t. m, extendability and symmetry of functions. In this case we can assume that $\mathbf{\Psi} <_{\mathrm{m-idcx-}\infty} \tilde{\mathbf{\Psi}}$ in order to obtain $H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(\mathbf{t}) <_{\mathrm{idcx}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(\mathbf{t})$ and $H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(\mathbf{I}) <_{\mathrm{idcx}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(\mathbf{I})$.

(iii) Assume in Proposition 3.2 that $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ are synchronized, i.e. $\Phi_i = \Phi$, $\tilde{\Phi}_i = \tilde{\Phi}$, $i = 1, \ldots, k$. Then we can relax independence assumptions by assuming that $\{U_n^i\}_{n\geq 1}$, $i = 1, \ldots, k$ or $\{\tilde{U}_n^i\}_{n\geq 1}$, $i = 1, \ldots, k$ are jointly stationary with $\mathbf{\Psi} <_{\mathrm{m-idcx-}\infty} \tilde{\mathbf{\Psi}}$ and $\Phi <_{\mathrm{cx-}N} \tilde{\Phi}$. Observe, however, that synchronized k-variate point process is not a *simple point process.*

(iv) Assume in Proposition 3.2 that $\mathbf{\Psi} = \tilde{\mathbf{\Psi}}$ and $\{U_n^i\}_{n\geq 1}$, $i = 1, \ldots, k$ are mutually independent renewal sequences. Then we can assume $\mathbf{\Phi} <_{\text{m-idcx-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ in order to obtain $H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(\mathbf{I}) <_{\text{idcx}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(\mathbf{I})$. As before, in the case of synchronization we do not need to assume that $\{U_n^i\}_{n\geq 1}$ is independent of $\{U_n^j\}_{n\geq 1}$, $j \neq i$.

(v) Obviously, we can assume $<_{\text{sm}}$-order in Proposition 3.2 instead of $<_{\text{idcx}}$. However, under stationarity assumptions, it is not possible to obtain

$$H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(t) <_{\text{sm}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(t)\,.$$

(vi) If in Proposition 3.2, $f_i^{(m)}(\{u_n^i\}) = \sum_{n=1}^m u_n^i$ then it is still valid with $<_{\text{idcx}}$ replaced by $<_{\text{dcx}}$.

(vii) In the special case, if $f_i^{(m)}$, $i = 1, \ldots, k$ have the form

$$f_i^{(m)}(\{u_n^i\}) = (h_i^{(m)}(u_1^i), \ldots, h_i^{(m)}(u_m^i))$$

for some functions $h_i^{(m)} : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, k$, $m \geq 1$, then we say that these functions are u-valued. For instance $f_i^{(m)}$ can be of the form

- $f_i^{(m)}(\{u_n^i\}) = u_m^i$
- $f_i^{(m)}(\{u_n^i\}) = (u_1^i, u_m^i)$

Because $<_{\text{sm}}$ is closed w.r.t. pointwise increasing transforms and $<_{\text{idcx}}$ is closed w.r.t. pointwise increasing convex transforms we have in Propositions 3.1 and 3.2 with $\mathbf{\Phi} = \tilde{\mathbf{\Phi}}$ that $\mathbf{\Psi} <_{\text{m-sm-}\infty} \tilde{\mathbf{\Psi}}$ ($\mathbf{\Psi} <_{\text{m-idcx-}\infty} \tilde{\mathbf{\Psi}}$) implies $H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(\mathbf{B}) <_{\text{sm}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(\mathbf{B})$ ($H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(\mathbf{B}) <_{\text{idcx}} H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Phi}})(\mathbf{B})$) for increasing (increasing and convex) functions $h_i^{(m)}$, where either $\mathbf{B} = \mathbf{t}$ or $\mathbf{B} = \mathbf{I}$.

4 Special cases

Recall firstly that $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ are stationary, $\mathbf{\Phi}$ is independent of $\mathbf{\Psi}$ and $\tilde{\mathbf{\Phi}}$ is independent of $\tilde{\mathbf{\Psi}}$. In this section we present some special cases of our results 3.1 and 3.2. However, if we compare $H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(B_1, \ldots, B_r)$ with $H_{\mathbf{f}}(\tilde{\mathbf{\Psi}}, \mathbf{\Phi})(B_1, \ldots, B_r)$ (or $H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(B)$ with $H_{\mathbf{f}}(\mathbf{\Psi}, \tilde{\mathbf{\Phi}})(B)$) we use foregoing remarks.

Example 4.1 (Comparison of multivariate sums and products) The following result is an easy consequence of Remark 3.3 (i) when $f_i^{(m)}(\{u_n^i\}) = \sum_{n=1}^m u_n^i$ or $f_i^{(m)}(\{u_n^i\}) = \prod_{n=1}^m (u_n^i)^d$. For $r = 1$ the result in (1) below was obtained in Denuit *et al.* [8].

Corollary 4.2 *Assume that for stationary $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ we have $\mathbf{\Phi} <_{\text{m-sm-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ ($\mathbf{\Phi} <_{\text{m-sm-D}} \tilde{\mathbf{\Phi}}$) and $\mathbf{\Psi} = \tilde{\mathbf{\Psi}}$.*

(i) *If* $\{U_n^i\}_{n\geq 1}$, $\{\tilde{U}_n^i\}_{n\geq 1}$, $i=1,\ldots,k$ *are sequences of nonnegative random variables then either for all* $B_j = I_j$ $(B_j=(0,t_j])$, $j=1,\ldots,r$, $r\geq 1$

$$\left(\sum_{n=1}^{N^i(B_1)} U_n^i,\ldots,\sum_{n=1}^{N^i(B_r)} U_n^i, i=1,\ldots,k\right) <_{\mathrm{sm}} \left(\sum_{n=1}^{\tilde{N}^i(B_1)} \tilde{U}_n^i,\ldots,\sum_{n=1}^{\tilde{N}^i(B_r)} \tilde{U}_n^i, i=1,\ldots,k\right). \quad (1)$$

(ii) *If* $\{U_n^i\}_{n\geq 1}$, $\{\tilde{U}_n^i\}_{n\geq 1}$, $i=1,\ldots,k$ *are sequences of random variables bounded below by 1, then for all* $d>0$ *and either for all* $B_j = I_j$ $(B_j=(0,t_j])$, $j=1,\ldots,r$, $r\geq 1$

$$\left(\prod_{n=1}^{N^i(B_1)} (U_n^i)^d,\ldots,\prod_{n=1}^{N^i(B_r)} (U_n^i)^d, i=1,\ldots,k\right) <_{\mathrm{sm}} \left(\prod_{n=1}^{\tilde{N}^i(B_1)} (\tilde{U}_n^i)^d,\ldots,\prod_{n=1}^{\tilde{N}^k(B_r)} (\tilde{U}_n^i)^d, i=1,\ldots,k\right).$$

Remark 4.3 All the results below can be formulated, as in Corollary 4.2, not only for $f_i^{(m)}(\{u_n^i\}) = \sum_{n=1}^m u_n^i$, but for $f_i^{(m)}(\{u_n^i\}) = \prod_{n=1}^m (u_n^i)^d$ as well. Moreover, Corollaries 4.2 and 4.4 can be formulated for example for $f_i^{(m)}(\{u_n^i\}) = h(\min\{u_1^i,\ldots,u_m^i\})$ and other functions (see Example 2.2).

In order to get in Corollary 4.2 more general comparison result we have to make some additional assumptions on $\mathbf{\Psi}$ and $\tilde{\mathbf{\Psi}}$, as in Proposition 3.1.

Corollary 4.4 *Assume that for stationary* $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ *we have* $\mathbf{\Phi} <_{\mathrm{m-sm}-\mathcal{N}} \tilde{\mathbf{\Phi}}$ $(\mathbf{\Phi} <_{\mathrm{m-sm-D}} \tilde{\mathbf{\Phi}})$ *and* $\mathbf{\Psi} <_{\mathrm{v-sm}-\infty} \tilde{\mathbf{\Psi}}$.
If $\{(U_n^1,\ldots,U_n^k)\}_{n\geq 1}$ *and* $\{(\tilde{U}_n^1,\ldots,\tilde{U}_n^k)\}_{n\geq 1}$ *are sequences of independent nonnegative random variables then either for all* $B_j = I_j$ *or* $B_j=(0,t_j]$, $j=1,\ldots,r$, $r\geq 1$

$$\left(\sum_{n=1}^{N^i(B_1)} U_n^i,\ldots,\sum_{n=1}^{N^i(B_r)} U_n^i, i=1,\ldots,k\right) <_{\mathrm{sm}} \left(\sum_{n=1}^{\tilde{N}^i(B_1)} \tilde{U}_n^i,\ldots,\sum_{n=1}^{\tilde{N}^i(B_r)} \tilde{U}_n^i, i=1,\ldots,k\right). \quad (2)$$

The above result (with $B_j=(0,t_j]$) was also obtained in Li and Xu [18]. Note, however, that we do not require boundness of supermodular functions. On the other hand, it follows from Müller and Stoyan [26] that it is sufficient to consider bounded supermodular functions in order to obtain $<_{\mathrm{sm}}$-order.

Because the function $f_i^{(m)}(\{u_n^i\}) = \sum_{n=1}^m u_n^i$ fulfills conditions of Proposition 3.2 and bearing in mind Remark 3.3 (vi) we have the following result.

Corollary 4.5 *Assume that* $\mathbf{\Psi} <_{\mathrm{h-dcx-}\infty} \tilde{\mathbf{\Psi}}$*,* $\mathbf{\Phi} <_{\mathrm{v-dcx-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ *and for all* $i = 1, \ldots, k$*,* $\{U_n^i\}_{n\geq 1}$ $(\{\tilde{U}_n^i\}_{n\geq 1})$ *is a stationary sequence independent of* $\{U_n^j\}_{n\geq 1}$ $(\{\tilde{U}_n^j\}_{n\geq 1})$*,* $j \neq i$*.*
If all random variables are nonnegative, then for all $t \geq 0$

$$\left(\sum_{n=1}^{N^1(t)} U_n^1, \ldots, \sum_{n=1}^{N^k(t)} U_n^k\right) <_{\mathrm{dcx}} \left(\sum_{n=1}^{\tilde{N}^1(t)} \tilde{U}_n^1, \ldots, \sum_{n=1}^{\tilde{N}^k(t)} \tilde{U}_n^k\right).$$

Bearing in mind the Remark 3.3 (iii) we have the next corollary.

Corollary 4.6 *Assume that* $\mathbf{\Phi}$ *and* $\tilde{\mathbf{\Phi}}$ *are synchronized point processes such that* $\Phi <_{\mathrm{v-cx-}N} \tilde{\Phi}$*. If* $\{U_n^i\}_{n\geq 1}$*,* $i = 1, \ldots, k$ *or* $\{\tilde{U}_n^i\}_{n\geq 1}$*,* $i = 1, \ldots, k$ *are jointly stationary and* $\mathbf{\Psi} <_{\mathrm{m-dcx-}\infty} \tilde{\mathbf{\Psi}}$ *then for all* $t \geq 0$

$$\left(\sum_{n=1}^{N(t)} U_n^1, \ldots, \sum_{n=1}^{N(t)} U_n^k\right) <_{\mathrm{dcx}} \left(\sum_{n=1}^{\tilde{N}(t)} \tilde{U}_n^1, \ldots, \sum_{n=1}^{\tilde{N}(t)} \tilde{U}_n^k\right).$$

The following result is not a direct corollary from Propositions 3.1 and 3.2. However, it will be useful in our applications (Section 7) and require a similar method of the proof.

Define for $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_r < b_r$ such that $b_j - a_j = b_l - a_l$, $l \neq j$, the intervals $I_j = (a_j, b_j]$, $j = 1, \ldots, r$.

Proposition 4.7 *Assume that one of the following holds.*

(i) $\mathbf{\Phi} <_{\mathrm{m-idcx-}\mathcal{N}} \tilde{\mathbf{\Phi}}$*,* $\mathbf{\Psi} = \tilde{\mathbf{\Psi}}$ *and* $\mathbf{\Psi}$ *consists of mutually independent iid nonnegative sequences,*
or

(ii) $\mathbf{\Psi} <_{\mathrm{m-idcx-}\infty} \tilde{\mathbf{\Psi}}$ *and* $\mathbf{\Phi} = \tilde{\mathbf{\Phi}}$*.*

Then for all $r \geq 1$

$$\left(\sum_{n=N^i(a_1)+1}^{N^i(b_1)} U_n^i, \ldots, \sum_{n=N^i(a_r)+1}^{N^i(b_r)} U_n^i, i = 1, \ldots, k\right) <_{\mathrm{idcx}} \left(\sum_{n=\tilde{N}^i(a_1)+1}^{\tilde{N}^i(b_1)} \tilde{U}_n^i, \ldots, \sum_{n=\tilde{N}^i(a_r)+1}^{\tilde{N}^i(b_r)} \tilde{U}_n^i, i = 1, \ldots, k\right).$$

Example 4.8 (Thinning of point processes) Our main results used with $f_i^{(m)}(\{u_n^i\}) = \sum_{n=1}^m u_n^i$ can be applied to compare thinned point processes. Assume that $\{U_n^i\}_{n\geq 1}$, $i = 1, \ldots, k$ are stationary $0-1$ valued sequences of random variables such that $\{U_n^i\}_{n\geq 1}$ is independent of $\{U_n^j\}_{n\geq 1}$ for all $i \neq j$. Note that this sequence can be seen as a realization of a discrete time k-variate point process. However, the results for random sums can be applied. Thinning of a point process $\mathbf{\Phi}$ with counting measures N^i, $i = 1, \dots, k$ is a point process $\mathbf{\Phi}_*$ which counting measures $N^i_{\mathbf{\Phi}_*}$, $i = 1, \dots, k$ can be represented for all Borel sets B as

$$(N^1_{\mathbf{\Phi}_*}(B), \dots, N^k_{\mathbf{\Phi}_*}(B)) = \left(\sum_{n=1}^{N^1(B)} U^1_n, \dots, \sum_{n=1}^{N^k(B)} U^k_n \right).$$

In the same way define a point process $\tilde{\mathbf{\Phi}}_*$. From Propositions 3.1, 3.2 and Remark 3.3 (vi) we have the following result.

Proposition 4.9 (i) *Assume that* $\{U^i_n\}_{n\geq 1}$, $i = 1, \dots, k$ *is independent of* $\{U^j_n\}_{n\geq 1}$ *for all* $i \neq j$. *If* $\mathbf{\Phi} <_{\text{v-dcx-D}} \tilde{\mathbf{\Phi}}$ $(\mathbf{\Phi} <_{\text{v-dcx-}\mathcal{N}} \tilde{\mathbf{\Phi}})$ *and* $\mathbf{\Psi} <_{\text{h-dcx-}\infty} \tilde{\mathbf{\Psi}}$ *then* $\mathbf{\Phi}_* <_{\text{v-dcx-D}} \tilde{\mathbf{\Phi}}_*$ $(\mathbf{\Phi}_* <_{\text{v-dcx-}\mathcal{N}} \tilde{\mathbf{\Phi}}_*)$.

(ii) *Assume that* $\{(U^1_n, \dots, U^k_n)\}_{n\geq 1}$, $\{(\tilde{U}^1_n, \dots, \tilde{U}^k_n)\}_{n\geq 1}$ *are sequences of independent random vectors. If* $\mathbf{\Phi} <_{\text{m-sm-D}} \tilde{\mathbf{\Phi}}$ $(\mathbf{\Phi} <_{\text{m-sm-}\mathcal{N}} \tilde{\mathbf{\Phi}})$ *and* $\mathbf{\Psi} <_{\text{v-sm-}\infty} \tilde{\mathbf{\Psi}}$ *then* $\mathbf{\Phi}_* <_{\text{m-sm-D}} \tilde{\mathbf{\Phi}}_*$ $(\mathbf{\Phi}_* <_{\text{m-sm-}\mathcal{N}} \tilde{\mathbf{\Phi}}_*)$.

The Remark 3.3 (iii) can be applied to compare markings of 1-variate point processes. Precisely, let $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ be 1-variate point processes with counting measures N and $\tilde{N}$, respectively. Consider a stationary sequence $\{V_n\}_{n\geq 1}$ of random variables with values in $\{1, \dots, k\}$. Define $U^i_n = \mathbb{I}(V_n = i)$, $n \geq 1$, $i = 1, \dots, k$. Then $\{U^1_n\}, \dots, \{U^k_n\}$ are jointly stationary. Define k-variate point processes $\mathbf{\Phi}^*$, $\tilde{\mathbf{\Phi}}^*$ by their counting measures $N^i_{\mathbf{\Phi}^*}$, $N^i_{\tilde{\mathbf{\Phi}}^*}$, $i = 1, \dots, k$ in the following way: $N^i_{\mathbf{\Phi}^*}(B) = \sum_{n=1}^{N(B)} U^i_n$, $i = 1, \dots, k$ and $N^i_{\tilde{\mathbf{\Phi}}^*}(B) = \sum_{n=1}^{\tilde{N}(B)} U^i_n$, $i = 1, \dots, k$. If $\mathbf{\Phi} <_{\text{cx-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ then $\mathbf{\Phi}^* <_{\text{v-dcx-}\mathcal{N}} \tilde{\mathbf{\Phi}}^*$.

The above results show how to increase (or hold) dependence and variability in arrival processes. Either multivariate arrivals are the same and, after suitable thinning, they can be compared in $<_{\text{sm}}$ or $<_{\text{idcx}}$ or 1-variate point processes are ordered and after the same marking k-variate point processes are ordered as well. Proposition 4.9 shows also that ordered arrivals are, after suitable thinning, ordered as well.

Example 4.10 (Comparison of multivariate arrival processes) The models in the previous example can be rewritten for multivariate batch arrival processes. Precisely, let $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ be point processes representing arrivals of the batches and $\{(U^1_n, \dots, U^k_n)\}_{n\geq 1}$, $\{(\tilde{U}^1_n, \dots, \tilde{U}^k_n)\}_{n\geq 1}$ be sequences representing the size of the batches, i.e. U^i_n is the size of the nth batch in queue i, $n \geq 1$, $i = 1, \dots, k$. Then we define batch arrival processes $\mathbf{\Phi}^{\text{`}}$, $\tilde{\mathbf{\Phi}}^{\text{`}}$ by their counting measures $N^i_{\mathbf{\Phi}^{\text{`}}}$, $N^i_{\tilde{\mathbf{\Phi}}^{\text{`}}}$, $i = 1, \dots, k$ in the following way: $N^i_{\mathbf{\Phi}^{\text{`}}}(B) = \sum_{n=1}^{N^i(B)} U^i_n$, $i = 1, \dots, k$ and $N^i_{\tilde{\mathbf{\Phi}}^{\text{`}}}(B) = \sum_{n=1}^{\tilde{N}^i(B)} \tilde{U}^i_n$, $i = 1, \dots, k$. The similar model was considered in Li and Xu [17]. Using the similar comment as in the previous example, dependence and variability in batch arrival process can be increased by dependence in batches or by dependence in non-batch arrival process.

5 Proofs of the main results

In this section we prove the main results. The proofs consist mainly of some technical lemmas. Subsequently, some regularity properties are easy consequence of these lemmas.

The proof of the first lemma follows directly from the definition of supermodular functions (see for instance Denuit *et al.* [8] where functions $f_i^{(n)}$ are given in the special form).

Lemma 5.1 *Let* $\{U_n^i\}_{n\geq 1}$, $i=1,\ldots,k$ *be sequences of random variables and* $\{f_i^{(m)}: \mathbb{R}^m \to \mathbb{R}^{d_i}\}_{m\geq 1}$, $1\leq d_i<\infty$, $i=1,\ldots,k$ *be increasing (decreasing) w.r.t. m. Then for all functions* $\varphi\in\mathcal{L}_{\mathrm{sm}}$

$$\begin{aligned}\psi((n_{1,1},\ldots,n_{1,r_1}),\ldots,(n_{k,1},\ldots,n_{k,r_k})) &= \\ \mathbb{E}\left[\varphi\left((f_1^{(n_{1,1})}(\{U_n^1\}),\ldots,f_1^{(n_{1,r_1})}(\{U_n^1\})),\ldots,(f_k^{(n_{k,1})}(\{U_n^k\}),\ldots,f_k^{(n_{k,r_k})}(\{U_n^k\}))\right)\right]\end{aligned}$$

is supermodular on $\mathbb{N}^{r_1+\cdots+r_k}$.

Recall now that $\mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$ is the class of all extendable functions on $\bar{\mathcal{J}}$ (see Definition 2.1) and $\mathcal{K}_m := \mathrm{supp}(U_1,\ldots,U_m)\subseteq\mathcal{J}_m$ is the support of $(U_1,\ldots,U_m)$.

Lemma 5.2 *For a fixed c, let* $\{U_n\}_{n\geq 1}$ *be a stationary sequence such that* $\sup_n U_n\leq c$ *or* $\inf_n U_n\geq c$. *If* $\{f^{(m)}:\mathbb{R}^m\to\mathbb{R}\}_{m\geq 1}\in\mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$ *and for all m,* $f^{(m)}\in\mathcal{L}_{\mathrm{ssm}}(\mathcal{J}_m)$, *then*

$$\phi(m)=\mathbb{E}[f^{(m)}(U_1,\ldots,U_m)]$$

is convex on $\mathbb{N}$.

Proof.

$$\begin{aligned}&\phi(m+1)+\phi(m-1)-2\phi(m)\\ &= \mathbb{E}[f^{(m+1)}(U_1,\ldots,U_{m+1})]+\mathbb{E}[f^{(m-1)}(U_1,\ldots,U_{m-1})]-2\mathbb{E}[f^{(m)}(U_1,\ldots,U_m)]\\ &= \mathbb{E}[f^{(m+1)}(U_1,\ldots,U_{m+1})]+\mathbb{E}[f^{(m+1)}(c,U_1,\ldots,U_{m-1},c)]\\ &\quad -\mathbb{E}[f^{(m+1)}(c,U_1,\ldots,U_m)]-\mathbb{E}[f^{(m+1)}(U_1,\ldots,U_m,c)]\\ &= \mathbb{E}[f^{(m+1)}(U_1,\ldots,U_{m+1})]+\mathbb{E}[f^{(m+1)}(c,U_2,\ldots,U_m,c)]\\ &\quad -\mathbb{E}[f^{(m+1)}(c,U_2,\ldots,U_{m+1})]-\mathbb{E}[f^{(m+1)}(U_1,\ldots,U_m,c)]\\ &\geq 0\,.\end{aligned}$$

In the second equation we used extendability and symmetry property, whereas in the third we applied stationarity. Inequality follows from the fact that $f^{(m+1)}$ is supermodular. Indeed, if $\sup_n U_n\leq c$ then

$$(U_1,\ldots,U_{m+1})=(\min\{c,U_1\},U_2,\ldots,U_m,\min\{c,U_{m+1}\})$$

and

$$(c,U_2,\ldots,U_m,c)=(\max\{c,U_1\},U_2,\ldots,U_m,\max\{c,U_{m+1}\})\,.$$

Analogously, in the case $\inf_n U_n\geq c$ we have to interchange min with max in the above expressions.

□

Combining ideas of Lemmas 5.1 and 5.2 we obtain the next result.

Lemma 5.3 *For a fixed c, let $\{U_n^i\}_{n\geq 1}$, $i = 1,\dots,k$ be stationary sequences such that for all $i \neq j$, $\{U_n^i\}_{n\geq 1}$ is independent of $\{U_n^j\}_{n\geq 1}$ and $\sup_{n,i} U_n^i \leq c$ or $\inf_{n,i} U_n^i \geq c$. If*

(i) $\{f_i^{(m)} : \mathbb{R}^m \to \mathbb{R}\}_{m\geq 1} \in \mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$, $i = 1,\dots,k$ *are monotone (increasing or decreasing) w.r.t m and increasing w.r.t. $\{u_n^i\}$*
and

(ii) *for all $i = 1,\dots,k$ and $m \geq 1$, $f_i^{(m)} \in \mathcal{L}_{\mathrm{ssm}}(\mathcal{J}_m)$,*

then for all functions $\varphi \in \mathcal{L}_{\mathrm{idcx}}$

$$\psi(n_1,\dots,n_k) = \mathbb{E}\left[\varphi\left(f_1^{(n_1)}(U_1^1,\dots,U_{n_1}^1),\dots,f_k^{(n_k)}(U_1^k,\dots,U_{n_k}^k)\right)\right]$$

is monotone (increasing or decreasing) and directionally convex on $\mathbb{N}^k$.

Proof. First, we can apply Lemma 5.1 in order to obtain that ψ is supermodular. We need only to show that ψ is convex w.r.t. n_i, $i = 1,\dots,k$. Let $\mathbf{U}_{n_i}^i = (U_1^i,\dots,U_{n_i}^i)$ and denote $\theta\mathbf{U}_{n_i}^i = (U_2^i,\dots,U_{n_i+1}^i)$. Then by independence assumption

$$\begin{aligned}
\psi(n_1+1,n_2,\dots,n_k) + \psi(n_1-1,n_2,\dots,n_k) - 2\psi(n_1,\dots,n_k) \quad =& \\
= \mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(\mathbf{U}_{n_1+1}^1), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
+\mathbb{E}\left[\varphi\left(f_1^{(n_1-1)}(\mathbf{U}_{n_1-1}^1), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
-2\mathbb{E}\left[\varphi\left(f_1^{(n_1)}(\mathbf{U}_{n_1}^1), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
= \mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(\mathbf{U}_{n_1+1}^1), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
+\mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(c,U_1^1,\dots,U_{n_1-1}^1,c), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
-\mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(c,U_1^1,\dots,U_{n_1}^1), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
-\mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(U_1^1,\dots,U_{n_1}^1,c), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
= \mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(\mathbf{U}_{n_1+1}^1), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
+\mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(c,U_2^1,\dots,U_{n_1}^1,c), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
-\mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(c,U_2^1,\dots,U_{n_1+1}^1), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& \\
-\mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(U_1^1,\dots,U_{n_1}^1,c), f_2^{(n_2)}(\mathbf{U}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{U}_{n_k}^k)\right)\right]& .
\end{aligned}$$

In the second equality we used extendability and symmetry properties of $f_i^{(n_i)}$, whereas in the third we used stationarity. Write the above equation in the form

$$\begin{aligned}
&\psi(n_1+1,n_2,\dots,n_k) + \psi(n_1-1,n_2,\dots,n_k) - 2\psi(n_1,\dots,n_k) = \\
&= \int \mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(\mathbf{U}_{n_1+1}^1), f_2^{(n_2)}(\mathbf{u}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{u}_{n_k}^k)\right)\right] d\mathbb{P}^{\mathbf{U}^1}(\mathbf{u}^1) \\
&+ \int \mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(c,U_2^1,\dots,U_{n_1}^1,c), f_2^{(n_2)}(\mathbf{u}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{u}_{n_k}^k)\right)\right] d\mathbb{P}^{\mathbf{U}^1}(\mathbf{u}^1) \\
&-\mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(c,U_2^1,\dots,U_{n_1+1}^1), f_2^{(n_2)}(\mathbf{u}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{u}_{n_k}^k)\right)\right] d\mathbb{P}^{\mathbf{U}^1}(\mathbf{u}^1) \\
&-\mathbb{E}\left[\varphi\left(f_1^{(n_1+1)}(U_1^1,\dots,U_{n_1}^1,c), f_2^{(n_2)}(\mathbf{u}_{n_2}^2),\dots,f_k^{(n_k)}(\mathbf{u}_{n_k}^k)\right)\right] d\mathbb{P}^{\mathbf{U}^1}(\mathbf{u}^1)
\end{aligned}$$

Here $\mathbb{P}^{\mathbf{U}^1}$ denotes the distribution of $(\mathbf{U}^2_{n_2}, \ldots, \mathbf{U}^k_{n_k})$ and $\mathbf{u}^1 \equiv (\mathbf{u}^2_{n_2}, \ldots, \mathbf{u}^k_{n_k})$.

Because for all $n \geq 1$, $f_1^{(n)}$ is supermodular and increasing w.r.t. $\{u^i_n\}$ and φ is increasing and convex w.r.t. the first coordinate we obtain that

$$\varphi\left(f_1^{(n_1+1)}(u^1_1, \ldots, u^1_{n_1+1}), f_2^{(n_2)}(\mathbf{u}^2_{n_2}), \ldots, f_k^{(n_k)}(\mathbf{u}^k_{n_k})\right)$$

is supermodular w.r.t. $(u^1_1, \ldots, u^1_{n_1+1})$. Therefore

$$\psi(n_1+1, n_2, \ldots, n_k) + \psi(n_1-1, n_2, \ldots, n_k) - 2\psi(n_1, \ldots, n_k) \geq 0$$

which ends the proof.

□

Using a similar technique as in Lemma 5.3 and observing that for $\varphi \in \mathcal{L}_{\text{idcx}}$ and $f_i^{(m)} : \mathbb{R}^m \to \mathbb{R} \in \mathcal{L}_{\text{idcx}}$, ψ defined by

$$\psi((u^1_1, \ldots, u^1_{n_1}), \ldots, (u^k_1, \ldots, u^k_{n_k})) = \varphi\left(f_1^{(n_1)}(u^1_1, \ldots, u^1_{n_1}), \ldots, f_k^{(n_k)}(u^k_1, \ldots, u^k_{n_k})\right)$$

is increasing and directionally convex (cf. Lemma 9.2) we obtain the following result.

Lemma 5.4 *For a fixed c, let $\{U^i_n\}_{n\geq 1}$, $i = 1, \ldots, k$ be jointly stationary sequences of random variables such that $\sup_{n,i} U^i_n \leq c$ or $\inf_{n,i} U^i_n \geq c$. If*

(i) $\{f_i^{(m)} : \mathbb{R}^m \to \mathbb{R}\}_{m\geq 1} \in \mathcal{E}_{\mathrm{c}}(\bar{\mathcal{J}})$, $i = 1, \ldots, k$, *and*

(ii) *for all $i = 1, \ldots, k$ and $m \geq 1$, $f_i^{(m)} \in \mathcal{L}_{\text{idcx}}(\mathcal{J}_m)$ are symmetric on $\mathcal{J}_m$ and increasing w.r.t. $\{u^i_n\}$,*

then for all functions $\varphi \in \mathcal{L}_{\text{idcx}}$

$$\psi(n) = \mathbb{E}\left[\varphi\left(f_1^{(n)}(U^1_1, \ldots, U^1_n), \ldots, f_k^{(n)}(U^k_1, \ldots, U^k_n)\right)\right]$$

is convex on $\mathbb{N}$.

Now, we establish comparison properties w.r.t. $\{U^1_n\}, \ldots, \{U^k_n\}$. The first result generalizes Theorem 2.7 in Li and Xu [17].

Lemma 5.5 *Assume that $\{(U^1_n, \ldots, U^k_n)\}_{n\geq 1}$, and $\{(\tilde{U}^1_n, \ldots, \tilde{U}^k_n)\}_{n\geq 1}$ are sequences of independent random vectors. If for all $n \geq 1$, $(U^1_n, \ldots, U^k_n) <_{\text{sm}} (\tilde{U}^1_n, \ldots, \tilde{U}^k_n)$ and $f_i^{(m)} : \mathbb{R}^m \to \mathbb{R}^{d_i}$, $m \geq 1$, $i = 1, \ldots, k$ are monotone in the same direction w.r.t. $\{u^i_n\}$ then for all $n_1, \ldots, n_k$,*

$$(f_1^{(n_1)}(\{U^1_n\}), \ldots, f_k^{(n_k)}(\{U^k_n\})) <_{\text{sm}} (f_1^{(n_1)}(\{\tilde{U}^1_n\}), \ldots, f_k^{(n_k)}(\{\tilde{U}^k_n\}))\,.$$

Proof. Without loss of generality we can assume that $\{(U^1_n, \ldots, U^k_n)\}_{n\geq 1}$ is independent of $\{(\tilde{U}^1_n, \ldots, \tilde{U}^k_n)\}_{n\geq 1}$. Clearly, the function ψ defined by

$$\psi((u^1_1, \ldots, u^1_{n_1}), \ldots, (u^k_1, \ldots, u^k_{n_k})) = \varphi\left(f_1^{(n_1)}(\{u^1_n\}), \ldots, f_k^{(n_k)}(\{u^k_n\})\right)$$

is supermodular as a function of $(u^1_{j_1}, \ldots, u^k_{j_k})$, $1 \leq j_i \leq n_i$, $i = 1, \ldots, k$, if $\varphi \in \mathcal{L}_{\text{sm}}$ and $f_i^{(n_i)}$ are monotone in the same direction w.r.t. $\{u^i_n\}$ (cf. Lemma

9.1). From our assumptions we have that $(U_1^1,\dots,U_1^k) <_{\rm sm} (\tilde U_1^1,\dots,\tilde U_1^k)$. Writing $f_i^{(n_i)}(\tilde u_1^i,\dots,\tilde u_l^i,\{u_{n_i}^i\})$ for $f_i^{(n_i)}(\tilde u_1^i,\dots,\tilde u_l^i,u_{l+1}^i,\dots,u_{n_i}^i)$, $l=0,\dots,n_i-1$, $i=1,\dots,k$ we have

$$\begin{aligned}
&\mathbb{E}[\varphi(f_1^{(n_1)}(\{U_n^1\}),\dots,f_k^{(n_k)}(\{U_n^k\})] = \\
&= \int \mathbb{E}[\varphi(f_1^{(n_1)}(U_1^1,\{u_n^1\}),\dots,f_k^{(n_k)}(U_1^k,\{u_n^k\}))]d\mathbb{P}^{\mathbf{U}_1}(\mathbf{u}_1)\\
&\le \int \mathbb{E}[\varphi(f_1^{(n_1)}(\tilde U_1^1,\{u_n^1\}),\dots,f_k^{(n_k)}(\tilde U_1^k,\{u_n^k\}))]d\mathbb{P}^{\mathbf{U}_1}(\mathbf{u}_1)\\
&= \mathbb{E}[\varphi(f_1^{(n_1)}(\tilde U_1^1,\{U_n^1\}),\dots,f_k^{(n_k)}(\tilde U_1^k,\{U_n^k\}))]\\
&= \int \mathbb{E}[\varphi(f^{(n_1)}(\tilde u_1^1,U_2^1,\{u_n^1\}),\dots,f^{(n_k)}(\tilde u_1^k,U_2^k,\{u_n^k\}))]d\mathbb{P}^{\mathbf{U}_2}(\mathbf{u}_2)\\
&\le \int \mathbb{E}[\varphi(f_1^{(n_1)}(\tilde u_1^1,\tilde U_2^1,\{u_n^1\}),\dots,f_k^{(n_k)}(\tilde u_1^k,\tilde U_2^k,\{u_n^k\}))]d\mathbb{P}^{\mathbf{U}_2}(\mathbf{u}_2)\,.
\end{aligned}$$

Here $\mathbb{P}^{\mathbf{U}_i}$ denotes the distribution of $(\{U_n^1\}_{n\neq i},\dots,\{U_n^k\}_{n\neq i})$ and

$$\mathbf{u}_1=(\{u_n^1\}_{n\ge 2},\dots,\{u_n^k\}_{n\ge 2})\,,$$

$$\mathbf{u}_2=(\tilde u_1^1,\{u_n^1\}_{n\ge 3},\dots,\tilde u_1^k,\{u_n^k\}_{n\ge 3})\,.$$

The second inequality follows from $(U_2^1,\dots,U_2^k) <_{\rm sm} (\tilde U_2^1,\dots,\tilde U_2^k)$. Now continuation of this operation completes the proof.

□

Assume now $n_i\ge \max\{n_{i,1},\dots,n_{i,r}\}$, $i=1,\dots,k$. As above, if φ is supermodular and $f_i^{(n_{i,j})}$ are monotone in the same direction then ψ defined as

$$\psi((u_1^1,\dots,u_{n_1}^1),\dots,(u_1^k,\dots,u_{n_k}^k))=\varphi\left(f_i^{(n_{i,1})}(\{u_n^i\}),\dots,f_i^{(n_{i,r})}(\{u_n^i\}),i=1,\dots,k\right)$$

is supermodular as a function of all vectors of the form $(u_{j_1}^1,\dots,u_{j_k}^k)$. Therefore, we have the following generalization of Lemma 5.5.

Lemma 5.6 *Under assumptions of Lemma 5.5 we have*

$$\begin{aligned}
(f_i^{(n_{i,1})}(\{U_n^i\}),\dots,f_i^{(n_{i,r})}(\{U_n^i\}),i=1,\dots,k) \quad &<_{\rm sm}\\
&(f_i^{(n_{i,1})}(\{\tilde U_n^i\}),\dots,f_i^{(n_{i,r})}(\{\tilde U_n^i\}),i=1,\dots,k)\,.
\end{aligned}$$

Since for $\varphi\in\mathcal{L}_{\rm idcx}$ and $f_i^{(m)}:\mathbb{R}^m\to\mathbb{R}\in\mathcal{L}_{\rm idcx}$ ψ defined as

$$\psi((u_1^1,\dots,u_{n_1}^1),\dots,(u_1^k,\dots,u_{n_k}^k))=\varphi\left(f_1^{(n_1)}(\{u_n^1\}),\dots,f_k^{(n_k)}(\{u_n^k\})\right)$$

is increasing and directionally convex (cf. Lemma 9.2) and using closure of $<_{\rm idcx}$ under marginalization we have the following result.

Lemma 5.7 *Assume that for all $n_i\ge 1$, $i=1,\dots,k$,*

$$((U_1^1,\dots,U_{n_1}^1),\dots,(U_1^k,\dots,U_{n_k}^k)) <_{\rm idcx} ((\tilde U_1^1,\dots,\tilde U_{n_1}^1),\dots,(\tilde U_1^k,\dots,\tilde U_{n_k}^k))\,.$$

(i) *If* $f_i^{(m)} : \mathbb{R}^m \to \mathbb{R} \in \mathcal{L}_{\rm idcx}$, $i = 1,\dots,k$, $m \geq 1$ *are increasing w.r.t.* $\{u_n^i\}$. *Then*

$$(f_1^{(n_1)}(U_1^1,\dots,U_{n_1}^1),\dots,f_k^{(n_k)}(U_1^k,\dots,U_{n_k}^k)) \quad <_{\rm idcx} \\ (f_1^{(n_1)}(\tilde{U}_1^1,\dots,\tilde{U}_{n_1}^1),\dots,f_k^{(n_k)}(\tilde{U}_1^k,\dots,\tilde{U}_{n_k}^k))\,.$$

(ii) *If* $f_i^{(m)} : \mathbb{R}^m \to \mathbb{R} \in \mathcal{L}_{\rm idcx}$, $i = 1,\dots,k$, $m \geq 1$ *are increasing w.r.t.* $\{u_n^i\}$. *Then*

$$(f_i^{(n_{i,1})}(U_1^i,\dots,U_{n_{i,1}}^i),\dots,f_i^{(n_{i,r}-n_{i,r-1})}(U_{n_{i,r-1}+1}^i,\dots,U_{n_{i,r}}^i), i = 1,\dots,k) <_{\rm idcx} \\ (f_i^{(n_{i,1})}(\tilde{U}_1^i,\dots,\tilde{U}_{n_{i,1}}^i),\dots,f_i^{(n_{i,r}-n_{i,r-1})}(\tilde{U}_{n_{i,r-1}+1}^i,\dots,\tilde{U}_{n_{i,r}}^i), i = 1,\dots,k)\,.$$

Now we are ready to prove our main results.
Proof of Proposition 3.1. Lemma 5.6 implies that for all $n_{i,r}$, $i = 1,\dots,k$, $r \geq 1$,

$$(f_i^{(n_{i,1})}(\{U_n^i\}),\dots,f_i^{(n_{i,r})}(\{U_n^i\}), i = 1,\dots,k) \quad <_{\rm sm} \\ (f_i^{(n_{i,1})}(\{\tilde{U}_n^i\}),\dots,f_i^{(n_{i,r})}(\{\tilde{U}_n^i\}), i = 1,\dots,k)\,.$$

Bearing in mind that $<_{\rm sm}$ is closed under mixture (cf. Lemma 9.4) we obtain that for sets $B_j = I_j$ ($B_j = (0,t_j]$), $j = 1,\dots,r$

$$(f_i^{(N^i(B_1))}(\{U_n^i\}),\dots,f_i^{(N^i(B_r))}(\{U_n^i\}), i = 1,\dots,k) \quad <_{\rm sm} \\ (f_i^{(N^i(B_1))}(\{\tilde{U}_n^i\}),\dots,f_i^{(N^i(B_r))}(\{\tilde{U}_n^i\}), i = 1,\dots,k)\,.$$

Now, using Lemma 5.1 and the assumption $\mathbf{\Phi} <_{\rm m-sm-\mathcal{N}} \tilde{\mathbf{\Phi}}$ ($\mathbf{\Phi} <_{\rm m-sm-D} \tilde{\mathbf{\Phi}}$) we obtain required result.

□

The proof of Proposition 3.2 is similar (we use Lemma 5.7 (i) and then Lemma 5.3). Remark 3.3 (ii) in case $\mathbf{\Psi} = \tilde{\mathbf{\Psi}}$ follows directly from Lemma 5.3, whereas the case $\mathbf{\Phi} = \tilde{\mathbf{\Phi}}$ from Lemma 5.7 (ii). For the Remark 3.3 (iii) we use Lemma 5.7 (i) and then Lemma 5.4.

□

6 Regularity properties

From technical Lemmas 5.1, 5.2, 5.3 and 5.4 we can easily get some regularity properties w.r.t. $(n_1,\dots,n_k)$.

Lemma 5.1, by taking sequences of functions $\{f_i^{(m)}(\{u_n^i\}) = \sum_{n=1}^m u_n^i \mathbb{I}(u_n^i \geq 0)\}_{m\geq 1}$ ($\{f_i^{(m)}(\{u_n^i\}) = \prod_{n=1}^m (u_n^i)^d \mathbb{I}(u_n^i \geq 1)\}_{m\geq 1}$, $\{f_i^{(m)}(\{u_n^i\}) = \min_{n=1,\dots,m} u_n^i\}_{m\geq 1}$), $i = 1,\dots,k$, implies the following results. The first one was obtained in Denuit *et al.* [8].

Corollary 6.1 (i) *Let $\{U_n^i\}_{n\geq 1}$, $i=1,\ldots,k$ be sequences of nonnegative random variables. Then for all functions $\varphi \in \mathcal{L}_{\text{sm}}$*

$$\psi(n_1,\ldots,n_k) = \mathbb{E}\left[\varphi\left(\sum_{n=1}^{n_1} U_n^1,\ldots,\sum_{n=1}^{n_k} U_n^k\right)\right]$$

is supermodular on $\mathbb{N}^k$.

(ii) *Let $\{U_n^i\}_{n\geq 1}$, $i=1,\ldots,k$ be sequences of random variables bounded below by 1. Then for all functions $\varphi \in \mathcal{L}_{\text{sm}}$*

$$\psi(n_1,\ldots,n_k) = \mathbb{E}\left[\varphi\left(\prod_{n=1}^{n_1} (U_n^1)^d,\ldots,\prod_{n=1}^{n_k} (U_n^k)^d\right)\right]$$

is supermodular on $\mathbb{N}^k$ for $d>0$.

(iii) *Let $\{U_n^i\}_{n\geq 1}$, $i=1,\ldots,k$ be sequences of random variables. Then for all functions $\varphi \in \mathcal{L}_{\text{sm}}$*

$$\psi(n_1,\ldots,n_k) = \mathbb{E}\left[\varphi\left(\min\{U_1^1,\ldots,U_{n_1}^1\},\ldots,\min\{U_1^k,\ldots,U_{n_k}^k\}\right)\right]$$

is supermodular on $\mathbb{N}^k$.

Observing that $\{f^{(m)}(\{u_n\}) = \psi(\prod_{n=1}^m u_n^d)\}_{m\geq 1}$, $\{f^{(m)}(\{u_n\}) = h(\min_{n=1,\ldots,m} u_n)\}_{m\geq 1}$ and $\{f^{(m)}(\{u_n\}) = \varphi(\sum_{n=1}^m u_n)\}_{m\geq 1}$, $\varphi \in \mathcal{L}_{\text{cx}}$, $\psi \in \mathcal{L}_{\text{icx}}$, $h \in \mathcal{L}_{\text{i}}$, $d \geq 0$ are extendable sequences of symmetric and supermodular functions and using Lemma 5.2 we have the next result.

Corollary 6.2 (i) *Let $\{U_n\}_{n\geq 1}$ be a stationary sequence of nonnegative random variables. Then for all convex functions φ*

$$\phi_1(m) = \mathbb{E}\left[\varphi\left(\sum_{n=1}^{m} U_n\right)\right]$$

is convex on $\mathbb{N}$.

(ii) *Let $\{U_n\}_{n\geq 1}$ be a stationary sequence of random variables bounded below by 1. Then for all increasing convex functions ψ and $d \geq 0$*

$$\phi_2(m) = \mathbb{E}\left[\psi\left(\prod_{n=1}^{m} (U_n)^d\right)\right]$$

is convex on $\mathbb{N}$.

(iii) *Let $\{U_n\}_{n\geq 1}$ be a stationary sequence of random variables bounded above by a constant c and h an increasing function. Then*

$$\phi_3(m) = \mathbb{E}\left[h(\min\{U_1,\ldots,U_m\})\right]$$

is convex on $\mathbb{N}$.

(iv) *Let $\{U_n\}_{n\geq 1}$ be a stationary sequence of random variables bounded below by a constant c and h a decreasing function. Then*

$$\phi_4(m) = \mathbb{E}\left[h(\max\{U_1, \ldots, U_m\})\right]$$

is convex on $\mathbb{N}$.

The convexity of ϕ_1 was proved in Ross [30, p. 278], in the case of $\{U_n\}_{n\geq 1}$ iid nonnegative random variables. Jean-Marie and Liu [11] showed it in the case of $\{U_n\}_{n\geq 1}$ - nonstationary sequences of independent nonnegative random variables such that $\mathbb{E}[\varphi(U_n)] \leq \mathbb{E}[\varphi(U_{n+1})]$ for all $\varphi \in \mathcal{L}_{\mathrm{cx}}$ ($\varphi \in \mathcal{L}_{\mathrm{icx}}$). Makowski and Phillips [20] showed that for iid nonnegative random variables $\{U_n\}_{n\geq 1}$, the function $\tilde{\phi}_1(m) = \psi_1(m)/m$ is increasing.

From Lemma 5.3 we obtain the following result.

Corollary 6.3 *Let $\{U_n^i\}_{n\geq 1}$, $i = 1, \ldots, k$ be stationary sequences such that for all $i \neq j$, $\{U_n^i\}_{n\geq 1}$ is independent of $\{U_n^j\}_{n\geq 1}$.*

(i) *If $\{U_n^i\}_{n\geq 1}$, $i = 1, \ldots, k$ are sequences of nonnegative random variables then for all functions $\varphi \in \mathcal{L}_{\mathrm{idcx}}$*

$$\psi(n_1, \ldots, n_k) = \mathbb{E}\left[\varphi\left(\sum_{n=1}^{n_1} U_n^1, \ldots, \sum_{n=1}^{n_k} U_n^k\right)\right]$$

is increasing and directionally convex on $\mathbb{N}^k$.

(ii) *If $\{U_n^i\}_{n\geq 1}$, $i = 1, \ldots, k$ sequences of random variables bounded below by 1. Then for all functions $\varphi \in \mathcal{L}_{\mathrm{idcx}}$ and $d > 0$*

$$\psi(n_1, \ldots, n_k) = \mathbb{E}\left[\varphi\left(\prod_{n=1}^{n_1} (U_n^1)^d, \ldots, \prod_{n=1}^{n_k} (U_n^k)^d\right)\right]$$

is increasing and directionally convex on $\mathbb{N}^k$.

The first result was obtained in Rolski [28] in the case of iid mutually independent sequences of nonnegative random variables.

7 Applied examples

In this section we present some applications of Propositions 3.1 and 3.2 to stochastic models.

Example 7.1 (Workload in parallel queues) Consider a queueing system of k parallel $G/G/1$ FIFO queues. The input is generated by k-variate point processes $\mathbf{\Phi}$ (interarrival times) and $\mathbf{\Psi}$ (service times), independent of $\mathbf{\Phi}$. For $t \geq 0$ and $I = (a, b]$ define

$$M^i(t) = \sum_{n=1}^{N^i(t)} U_n^i, \quad i = 1, \ldots, k$$

and

$$M^i(I) = \sum_{n=N^i(a)+1}^{N^i(b)} U_n^i, \quad i = 1, \dots, k\,.$$

Call M^i, $i = 1, \dots, k$ cumulative processes. Denote by

$$\mathbf{V}(t) \equiv (V^1(t), \dots, V^k(t))$$

the vector of transient workloads, which is known to fulfill

$$V^i(t) = \max_{0 \le u \le t} (0, M^i(t) - M^i(u) - (t-u))$$

(Borovkov [6, p. 23]). Similarly, for k-variate point processes $\tilde{\mathbf{\Phi}}$, $\tilde{\mathbf{\Psi}}$ define

$$\tilde{M}^i(t) = \sum_{n=1}^{\tilde{N}_i(t)} \tilde{U}_n^i, \quad i = 1, \dots, k$$

and as above $\tilde{M}^i(I)$ and $\tilde{\mathbf{V}}(t)$.

Using a similar argument as in Meester and Shanthikumar [22] we obtain the following Lemma.

Lemma 7.2 *Assume that for all $r \ge 1$,*

$$(M^i(I_1), \dots, M^i(I_r), i = 1, \dots, k) <_{\mathrm{idcx}} (\tilde{M}^i(I_1), \dots, \tilde{M}^i(I_r), i = 1, \dots, k)\,. \tag{3}$$

If for every $t \ge 0$, $\mathbb{P}(M^i(t) < \infty) = \tilde{\mathbb{P}}(\tilde{M}^i(t) < \infty) = 1$, $i = 1, \dots, k$ and for all $t > 0$, $\mathbb{P}(M^i$ is discontinuous at $t) = \tilde{\mathbb{P}}(\tilde{M}^i$ is discontinuous at $t) = 0$ then for all $0 < t_1 < \dots < t_r$,

$$(\mathbf{V}(t_1), \dots, \mathbf{V}(t_r)) <_{\mathrm{idcx}} (\tilde{\mathbf{V}}(t_1), \dots, \tilde{\mathbf{V}}(t_r))\,.$$

Proposition 7.3 (i) *Assume that $\mathbf{\Phi} <_{\mathrm{m-idcx-}\mathcal{N}} \tilde{\mathbf{\Phi}}$, $\mathbf{\Psi} = \tilde{\mathbf{\Psi}}$ and $\mathbf{\Psi}$ consists of mutually independent iid sequences. Then for all $0 < t_1 < \dots < t_r$,*

$$(\mathbf{V}(t_1), \dots, \mathbf{V}(t_r)) <_{\mathrm{idcx}} (\tilde{\mathbf{V}}(t_1), \dots, \tilde{\mathbf{V}}(t_r))\,.$$

(ii) *Assume that $\mathbf{\Psi} <_{\mathrm{m-idcx-}\infty} \tilde{\mathbf{\Psi}}$, $\mathbf{\Phi} = \tilde{\mathbf{\Phi}}$. Then for all $0 < t_1 < \dots < t_r$,*

$$(\mathbf{V}(t_1), \dots, \mathbf{V}(t_r)) <_{\mathrm{idcx}} (\tilde{\mathbf{V}}(t_1), \dots, \tilde{\mathbf{V}}(t_r))\,.$$

Proof. In both cases we have from Proposition 4.7 that for all $r \ge 1$ and disjoint intervals $I_1, \dots, I_r$ of equal lengths,

$$\begin{aligned}(M^i(I_1), \dots, M^i(I_r), i = 1, \dots, k) \quad &<_{\mathrm{idcx}} \\ &(\tilde{M}^i(I_1), \dots, \tilde{M}^i(I_r), i = 1, \dots, k)\end{aligned}$$

which means that (3) holds. Now, the result follows from Lemma 7.2.

□

Example 7.4 (Workload in batch queues) Consider a queueing system of k parallel $G/GI/1$ FIFO queues. The input is generated by k-variate point processes $\mathbf{\Phi}$ (arrival times) and $\mathbf{\Psi}$ (batch sizes), independent of $\mathbf{\Phi}$. For $t \geq 0$ and $I = (a, b]$ define

$$K^i(t) = \sum_{n=1}^{N^i(t)} U_n^i, \quad i = 1, \ldots, k\,,$$

and

$$K^i(I) = \sum_{n=N^i(a)+1}^{N^i(b)} U_n^i, \quad i = 1, \ldots, k\,.$$

Here, $K^i(t)$ represents the number of jobs brought to a queue i up to time t. For $\{S_n^i\}_{n\geq 1}$, $i = 1, \ldots, k$, iid mutually independent service times, independent of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ define cumulative processes

$$M^i(t) = \sum_{n=1}^{K^i(t)} S_n^i, \quad i = 1, \ldots, k\,,$$

and

$$M^i(I) = \sum_{n=K^i(a)+1}^{K^i(b)} S_n^i, \quad i = 1, \ldots, k\,.$$

Then the transient workload is given by

$$V^i(t) = \max_{0\leq u\leq t}(0, M^i(t) - M^i(u) - (t-u))\,.$$

Denote by

$$\mathbf{V}(t) \equiv (V^1(t), \ldots, V^k(t))$$

the vector of transient workload. Similarly, having arrival process $\tilde{\mathbf{\Phi}} = \mathbf{\Phi}$, batch size process $\tilde{\mathbf{\Psi}}$ and the same service times, we define $\tilde{K}^i(t)$, $\tilde{K}^i(I)$, $\tilde{M}^i(t)$, $\tilde{M}^i(I)$, $\tilde{V}^i(t)$ and $\tilde{\mathbf{V}}(t)$.

Proposition 7.5 *Assume that* $\{(U_n^1, \ldots, U_n^k)\}_{n\geq 1}$, $\{(\tilde{U}_n^1, \ldots, \tilde{U}_n^k)\}_{n\geq 1}$ *are sequences of independent random variables such that for all* $n \geq 1$, $(U_n^1, \ldots, U_n^k) <_{\mathrm{sm}} (\tilde{U}_n^1, \ldots, \tilde{U}_n^k)$. *Then for all* $0 < t_1 < \cdots < t_r$,

$$(\mathbf{V}(t_1), \ldots, \mathbf{V}(t_r)) <_{\mathrm{idcx}} (\tilde{\mathbf{V}}(t_1), \ldots, \tilde{\mathbf{V}}(t_r))\,.$$

Proof. Note that from assumption we have (cf. Lemma 5.5 with functions $f_i^{(n_i)}(\{u_n^i\}) = (u_1^i, \ldots, u_{n_i}^i)$, $i = 1, \ldots, k$) $\mathbf{\Psi} <_{\mathrm{m-sm-\infty}} \tilde{\mathbf{\Psi}}$ and hence $\mathbf{\Psi} <_{\mathrm{m-idcx-\infty}} \tilde{\mathbf{\Psi}}$. From Proposition 4.7 we have

$$(K^i(I_1), \ldots, K^i(I_r), i = 1, \ldots, k) <_{\mathrm{idcx}} (\tilde{K}^i(I_1), \ldots, \tilde{K}^i(I_r), i = 1, \ldots, k)\,.$$

Replacing in Proposition 4.7 N by K and U by S we obtain

$$(M^i(I_1), \ldots, M^i(I_r), i = 1, \ldots, k) <_{\mathrm{idcx}} (\tilde{M}^i(I_1), \ldots, \tilde{M}^i(I_r), i = 1, \ldots, k)\,.$$

The conclusion follows from Lemma 7.2.

□

Example 7.6 (Workload in synchronized queues) Let $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ be k-variate arrival processes with interarrival times X_n^i, $\tilde{X}_n^i$, $i = 1, \ldots, k$. If $\{X_n^1, \ldots, X_n^k\}_{n\geq 1}$ and $\{\tilde{X}_n^1, \ldots, \tilde{X}_n^k\}_{n\geq 1}$ are sequences of independent random vectors and for all $n \geq 1$, $(X_n^1, \ldots, X_n^k) <_{\mathrm{sm}} (\tilde{X}_n^1, \ldots, \tilde{X}_n^k)$, (i.e. $\mathbf{\Phi} <_{\mathrm{v-sm-}\infty} \tilde{\mathbf{\Phi}}$) then $\mathbf{\Phi} <_{\mathrm{m-sm-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ (Li and Xu [17]). Assume that $X_n \stackrel{\mathrm{d}}{=} X_n^i \stackrel{\mathrm{d}}{=} X_n^j$, $i, j = 1, \ldots, k$, $n \geq 1$. From Lorentz inequality (cf. Lemma 9.5) we obtain that $(X_n^1, \ldots, X_n^k) <_{\mathrm{sm}} (X_n, \ldots, X_n)$. Therefore, synchronization give the upper bound (in $<_{\mathrm{sm}}$ and hence in $<_{\mathrm{idcx}}$-order) for arrival processes and hence, using previous results, for workload in parallel queues.

Example 7.7 (Multivariate shock models) The results for random sums can be used to compare multivariate shock models. Precisely, consider two multicomponent systems in which k components of each system are subject to shocks. Let $\mathbf{M} = (M^1, \ldots, M^k)$ be a vector of random number of shocks until failure of the components. Interarrival between shocks are described by k-variate point process $\mathbf{\Psi}$. Since the vector of lifetimes $\mathbf{Z} = (Z_1, \ldots, Z_k)$ is defined by

$$\mathbf{Z} = \left(\sum_{n=1}^{M^1} U_n^1, \ldots, \sum_{n=1}^{M^k} U_n^k \right),$$

we have the following result (cf. Corollaries 4.2, 4.4, 4.5, 4.6).

Proposition 7.8 (i) *If $\mathbf{M} <_{\mathrm{sm}} \tilde{\mathbf{M}}$ and $\mathbf{\Psi} \stackrel{\mathrm{d}}{=} \tilde{\mathbf{\Psi}}$ then $\mathbf{Z} <_{\mathrm{sm}} \tilde{\mathbf{Z}}$.*

(ii) *If $\mathbf{M} <_{\mathrm{sm}} \tilde{\mathbf{M}}$ and $\{(U_n^1, \ldots, U_n^k)\}_{n\geq 1}$, $\{(\tilde{U}_n^1, \ldots, \tilde{U}_n^k)\}_{n\geq 1}$ are sequences of independent random variables such that $\mathbf{\Psi} <_{\mathrm{v-sm-}\infty} \tilde{\mathbf{\Psi}}$ then $\mathbf{Z} <_{\mathrm{sm}} \tilde{\mathbf{Z}}$.*

(iii) *If $\mathbf{M} <_{\mathrm{dcx}} \tilde{\mathbf{M}}$, $\mathbf{\Psi} <_{\mathrm{h-dcx-}\infty} \tilde{\mathbf{\Psi}}$ and for all $i = 1, \ldots, k$, $\{U_n^i\}_{n\geq 1}$ ($\{\tilde{U}_n^i\}_{n\geq 1}$) is a stationary sequence independent of $\{U_n^j\}_{n\geq 1}$ ($\{\tilde{U}_n^j\}_{n\geq 1}$), $j \neq i$ then $\mathbf{Z} <_{\mathrm{dcx}} \tilde{\mathbf{Z}}$.*

(iv) *If $\mathbf{M} = (M^0, \ldots, M^0)$, $\tilde{\mathbf{M}} = (\tilde{M}^0, \ldots, \tilde{M}^0)$ and $M^0 <_{\mathrm{cx}} \tilde{M}^0$ then for jointly stationary sequences $\{U_n^i\}_{n\geq 1}$, $\{\tilde{U}_n^i\}_{n\geq 1}$ we have $\mathbf{Z} <_{\mathrm{dcx}} \tilde{\mathbf{Z}}$.*

Pellerey [27] considered sequences of nonnegative random $\{U_n^i\}_{n\geq 1}$ such that:

(i) $\{U_n^i\}_{n\geq 1}$ are sequences of independent random variables, $i = 1, \ldots, k$;

(ii) $U_n^i <_{\mathrm{a}} U_{n+1}^i$ for all $n \geq 1$, $i = 1, \ldots, k$;

(iii) $\{U_n^i\}_{n\geq 1}$ is independent of $\{U_n^j\}_{n\geq 1}$, $i \neq j$.

It was stated that $\mathbf{\Psi} = \tilde{\mathbf{\Psi}}$ and $\mathbf{M} <_{\mathrm{a}} \tilde{\mathbf{M}}$ implies $\mathbf{Z} <_{\mathrm{b}} \tilde{\mathbf{Z}}$, where $<_{\mathrm{a}}$ and $<_{\mathrm{b}}$ are the following pairs of orderings: $<_{\mathrm{icx}}$ and $<_{\mathrm{icx}}$, $<_{\mathrm{cx}}$ and $<_{\mathrm{ccx}}$, $<_{\mathrm{icx}}$ and $<_{\mathrm{iccx}}$, $<_{\mathrm{cx}}$ and $<_{\mathrm{symcx}}$ or $<_{\mathrm{icx}}$ and $<_{\mathrm{symcx}}$, where $<_{\mathrm{ccx}}$, $<_{\mathrm{iccx}}$, $<_{\mathrm{symcx}}$ are coordinatewise convex, increasing coordinatewise convex and symmetric convex orderings, respectively. However, it was mentioned in Li and Xu [19] that these results are inaccurate, since it was really proved a closure property of directionally convex order. On the other hand, Pellerey showed that for $\mathbf{M} <_{\mathrm{uo}}$ $(<_{\mathrm{lo}})\tilde{\mathbf{M}}$ and for sequences of arbitrary nonnegative random sequences $\mathbf{Z} <_{\mathrm{uo}}$ $(<_{\mathrm{lo}})\tilde{\mathbf{Z}}$ holds. This case was also considered in Li and Xu [18]. The similar models was considered in Wong [36].

Another shock model was considered in Shanthikumar and Sumita [33]. Let $\mathbf{\Phi}$ and $\mathbf{\Psi}$ be 1-variate point processes such that $\{(X_n, U_n)\}_{n\geq 1}$ is an iid sequence. Observe that in their case $\mathbf{\Phi}$ is not independent of $\mathbf{\Psi}$. They considered

$$Z(t) = \max\{U_1, \ldots, U_{N(t)}\}$$

and

$$z(t) = \min\{U_1, \ldots, U_{N(t)}\}\,.$$

Their aim was to established properties of $Z(t)$ and $z(t)$. We modify this model in the following way. We assume that $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are independent and consist of stationary sequences. Using Corollary 6.2 we have the following result.

Proposition 7.9 *If* $\mathbf{\Psi} <_{1-\mathrm{sm}-\infty} \tilde{\mathbf{\Psi}}$ *and* $\mathbf{\Phi} <_{\mathrm{v-cx}-\mathcal{N}} \tilde{\mathbf{\Phi}}$ *then*

$$z(t) <_{\mathrm{st}} \tilde{z}(t)$$

and

$$Z(t) <_{\mathrm{st}} \tilde{Z}(t)\,.$$

Example 7.10 (Premium calculation principle) In many actuarial applications it is important to consider so called stop-loss and stop-excess orders, i.e. $V <_{\mathrm{sl}} \tilde{V}$ ($V <_{\mathrm{se}} \tilde{V}$) if for all $x > 0$, $\mathbb{E}[V - x]_+ \leq \mathbb{E}[\tilde{V} - x]_+$ ($\mathbb{E}[x - V]_+ \leq \mathbb{E}[x - \tilde{V}]_+$). It is easy to observe that $<_{\mathrm{cx}}$-order for random variables V and $\tilde{V}$ implies both of the above orderings. In many cases there are known results for a stop-loss order for partial sums, i.e. $\sum_{n=1}^{n} U_i <_{\mathrm{sl}} \sum_{n=1}^{n} \tilde{U}_i$ (see e.g. Müller [23]). Our results can be applied for comparison of partial random sums in stop-loss and stop-excess orders as well.

Consider a premium $H[.]$ which assigns premium amount $H[V]$ to a risk V. We will assume that $H[.]$ preserves stop-loss or stop-excess order, i.e.

$$V <_{\mathrm{sl}} \tilde{V} \Longrightarrow H[V] \leq H[\tilde{U}] \quad V <_{\mathrm{se}} \tilde{V} \Longrightarrow H[V] \leq H[\tilde{V}]\,.$$

Assume that risk process is described by 1-variate point processes $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ (arrivals) and $\mathbf{\Psi}$, $\tilde{\mathbf{\Psi}}$ (risks). The premium is calculated w.r.t. all risks up to time t. If for all t, $N(t) <_{\mathrm{cx}} \tilde{N}(t)$ and $\{U_n\} <_{\mathrm{idcx}} \{\tilde{U}_n\}$, where $\{U_n\}_{n\geq 1}$ and $\{\tilde{U}_n\}_{n\geq 1}$ are stationary sequences then using Proposition 3.2 with $k = 1$ and $f_1^{(m)}(\{u_n\}) = \sum_{n=1}^{m} u_n$,

$$H\left[\sum_{i=1}^{N(t)} U_i\right] \leq H\left[\sum_{i=1}^{\tilde{N}(t)} \tilde{U}_i\right]\,,$$

i.e. roughly speaking, more dependent claims and more dependent point process give higher premiums (cf. Denuit *et al.* [7]).

8 Comments and extensions

Example 8.1 (Number of events in random intervals) Let N^i, $\tilde{N}^i$, $i = 1, \ldots, k$ be counting processes and let U^i, $\tilde{U}^i$, $i = 1, \ldots, k$ be nonnegative random variables independent of N^i and $\tilde{N}^i$, respectively. Denote $N^i(U^i) = N^i((0, U^i])$ and

$\tilde{N}^i(\tilde{U}^i) = \tilde{N}^i((0, \tilde{U}^i])$. Observe that, conditionally on $\{T_n^i\} = \{t_n^i\}$, a function $g_i^{(m)}(u^i) = \max\{n : t_n^i < u^i\}$ does not depend on m and is monotone w.r.t. u^i. Observe that these functions are not the same as considered in Section 3. Indeed, those functions, for given m, does not depend on realizations of $\mathbf{\Phi}$. They depend only on $\mathbf{\Psi}$. It is not the case for $g_i^{(m)}$. However, the same technique as within the proof of Proposition 3.1 can be applied and therefore we get the following result.

Proposition 8.2 *Suppose that N^i and $\tilde{N}^i$, $i = 1, \ldots, k$ are independent of U^i and $\tilde{U}^i$, $i = 1, \ldots, k$, respectively. Assume that for all $0 < t_1 \leq \cdots \leq t_k$, $(N^1(t_1), \ldots, N^k(t_k)) <_{\mathrm{sm}} (\tilde{N}^1(t_1), \ldots, \tilde{N}^k(t_k))$ and $(U^1, \ldots, U^k) <_{\mathrm{sm}} (\tilde{U}^1, \ldots, \tilde{U}^k)$. Then*

$$(N^1(U^1), \ldots, N^k(U^k)) <_{\mathrm{sm}} (\tilde{N}^1(\tilde{U}^1), \ldots, \tilde{N}^k(\tilde{U}^k))$$

From Baccelli and Brémaud [2, p. 231] we know that for all $\varphi \in \mathcal{L}_{\mathrm{cx}}$ the function $\phi(x) = \mathbb{E}[\varphi(N(x))]$ is convex. Hence $N(U) <_{\mathrm{cx}} N(\tilde{U})$ provided $U <_{\mathrm{cx}} \tilde{U}$. Conditionally on $\tilde{U}$, we have $N(\tilde{U}) <_{\mathrm{cx}} \tilde{N}(\tilde{U})$. Unconditioning gives $N(U) <_{\mathrm{cx}} \tilde{N}(\tilde{U})$. Shaked and Wong [32] got comparisons for $N(U)$ and $N(\tilde{U})$ under suitable assumptions on U and $\tilde{U}$.

Comment 8.3 The results for the $<_{\mathrm{sm}}$-order can be rewritten for other dependence orderings. Indeed, we can consider every ordering $<_{\mathrm{a}}$ which has (MA), (ID), (MI), (IN) and (IT) property (We refer for these properties to Appendix below). For example, we can take concordance ordering $<_{\mathrm{c}}$, upper orthant ordering $<_{\mathrm{uo}}$ or lower orthant ordering $<_{\mathrm{lo}}$. The main result of Proposition 3.1 is still valid (with $f_i^{(m)}$ being increasing, not monotone). Moreover, Lemma 5.5 (with $f_i^{(m)}$ being increasing), Corollaries 4.2, 4.4 and Proposition 8.2 can be rewritten using one of the above orderings instead of $<_{\mathrm{sm}}$. The results concerning comparison of shock models and arrival processes obtained in Li and Xu [17], [18] can be obtained using our results.

Comment 8.4 From the discussion in the previous comment the above mentioned results can be rewritten in terms of positive (negative) dependence, i.e. orthant dependence (PUOD, PLOD, NUOD, NLOD, more concordant dependence) or association. For example, assuming that for all $t \geq 0$, $(N^1(t), \ldots, N^k(t))$ and $(\{U_n^1\}, \ldots, \{U_n^k\})$ are associated then, under assumptions of Proposition 3.1 and for increasing function $f_i^{(m)}$, $i = 1, \ldots, k$, $H_{\mathbf{f}}(\mathbf{\Psi}, \mathbf{\Phi})(t)$ is associated. The similar results can be established in Lemma 5.5 (with $f_i^{(m)}$ being increasing), Corollaries 4.2, 4.4 and Proposition 8.2.

Comment 8.5 Our results can be formulated not only for functions $f^{(m)}(\{u_n\})$ which are defined on the first m variables $(u_1, \ldots, u_m)$, but also for functions which depend on arbitrary subsequence of $\{u_n\}$ of the length m, $u_{r_1}, \ldots, u_{r_m}$, say.

9 Appendix

We recall some well known closure properties of supermodular and directionally convex functions (Shaked and Shanthikumar [31], Meester and Shanthikumar [22], Marshall and Olkin [21]). Additionally, we prove some needed new technical results.

Lemma 9.1 *Let $\mathbf{u} = (u_1, \ldots, u_k)$.*

(i) *Assume that* $\varphi : \mathbb{R}^k \to \mathbb{R} \in \mathcal{L}_{\text{sm}}$ *and* $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, k$ *are monotone in the same direction. Then* $\psi(\mathbf{u}) = \varphi(f_1(u_1), \ldots, f_k(u_k))$ *is supermodular on* $\mathbb{R}^k$.

(ii) *Let* $\varphi : \mathbb{R}^k \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$. *If* $\varphi \in \mathcal{L}_{\text{ism}}$ *and* $f \in \mathcal{L}_{\text{icx}}$, *then* $f \circ \varphi \in \mathcal{L}_{\text{ism}}$.

(iii) *Let* $\varphi : \mathbb{R}^k \to \mathbb{R} \in \mathcal{L}_{\text{sm}}$. *Assume that functions* $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$, $i = 1, \ldots, k$ *are monotone in the same direction. Then* ψ *defined as*

$$\psi((u_1^1, \ldots, u_{n_1}^1), \ldots, (u_1^k, \ldots, u_{n_k}^k)) = \varphi\left(f_1(u_1^1, \ldots, u_{n_1}^1), \ldots, f_k(u_1^1, \ldots, u_{n_k}^1)\right)$$

is supermodular w.r.t. all vectors of variables of the form $(u_{j_1}^1, \ldots, u_{j_k}^k)$, $1 \leq j_i \leq n_i$, $i = 1, \ldots, k$.

(iv) *Let* $\varphi : \mathbb{R}^k \to \mathbb{R} \in \mathcal{L}_{\text{sm}}$. *Assume that functions* $f_i^{(m)} : \mathbb{R}^m \to \mathbb{R}$ *are monotone in the same direction w.r.t.* m, $1 \leq m \leq \infty$. *Then* ψ *defined as*

$$\psi(n_1, \ldots, n_k) = \varphi(f_1^{(n_1)}(\{u_n^1\}_{n=1}^{n_1}), \ldots, f_k^{(n_k)}(\{u_n^k\}_{n=1}^{n_k}))$$

is supermodular on $\mathbb{N}^k$.

Proof. The proof of (i)-(ii) can be found in Marshall and Olkin [21], (p. 151). In order to obtain (iv) observe that

$$\begin{aligned}
\Delta_1^1 \Delta_2^1 \psi(n_1, \ldots, n_k) &= \varphi(f_1^{(n_1+1)}(\{u_n^1\}), f_2^{(n_2+1)}(\{u_n^2\}) \ldots, f_k^{(n_k)}(\{u_n^k\})) \\
&\quad + \varphi(f_1^{(n_1)}(\{u_n^1\}), f_2^{(n_2)}(\{u_n^2\}) \ldots, f_k^{(n_k)}(\{u_n^k\})) \\
&\quad - \varphi(f_1^{(n_1+1)}(\{u_n^1\}), f_2^{(n_2)}(\{u_n^2\}) \ldots, f_k^{(n_k)}(\{u_n^k\})) \\
&\quad - \varphi(f_1^{(n_1)}(\{u_n^1\}), f_2^{(n_2+1)}(\{u_n^2\}) \ldots, f_k^{(n_k)}(\{u_n^k\})) \\
&\geq 0
\end{aligned}$$

because $f_i^{(n_i+1)}(\{u_n^i\}) \geq (\leq) f_i^{(n_i)}(\{u_n^i\})$. Here, we write shortly $\{u_n^i\}$ for $\{u_n^i\}_{n=1}^{n_i}$. The (iii) result can be proved in a similar way.

□

Lemma 9.2 *Let* $\mathbf{u} = (u_1, \ldots, u_k)$.

(i) *Let* $\varphi : \mathbb{R}^k \to \mathbb{R} \in \mathcal{L}_{\text{idcx}}$ *and* $f_i : \mathbb{R}^k \to \mathbb{R} \in \mathcal{L}_{\text{idcx}}$ *for all* $i = 1, \ldots, k$. *Then* ψ *defined as*

$$\psi(\mathbf{u}) = \varphi\left(f_1(\mathbf{u}), \ldots, f_k(\mathbf{u})\right) \tag{4}$$

is increasing and directionally convex on $\mathbb{R}_+^k$.

(ii) *Let* $\varphi : \mathbb{R}^k \to \mathbb{R} \in \mathcal{L}_{\text{idcx}}$ *and* $f_i : \mathbb{R}^{n_i} \to \mathbb{R} \in \mathcal{L}_{\text{idcx}}$ *for all* $i = 1, \ldots, k$. *Then* ψ *defined as*

$$\psi((u_1^1, \ldots, u_{n_1}^1), \ldots, (u_1^k, \ldots, u_{n_k}^k)) = \varphi\left(f_1(u_1^1, \ldots, u_{n_1}^1), \ldots, f_k(u_1^k, \ldots, u_{n_k}^k)\right) \tag{5}$$

is increasing and directionally convex on $\mathbb{R}_+^{n_1+\cdots+n_k}$.

(iii) *Let* $\varphi : \mathbb{R}^k \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$. *If* $\varphi \in \mathcal{L}_{\text{idcx}}$ *and* $f \in \mathcal{L}_{\text{icx}}$, *then* $f \circ \varphi \in \mathcal{L}_{\text{idcx}}$.

(iv) *Let* $f : \mathbb{R} \to \mathbb{R} \in \mathcal{L}_{\mathrm{cx}}$. *Then* $\varphi : \mathbb{R}^k \to \mathbb{R}$ *defined as*

$$\varphi(\mathbf{u}) = f\left(\sum_{n=1}^{k} u_n\right)$$

is directionally convex on $\mathbb{R}_+^k$.

(v) *Let* $\varphi : \mathbb{R}^k \to \mathbb{R} \in \mathcal{L}_{\mathrm{dcx}}$ *and* $u_j^i \geq 0$, $1 \leq i \leq k$, $1 \leq j \leq n_i$. *Then* ψ *defined as*

$$\psi((u_1^1, \ldots, u_{n_1}^1), \ldots, (u_1^k, \ldots, u_{n_k}^k)) = \varphi\left(\sum_{l=1}^{n_1} u_l^1, \ldots, \sum_{l=1}^{n_k} u_l^k\right)$$

is directionally convex on $\mathbb{R}_+^{n_1+\cdots+n_k}$.

Proof. The (i) result was obtained in Meester and Shanthikumar [22] and from this we easily have (ii) and (iii). The (iv) result is taken from Marshall and Olkin [21, p. 152]. In order to obtain (v) result we proceed as follows.

Let $n = n_1 + \cdots + n_k$. We need to show that for all $1 \leq j_1 \leq j_2 \leq n$ and $\epsilon_{j_1}, \epsilon_{j_2} > 0$, $\Delta_{j_1}^{\epsilon_{j_1}} \Delta_{j_2}^{\epsilon_{j_2}} \psi \geq 0$. Observe that for $1 \leq j_1 \leq j_2 \leq n_1$

$$\begin{aligned}
\Delta_{j_1}^{\epsilon_{j_1}} \Delta_{j_2}^{\epsilon_{j_2}} \psi((u_1^1, \ldots, u_{n_1}^1), \ldots, (u_1^k, \ldots, u_{n_k}^k)) &= \varphi\left(\sum_{l=1}^{n_1} u_l^1 + \epsilon_{j_1} + \epsilon_{j_2}, \ldots, \sum_{l=1}^{n_k} u_l^k\right) \\
&\quad + \varphi\left(\sum_{l=1}^{n_1} u_l^1, \ldots, \sum_{l=1}^{n_k} u_l^k\right) \\
&\quad - \varphi\left(\sum_{l=1}^{n_1} u_l^1 + \epsilon_{j_1}, \ldots, \sum_{l=1}^{n_k} u_l^k\right) \\
&\quad - \varphi\left(\sum_{l=1}^{n_1} u_l^1 + \epsilon_{j_2}, \ldots, \sum_{l=1}^{n_k} u_l^k\right) \\
&\geq 0
\end{aligned}$$

from the convexity of φ w.r.t. first coordinate. Similarly for $n_1 + \cdots + n_{r-1} < j_1 \leq n_1 + \cdots + n_r$, $n_1 + \cdots + n_{s-1} < j_2 \leq n_1 + \cdots + n_s$, $r < s$

$$\begin{aligned}
\Delta_{j_1}^{\epsilon_{j_1}} \Delta_{j_2}^{\epsilon_{j_2}} \psi((u_1^1, \ldots, u_{n_1}^1), \ldots, (u_1^k, \ldots, u_{n_k}^k)) &= \varphi\left(\ldots, \sum_{l=1}^{n_r} u_l^r + \epsilon_{j_1}, \ldots, \sum_{l=1}^{n_s} u_l^s + \epsilon_{j_2}, \ldots\right) \\
&\quad + \varphi\left(\ldots, \sum_{l=1}^{n_r} u_l^r, \ldots, \sum_{l=1}^{n_s} u_l^s, \ldots\right) \\
&\quad - \varphi\left(\ldots, \sum_{l=1}^{n_r} u_l^r + \epsilon_{j_1}, \ldots, \sum_{l=1}^{n_s} u_l^s, \ldots\right) \\
&\quad - \varphi\left(\ldots, \sum_{l=1}^{n_r} u_l^r, \ldots, \sum_{l=1}^{n_s} u_l^s + \epsilon_{j_2}, \ldots\right) \\
&\geq 0
\end{aligned}$$

from supermodularity of φ.

□

Now, we recall needed closure properties of supermodular and directionally convex orderings.

Definition 9.3 Let $K = \{k_1, \dots, k_n\} \subseteq \{1, \dots, n\}$. We write $\mathbf{u}_K$ for the vector $(u_{k_1}, \dots, u_{k_n})$. A stochastic order $<_{\mathrm{a}}$ has a property

(i) (MA - closure under marginalization): if $(Y_1, \dots, Y_n) <_{\mathrm{a}} (\tilde{Y}_1, \dots, \tilde{Y}_n)$ implies $\mathbf{Y}_K \le_{\mathrm{a}} \tilde{\mathbf{Y}}_K$ for all $K \subseteq \{1, \dots, n\}$;

(ii) (ID - closed under identical concatenation): if $(Y_1, \dots, Y_n) <_{\mathrm{a}} (\tilde{Y}_1, \dots, \tilde{Y}_n)$ implies $(\mathbf{Y}_K, \mathbf{Y}_L) <_{\mathrm{a}} (\tilde{\mathbf{Y}}_K, \tilde{\mathbf{Y}}_L)$ for all K and $L \subseteq \{1, \dots, n\}$;

(iii) (MI - closed under mixture): if $[(Y_1, \dots, Y_n)|\Theta = \theta] <_{\mathrm{a}} [(\tilde{Y}_1, \dots, \tilde{Y}_n)|\tilde{\Theta} = \theta]$ implies $(Y_1, \dots, Y_n) <_{\mathrm{a}} (\tilde{Y}_1, \dots, \tilde{Y}_n)$ where $(Y_1, \dots, Y_n)$ and a random element Θ ($(\tilde{Y}_1, \dots, \tilde{Y}_n)$ and $\tilde{\Theta}$) are defined on the same probability space and $\Theta \stackrel{\mathrm{d}}{=} \tilde{\Theta}$;

(iv) (IT - closed under increasing transform): if $(Y_1, \dots, Y_n) <_{\mathrm{a}} (\tilde{Y}_1, \dots, \tilde{Y}_n)$ implies $(f_1(Y_1), \dots, f_n(Y_n)) <_{\mathrm{a}} (f_1(\tilde{Y}_1), \dots, f_n(\tilde{Y}_n))$ for all increasing functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$;

The following lemma is a corollary of some results of Müller and Stoyan [26, chapter 3].

Lemma 9.4 (i) *The orders* $<_{\mathrm{sm}}$, $<_{\mathrm{c}}$, $<_{\mathrm{uo}}$ *and* $<_{\mathrm{lo}}$ *have the properties* (*MA*), (*ID*), (*MI*) *and* (*IT*).

(ii) *The orders* $<_{\mathrm{dcx}}$ *and* $<_{\mathrm{idcx}}$ *have the properties* (*MA*), (*ID*), (*MI*).

Note however, that neither $<_{\mathrm{dcx}}$ nor $<_{\mathrm{idcx}}$ are closed under increasing transforms.

Lemma 9.5 (Lorentz inequality) *Assume that* $U_1 \stackrel{\mathrm{d}}{=} \cdots \stackrel{\mathrm{d}}{=} U_n$. *Then*

$$(U_1, \dots, U_n) <_{\mathrm{sm}} (U_1, \dots, U_1)$$

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