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SUFFICIENT CONDITIONS FOR LONG-RANGE COUNT DEPENDENCE OF STATIONARY POINT PROCESSES ON THE REAL LINE

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Abstract

Daley and Vesilo (1997) introduced long-range count dependence (LRcD) for stationary point processes on the real line as a natural augmentation of the classical long-range dependence of the corresponding interpoint sequence. They studied LRcD for some renewal processes and some output processes of queueing systems, continuing the previous research on such processes of Daley (1968), (1975). Subsequently, Daley (1999) showed that a necessary and sufficient condition for a stationary renewal process to be LRcD is that under its Palm measure the generic lifetime distribution has infinite second moment. We show that point processes dominating, in a sense of stochastic ordering, LRcD point processes are LRcD, and as a corollary we obtain that for arbitrary stationary point processes with finite intensity a sufficient condition for LRcD is that under Palm measure the interpoint distances are positively dependent (associated) with infinite second moment. We give many examples of LRcD point processes, among them exchangeable, cluster, moving average, Wold, semi-Markov processes and some examples of LRcD point processes with finite second Palm moment of interpoint distances. These examples show that, in general, the condition of infiniteness of the second moment is not necessary for LRcD. It is an open question whether the infinite second Palm moment of interpoint distances suffices to make a stationary point process LRcD.

Keywords: Long-range count dependence; Palm measure; semi-Markov processes

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1. Introduction

The long-memory phenomenon of stochastic processes was discovered in various applied models, for example of Markov, ARMA and self-similar types. This is a rapidly developing subject in statistics; a summary is given by Beran (1994). For stationary point processes the usual setting applies to sequences of interpoint distances when in fact stationary point processes are more often described via their counting properties, as in the case of traffic or population data for example. The situation cannot be better described than by D. R. Cox in his comment in the discussion of Bartlett's classical paper on the spectral analysis of point processes (Bartlett (1963)): 'Although there are elegant general formulae that relate the correlation function of intervals to the distributional properties of numbers of events, and the covariance density of numbers to the distributional properties of intervals, the two correlation functions are mathematically independent. . . . a searching analysis of a series of events, assumed stationary,

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requires, in general, simultaneous consideration at least of some second-degree properties of numbers, of the distribution of intervals between successive events, and of some second-degree properties of intervals. . . .'

In view of the above comment, long-range count dependence (LRcD) for stationary point processes on the real line, defined via the variance function of the number of events, introduced by Daley and Vesilo (1997), is a natural augmentation of the classical long-range dependence of the corresponding interpoint sequence. To be more specific, let $\cdots < T_{-1} < T_0 \leq 0 < T_1 < \cdots$ be a sequence of points of a simple non-explosive point process N, and $(N(t),\ t \geq 0)$ be a corresponding *counting process* defined by

$$N(t) = \sum_{n \geq 1} \mathbf{1}(T_n \leq t).$$

The process N is (time) stationary with finite intensity $0 < \lambda = \mathrm{E}[N(0, 1]] < \infty$ under a probability measure P, if for any bounded Borel sets $B_1, \dots, B_k, n_1, \dots, n_k \in \mathbb{N}$, and $k \in \mathbb{N}$

$$\mathrm{P}(N(B_1 + t) = n_1, \dots, N(B_k + t) = n_k)$$

is independent of $t \in \mathbb{R}$, where $B_i + t = \{x + t : x \in B_i\},\ i = 1, \dots, k$.

Let $(X_n = T_n - T_{n-1},\ n \in \mathbb{Z})$ denote the interpoint distances sequence. There is a one–one correspondence between those P which make N time stationary and those P^0 (called the corresponding Palm measure) which make $(X_n = T_n - T_{n-1},\ n \in \mathbb{Z})$ a (Palm) stationary sequence (see e.g. Baccelli and Brémaud (1994)).

Distributions with respect to P^0 will be written with the superscript 0, for example the marginal distribution function of X_n will be denoted by F^0. The first (Palm) moment of X_n, m_F say, is finite and $m_F = 1/\lambda$. We denote $(X_1, \dots, X_n)$ by $\boldsymbol{X}$.

Definition 1.1. *(Daley and Vesilo (1997).)* A stationary point process N is long-range count dependent (LRcD) if

$$\limsup_{x \to \infty} \frac{\operatorname{var}[N(x)]}{x} = \infty.$$

Daley and Vesilo studied LRcD for some renewal processes and some output processes of queueing systems, continuing the previous research on such processes of Daley (1968), (1975). Subsequently, Daley (1999) showed that the Hurst index

$$H = \inf\left\{h : \limsup_{t \to \infty} t^{-2h} \operatorname{var}[N(t)] < \infty\right\}$$

of a stationary renewal process with finite intensity is related to the corresponding *critical moment index*

$$\kappa = \sup\{k : \mathrm{E}^0[X_n^k] < \infty\}$$

by the equation $H = \frac{1}{2}(3 - \kappa)$, which implies that a necessary and sufficient condition for a stationary renewal process to be LRcD is that under its Palm measure the generic lifetime distribution has infinite second moment.

Recently, Daley *et al.* (2000) showed that, if the moment index of a stationary ergodic point process N is $\kappa < 2$, then N is LRcD. Therefore, searching for LRcD processes in the class of stationary processes can be limited to processes with $\mathrm{E}^0[X_n^2] < \infty$ or $\mathrm{E}^0[X_n^2] = \infty$ but $\mathrm{E}^0[X_n^{2-\delta}] < \infty$ for all $\delta \in (0, 1)$.

We show that point processes stochastically dominating (in some sense) LRcD point processes are LRcD too. The stochastic orderings for point processes we use are related to the dependence structure of the corresponding sequences of interarrival times. Therefore, an intuitive meaning of our results is that 'more dependence in the interpoint distances of point processes implies more variability in the corresponding counting processes'. In particular, the supermodular ordering is used to compare a random sequence with its independent version and we obtain that, for arbitrary stationary point processes with finite intensity, a sufficient condition for LRcD is that under Palm measure the interpoint distances are positively dependent (associated) with infinite second moment. We give many examples of LRcD point processes, among them exchangeable, Cox, cluster, moving average processes as well as some examples of LRcD point processes with finite second Palm moment of interpoint distances. These examples show that, in general, the infiniteness condition of the second moment is not necessary for LRcD. It seems to be an open question whether the infinite second Palm moment of interpoint distances suffices to make a stationary point process LRcD.

The paper is organized as follows. In Section 2 we show the above mentioned result using stochastic orderings. This result implies that stochastically monotone Wold processes with infinite variance generate LRcD processes. We give also some semi-Markov point processes which possess a stronger property than association (conditionally increasing in sequence) and have infinite second moment which are therefore LRcD. Next, using MTP_2, another stronger property than association, we give another class of semi-Markov LRcD processes. Using association, we show that point processes generated by moving averages based on the gamma process, introduced by Resnick and Samorodnitsky (1997), are LRcD. In Section 3, we gather a variety of LRcD stationary point processes: related to exchangeable interpoint sequences, Cox, cluster, queueing output, superposition of renewal, and finite state semi-Markov processes.

2. Comparison result

Following arguments of Baccelli and Brémaud (1994, pp. 229–230), assume that for the respective interarrival times (X_n), (X'_n) of stationary point processes N and N'

$$\mathrm{E}^0[\phi(X_1 + \cdots + X_n)] \le \mathrm{E}^0[\phi(X'_1 + \cdots + X'_n)] \tag{2.1}$$

holds for all convex ϕ and all $n \in \mathbb{N}$. Taking $\phi(u) = [t-u]_+$ (which is convex for $u \ge 0$) gives

$$\mathrm{E}^0[t - (X_1 + \cdots + X_n)]_+ \le \mathrm{E}^0[t - (X'_1 + \cdots + X'_n)]_+,$$

or, equivalently,

$$\mathrm{E}^0[t - T_n]_+ \le \mathrm{E}^0[t - T'_n]_+,$$

since for any function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(0) = 0$ and for all $t \in \mathbb{R}_+$ it holds that

$$\mathrm{E}[f(N(t))] = \sum_{n=0}^{\infty} (f(n+1) - 2f(n) + f((n-1)_+))\lambda \mathrm{E}^0[(t - T_n)_+].$$

Thus, for $f(x) = x^2,\ x \ge 0$, using the fact that from the assumption (2.1) N and N' have equal intensities, we get

$$\operatorname{var}[N(t)] \le \operatorname{var}[N'(t)].$$

Consequently we obtain the following result.

Proposition 2.1. *Assume that N, N' are two stationary point processes with finite intensities. If N is LRcD and for the corresponding interarrival times*

$$\mathrm{E}^0[\phi(X_1 + \cdots + X_n)] \leq \mathrm{E}^0[\phi(X'_1 + \cdots + X'_n)] \tag{2.2}$$

for all convex ϕ and all $n \in \mathbb{N}$, then N' is LRcD.

An interesting case when (2.2) holds is related to the supermodular ordering (for the supermodular ordering see Shaked and Shanthikumar (1997)). More specifically, recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *supermodular*, if for any $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$

$$f(\boldsymbol{x} \vee \boldsymbol{y}) + f(\boldsymbol{x} \wedge \boldsymbol{y}) \geq f(\boldsymbol{x}) + f(\boldsymbol{y}),$$

where $\vee$ and $\wedge$ denote coordinate-wise *maximum* and *minimum* operations.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *directionally convex*, if it is supermodular and coordinate-wise convex (see Meester and Shanthikumar (1993)). These classes of functions are used to define the following orderings.

Definition 2.1. We say that $\boldsymbol{X}$ is smaller than $\boldsymbol{X'}$ in the supermodular ($\boldsymbol{X} <_{\mathrm{sm}} \boldsymbol{X'}$), directionally convex ($\boldsymbol{X} <_{\mathrm{dcx}} \boldsymbol{X'}$) or convex ($\boldsymbol{X} <_{\mathrm{cx}} \boldsymbol{X'}$) ordering if $\mathrm{E}^0[f(\boldsymbol{X})] \leq \mathrm{E}^0[f(\boldsymbol{X'})]$ for, respectively, all supermodular, all directionally convex or all convex functions f for which the expectations exist. Moreover $\boldsymbol{X} <_{\mathrm{plcx}} \boldsymbol{X'}$ if $\boldsymbol{a}^\top \boldsymbol{X} <_{\mathrm{cx}} \boldsymbol{a}^\top \boldsymbol{X'}$ for any $\boldsymbol{a} \in \mathbb{R}^n_+$ (see Scarsini (1998)).

Lemma 2.1. *If $\boldsymbol{X} < \boldsymbol{X'}$, where $<$ denotes one of the orders $<_{\mathrm{sm}}$, $<_{\mathrm{dcx}}$, $<_{\mathrm{plcx}}$, then (2.2) holds.*

Proof. Since the class of directionally convex functions is a subclass of the class of supermodular functions, the $<_{\mathrm{sm}}$ ordering implies the $<_{\mathrm{dcx}}$ ordering. Moreover, if ϕ is convex, then a function ψ defined by $\psi(x_1, \ldots, x_n) = \phi(a_1 x_1 + \cdots + a_n x_n)$, where $a_i \geq 0$, is supermodular and coordinate-wise convex, and hence directionally convex. This fact implies that $\mathrm{E}\psi(\boldsymbol{X}) \leq \mathrm{E}\psi(\boldsymbol{X'})$ provided $\boldsymbol{X} <_{\mathrm{dcx}} \boldsymbol{X'}$. Hence, $\boldsymbol{X} <_{\mathrm{dcx}} \boldsymbol{X'}$ implies that $\boldsymbol{X} <_{\mathrm{plcx}} \boldsymbol{X'}$. Summing up we have

$$<_{\mathrm{sm}} \implies <_{\mathrm{dcx}} \implies <_{\mathrm{plcx}},$$

and it is clear that each of the above orderings implies (2.2).

Definition 2.2. A sequence (X_n) is said to be *conditionally increasing in sequence* (CIS) if for any $n \geq 1$ and any $x_n \in \mathbb{R}$, $\mathrm{P}^0(X_n \geq x_n \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1})$ is increasing coordinate-wise with respect to $(x_1, \ldots, x_{n-1})$.

The CIS dependence and the supermodular ordering are linked by the following lemma (see e.g. Meester and Shanthikumar (1993)).

Lemma 2.2. *If $(X_n)_{n\geq 1}$ is CIS, then for each n*

$$\hat{\boldsymbol{X}} <_{\mathrm{sm}} \boldsymbol{X},$$

where $\hat{\boldsymbol{X}} = (\hat{X}_1, \ldots, \hat{X}_n)$, and the $\hat{X}_i$ are i.i.d. with the same marginal distribution F^0 as X_i $(i = 1, \ldots, n)$.

Now, combining Lemma 2.2, Lemma 2.1 and Proposition 2.1 it is possible to relate the LRcD property of point processes with positively dependent interpoint sequences with the LRcD property of renewal processes. Recall that Daley (1999) obtained that a stationary renewal point process with finite intensity is LRcD if and only if the second moment with respect to the corresponding Palm measure of the interrenewal times is infinite. Thus we have the following corollary.

Corollary 2.1. *If N is a stationary point process such that under the Palm measure the sequence of interarrival times $(X_n)_{n\geq 1}$ is CIS and the marginal (Palm) distribution of this sequence has infinite second moment, then N is LRcD.*

Remark 2.1. Alfred Müller pointed out (personal communication) that in Corollary 2.1 it is enough to assume the association of $(X_n)_{n\geq 1}$ instead of the stronger CIS property. Indeed, in Proposition 2.1 it is enough to have $T_n <_{\mathrm{cx}} T_n'$ in order to have $\mathrm{var}[N(t)] \leq \mathrm{var}[N'(t)]$. Assuming that $T_n = \hat{T}_n$, where $\hat{T}_n$ denotes points of the 'independent' version, the conclusion in Corollary 2.1 for associated (X_n') follows immediately from Theorem 3.7 in Müller and Pflug (2001).

Example 2.1. *(Stochastically monotone Wold processes.)* Let (X_n) be a stationary Markov chain on $\mathbb{R}_+$ with the transition probability density $h(y \mid x)$ and the corresponding distribution function $H(y \mid x)$. Assume that it is stochastically monotone, i.e. $1 - H(\cdot \mid x)$ is increasing in x. Then obviously it is CIS. If $\mathrm{E}^0[X_1^2] = \infty$, then the point process with interarrival times (X_n) is LRcD.

Daley *et al.* (2000) studied some special semi-Markov point processes with finite second moment of intervals between points under Palm distribution and showed that the LRcD and LRiD properties hold for them, where a process is LRiD if

$$\limsup_{n\to\infty} \frac{\mathrm{var}^0[\sum_{i=1}^n X_i]}{n} = +\infty.$$

This study suggests two possible scenarios for a stationary point process to be LRcD. The first one is that the LRcD and LRiD properties of a point process coexist, the second one is that the second moment of interpoint intervals under the Palm distribution is infinite and the sequence of interpoint distances has some additional properties. In the next example we show the LRcD property of semi-Markov processes with infinite second moment and additional dependency structures (MTP_2, CIS) of interpoint distances. Here we consider a semi-Markov process with infinite state space because the finite state case is described in Proposition 3.3.

Example 2.2. *(Infinite state space, semi-Markov process with additional dependency structures.)* Take $p \in (0, 1)$ and let (y_n) be an increasing sequence of positive real numbers such that $\sum_{i=1}^{\infty} y_i (1-p)^{i-1} < \infty$. For the transition probability matrix

$$q_{ij} = \begin{cases} p & \text{if } j = 1,\ i \geq 1, \\ 1-p & \text{if } i = j-1, \\ 0 & \text{otherwise,} \end{cases}$$

consider the stationary Markov chain $(Z_n,\ n \geq 0)$ with the stationary distribution $\pi_i = \mathrm{P}(Z_0 = i) = p(1-p)^{i-1},\ i \geq 1$. Then $(X_n,\ n \geq 0)$, defined by $X_n = y_{Z_n}$, forms a (Palm) stationary sequence of interpoint distances of the corresponding semi-Markov point process with finite mean value, but with infinite variance. Because (y_n) is increasing it is easily seen that (X_n) is CIS and therefore the corresponding point process is LRcD. A stronger property than CIS, but sometimes easier to show, is MTP_2. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is MTP_2 (multidimensional totally positive) if $f(\boldsymbol{x})f(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{y}) f(\boldsymbol{x} \vee \boldsymbol{y})$ for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. We say that a random vector $\boldsymbol{X}$ is MTP_2 if its density is an MTP_2 function. Recall also that a matrix $\boldsymbol{Q} = (q_{ij})$ is TP_2 if for any $i \leq i',\ j \leq j'$

$$\begin{vmatrix} q_{ij} & q_{ij'} \\ q_{i'j} & q_{i'j'} \end{vmatrix} \geq 0.$$

Consider semi-Markov point process such that $X_k = Y^{(k)}_{Z_k}$, $k \in \mathbb{N}$, where $(Y^{(j)}_k, \ j \geq 1)$ is for each k an i.i.d. sequence of variables with the respective distribution function F^0_k. It is known that, if for all $k \geq 1$, $F^0_k <_{\mathrm{lr}} F^0_{k+1}$ (which means that for the corresponding density functions $f^0_k(x)/f^0_{k+1}(x)$ is a non-decreasing function in x) and if the transition matrix of (Z_k) is TP_2 then for each n the vector of interpoint distances $(X_1, \dots, X_n)$ is MTP_2 (see Szekli (1995, p. 166)). It is known from Karlin and Rinott (1980) that MTP_2 implies CIS, hence all semi-Markov process with the above properties, and in addition having interpoint distances with infinite second moment, are LRcD.

Example 2.3. *(Moving average generated by Gamma process.)* Resnick and Samorodnitsky (1997) used a stationary sequence of marginally exponentially distributed random variables with a very flexible correlation function. More precisely, denote by $(Y(t), \ t \in \mathbb{R})$ the Lévy Gamma process. Consider

$$\xi_n = \sum_{j=1}^{\infty} Y(j-1-n, j-1-n+a_j],$$

where $Y(a, b] = Y(b) - Y(a)$ and $(a_j)_{j \geq 1}$ is a probability sequence. Then $\mathrm{corr}(\xi_0, \xi_n) = \sum_{i=n+1}^{\infty} a_i$. It is then possible to have (ξ_n) long-range dependent (LRiD). If we apply a standard inversion procedure on the ξ_n, we obtain a stationary sequence (X_n) with arbitrary marginals and a similar dependency structure. Note that for each sequence $(a_j)_{j\geq 1}$, independently of (ξ_n) being long-range dependent or not, the sequence (ξ_n) is always associated (as a monotone transformation of an i.i.d. sequence), moreover (X_n) obtained by a pointwise monotone transformation of (ξ_n) is also associated, and therefore if marginals are selected to have infinite variance the generated point process will be LRcD.

3. Some further classes of LRcD processes

We proceed now with collecting other classes of LRcD point processes.

3.1. Queueing output processes

The study of the correlation structure of output processes from queues began with Daley (1968) and continued in Daley (1975). This structure turned out to be surprisingly complicated. Using this background the main results of Daley and Vesilo (1997) on the LRcD property are as follows.

1. If the renewal input in a $GI/M/1$ stable system has interpoint distances distribution with regularly varying tail $\mathrm{P}^0(X > x) \sim x^{-c}L(x)$ for $c \in (1, 2)$ and L slowly varying, then the output point process is LRcD.

2. If the renewal sequence of service times in a $M/GI/1$ stable system has distribution with regularly varying tail $\mathrm{P}^0(S > x) \sim x^{-c}L(x)$ for $c \in (1, 2)$ and L slowly varying, then the output point process is LRcD.

3. In a $G/GI/\infty$ system with finite average service time, the output process is LRcD if and only if the input process is LRcD.

Some generalizations, including loss systems, are contained in Daley and Vesilo (2000). For example:

4. The output of a stable $GI/M/k$ system is LRcD if and only if the input is LRcD, i.e. $\mathrm{E}^0[X^2] = \infty$.

5. If a stable queueing system without any loss of arrivals, in which the arrival point process is second order stationary and for which the stationary number of customers in the system has finite second moment, the departure process has the same Hurst index as the arrival process, and is LRcD if and only if the arrival process is LRcD.

3.2. Exchangeable sequences

Consider a sequence (X_n) of positive random variables such that for any $n \in \mathbb{N}$, the n-dimensional distribution function $\mathbb{F}^0$ of $(X_1, \dots, X_n)$ (under P^0) has the form

$$\mathbb{F}^0(\boldsymbol{x}) = \int_\Theta \prod_{k=1}^n F^0(x_k \mid \theta) G(\mathrm{d}\theta),$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$, G is a probability measure on a subset Θ of a finite-dimensional Euclidean space and for each $\theta \in \Theta$, and $F^0(\cdot \mid \theta)$ is a one-dimensional distribution function. Such random vectors or, equivalently, distribution functions are called by Shaked (1977) *positively dependent by mixture* (PDM). It is well known (Hewitt and Savage (1955)) that a vector $(X_1, \dots, X_n)$ can be embedded in an infinite sequence of exchangeable random variables if and only if $(X_1, \dots, X_n)$ is PDM. We shall interpret X_n as an interpoint distance, therefore we consider positive random variables X_n. Because of the de Finetti representation, it is clear that the sequence $(X_n,\ n \in \mathbb{Z})$ with PDM finite dimensional distributions has a long-range memory, however it is not immediate that the corresponding stationary point process is LRcD. Denote by N the stationary point process corresponding to the (Palm) stationary PDM sequence (X_n). For each $\theta \in \Theta$ consider the stationary renewal process N_θ with the corresponding Palm distribution of interpoint distances $F^0(\cdot \mid \theta)$. Assume that

$$0 < \lambda = (\mathrm{E}^0[X_1])^{-1} < \infty$$

and

$$0 < \lambda_\theta := (\mathrm{E}^0[X_1 \mid \theta])^{-1} < \infty,$$

where $\mathrm{E}^0[X_1 \mid \theta]$ denotes the expectation of the distribution $F^0(\cdot \mid \theta)$ and $\mathrm{E}^0[X_1]$ denotes the expectation under $F(x) = \int_\Theta F^0(x \mid \theta) G(\mathrm{d}\theta)$.

Proposition 3.1. *Under above assumptions, if G is not degenerate, then N is LRcD.*

Proof. Using a known formula for variance with respect to the stationary measure of point processes (see e.g. Daley and Vere-Jones (1988, Chapter 3)), we have

$$\begin{aligned}
&\mathrm{var}[N(t)] \\
&\quad = \lambda \int_0^t \{2(\mathrm{E}^0[N(u)] - \lambda u) + 1\}\, \mathrm{d}u \\
&\quad = \lambda \int_\Theta \biggl[\int_0^t \{2(\mathrm{E}^0[N_\theta(u)] - \lambda_\theta u) + 1\}\, \mathrm{d}u + 2\lambda\biggl(\lambda_\theta \int_0^t u\, \mathrm{d}u\biggr)\biggr] G(\mathrm{d}\theta) - 2\lambda^2 \int_0^t u\, \mathrm{d}u \\
&\quad = \lambda \int_\Theta \frac{\mathrm{var}[N_\Theta(t)]}{\lambda_\theta} G(\mathrm{d}\theta) + \lambda t^2 \biggl(\int_\Theta \lambda_\theta G(\mathrm{d}\theta) - \lambda\biggr).
\end{aligned}$$

In the second equality we used Fubini's theorem and in the last equality we used for the renewal process N_θ the representation of the variance once more. It remains to show that

$$\int_\Theta \lambda_\theta G(\mathrm{d}\theta) - \frac{1}{\int_\Theta \lambda_\theta^{-1} G(\mathrm{d}\theta)} \geq 0. \tag{3.1}$$

Notice that

$$\int_\Theta \lambda_\theta^{1/2} \lambda_\theta^{-1/2} G(\mathrm{d}\theta) = 1.$$

From the Schwartz inequality,

$$\left(\int_\Theta \lambda_\theta G(\mathrm{d}\theta)\right)^{1/2} \left(\int_\Theta \lambda_\theta^{-1} G(\mathrm{d}\theta)\right)^{1/2} \geq \int_\Theta \lambda_\theta^{1/2} \lambda_\theta^{-1/2} G(\mathrm{d}\theta) = 1.$$

Finally, we get

$$\left(\int_\Theta \lambda_\theta G(\mathrm{d}\theta)\right)\left(\int_\Theta \lambda_\theta^{-1} G(\mathrm{d}\theta)\right) \geq 1$$

which is equivalent to (3.1). Equality in (3.1) holds only in the case when G is degenerate.

There are a number of known random vectors which are PDM, for example, multivariate logistic, multivariate Gamma, multivariate Cauchy, Marshall–Olkin multivariate exponential, Gumbel distribution, Burr–Pareto–logistic distribution, etc. It is worth noting that constructing point processes with finite-dimensional distributions having the PDM property is connected with the theory of so-called extendable copulas (the family of finite dimensional distributions of interpoint distances should be consistent). The classes of distributions studied by Nelsen (1997) can be used to generate LRcD point processes.

3.3. Cox processes

The point process N is a mixed Poisson process, if

$$\mathrm{P}\left(\bigcap_{i=1}^{n} \{N(t_i) = k_i\}\right) = \int_0^\infty \prod_{i=1}^{n} \exp(-\lambda t_i) \frac{(\lambda t_i)^{k_i}}{k_i!} G(\mathrm{d}\lambda),$$

where G is a distribution of a random variable Λ. We have $\mathrm{E}[N(t) \mid \Lambda = \lambda] = \lambda t$ and $\mathrm{var}[N(t)] = t^2 \,\mathrm{var}[\Lambda] + t\mathrm{E}[\Lambda]$, so this process is always LRcD if G is not trivial. This process is a special case of the PDM case, where mixed distributions are exponentially distributed. Note that, in contrast to the renewal case, in this process the interpoint distances can have finite or infinite second (Palm) moment.

Let $(\Lambda(t),\ t \geq 0)$ be a stochastic process with absolutely continuous trajectories, that is

$$\Lambda(A) = \int_A \lambda(x)\,\mathrm{d}x, \qquad A \in \mathcal{B},$$

where $(\lambda(t),\ t \geq 0)$ is a stationary, non-negative random process, called the *intensity process*. Suppose, that N is a doubly stochastic Poisson process driven by $(\Lambda(t))$. Recall (Daley and Vere-Jones (1988, p. 263)), that

$$\mathrm{var}[N(A)] = \mathrm{E}[\Lambda(A)] + \mathrm{var}[\Lambda(A)].$$

Therefore N is LRcD if and only if Λ is long-range dependent in the sense that

$$\limsup_{t\to\infty} \frac{\operatorname{var}[\Lambda((0,t])]}{t} = \infty.$$

An interesting example of a Cox process which is LRcD is the process driven by the so-called (see Lowen and Teich (1991)) fractal shot noise, which is given by

$$\lambda(t) = \sum_{j=-\infty}^{\infty} h(t - t_j),$$

where $h(t) = t^{-\alpha},\ t \geq 0,\ \alpha \in (0,1)$ and (t_j) denote the points of a homogeneous Poisson process. In this context, $\operatorname{var}[N(t)]/t$ is called the *Fano factor*.

3.4. Cluster processes

Let N be a Neyman–Scott process on $\mathbb{R}$ with a general cluster centre stationary point process N_{c} which has finite intensity μ and the Bartlett spectrum Γ_{c}. Denote by $m_{[1]}$, $m_{[2]}$ the first two factorial moments of the cluster size distribution G.

It is known that the *Bartlett spectrum* of N is of the following form (Daley and Vere-Jones (1988, p. 417)):

$$\Gamma(\mathrm{d}\omega) = |m_{[1]}\phi(\omega)|^2\Gamma_{\mathrm{c}}(\mathrm{d}\omega) + \frac{\mu}{2\pi}[m_{[1]} + (m_{[2]} - m_{[1]}^2)|\phi(\omega)|^2]\ell(\mathrm{d}\omega),$$

where ϕ is the Fourier transform of G. The spectra of cluster processes were originally studied by Daley (1971) and used for the output processes of $G/GI/\infty$ by Daley and Vesilo (1997). This idea can be slightly generalized as follows.

Proposition 3.2. *Assume that the cluster size distribution has finite second moment. Then N is LRcD if and only if N_{c} is LRcD.*

Proof. We have

$$\begin{aligned}
\operatorname{var}[N(t)] &= \int_{\mathbb{R}} \bigg(\frac{\sin(\frac{1}{2}t\omega)}{\frac{1}{2}\omega}\bigg)^2 |\psi(\omega)|^2 \Gamma(\mathrm{d}\omega) \\
&= \int_{\mathbb{R}} \bigg(\frac{\sin(\frac{1}{2}t\omega)}{\frac{1}{2}\omega}\bigg)^2 |\psi(\omega)|^2 m_{[1]}^2 \Gamma_{\mathrm{c}}(\mathrm{d}\omega) \\
&\quad + \frac{\mu}{2\pi}\bigg[2\pi m_{[1]}t + (m_{[2]} - m_{[1]}^2)\int_{\mathbb{R}} \bigg(\frac{\sin(\frac{1}{2}t\omega)}{\frac{1}{2}\omega}\bigg)^2 |\psi(\omega)|^2\,\mathrm{d}\omega\bigg] \\
&\leq m_{[1]}^2 \operatorname{var}[N_{\mathrm{c}}(t)] + \frac{\mu}{2\pi}\bigg[2\pi m_{[1]}t + (m_{[2]} - m_{[1]}^2)\int_{\mathbb{R}} \bigg(\frac{\sin(\frac{1}{2}t\omega)}{\frac{1}{2}\omega}\bigg)^2 \mathrm{d}\omega\bigg] \\
&= m_{[1]}^2 \operatorname{var}[N_{\mathrm{c}}(t)] + \mu(m_{[1]} + m_{[2]} - m_{[1]}^2)t,
\end{aligned}$$

so, if N is LRcD, then N_{c} is LRcD as well, since $m_{[1]} + m_{[2]} - m_{[1]}^2 > 0$. On the other hand, because

$$\operatorname{var}[N(t)] \geq m_{[1]}^2 \int_{\mathbb{R}} \bigg(\frac{\sin(\frac{1}{2}t\omega)}{\frac{1}{2}\omega}\bigg)^2 |\psi(\omega)|^2 \Gamma_{\mathrm{c}}(\mathrm{d}\omega),$$

the rest of the proof can be completed exactly as in Theorem 5 of Daley and Vesilo (1997).

3.5. Auxiliary examples

Example 3.1. *(Superposition of renewal processes.)* Let $N_1, N_2, \dots, N_k$ be independent stationary renewal processes with intensities $\lambda_1, \dots, \lambda_k$. Denote by F_i, F_i^0 the distribution function of the distance to the first point of the ith process, $i = 1, \dots, k$, under stationary and Palm distributions, respectively. Define

$$N = \sum_{i=1}^{k} N_i.$$

Then N is called the *superposition of renewal processes*. Because

$$\mathrm{var}[N(t)] = \sum_{i=1}^{k} \mathrm{var}[N_i(t)],$$

it is clear that N is LRcD if and only if there exist $i,\ i = 1, \dots, k$ such that N_i is LRcD.

Let $G^0(x)$ be the Palm distribution function of the distance to the first point of N. Then (Baccelli and Brémaud (1994, p. 33))

$$1 - G^0(x) = \sum_{i=1}^{k} \frac{\lambda_i}{\lambda}(1 - F_i^0(x)) \prod_{i \neq j}(1 - F_j(x)),$$

where $\lambda = \sum_{i=1}^{k} \lambda_i$. Therefore, taking for example $k = 2$, $1 - F_1^0(x) = x^{-\alpha}\mathbf{1}(x > 1)$, $\alpha \in (1, 2)$, $1 - F_2^0(x) = x^{-\beta}\mathbf{1}(x > 1)$, $\beta > 2$, and $1 - F_1(x) = x^{-\alpha+1}\mathbf{1}(x > 1)$, $1 - F_2(x) = x^{-\beta+1}\mathbf{1}(x > 1)$, we have

$$1 - G^0(x) = x^{-(\alpha+\beta)+1}\mathbf{1}(x > 1) = x^{-\gamma}\mathbf{1}(x > 1),$$

where $\gamma > 2$. In this case, N is LRcD but the second moment of interpoint distances is finite.

Example 3.2. *(Finite state semi-Markow processes.)* Consider a semi-Markov process $X_n = Y_{Z_n}^{(n)}$, $n \geq 1$ for (Z_n) a stationary governing Markov chain with the state space $E = \{1, \dots, m\}$, $m \geq 1$ and $((Y_n^{(k)},\ k \geq 1),\ n = 1, \dots, m)$ a table of independent random variables such that $(Y_n^{(k)},\ k \geq 0)$ are i.i.d. with distribution function F_n. Let

$$N_k(t) = \#\{n : Z_n = k,\ T_n \leq t\}, \qquad k = 1, \dots, m,$$

where $T_n = X_1 + \cdots + X_n$. It is known that for each $k = 1, \dots, m$, N_k is a renewal process.

Proposition 3.3. *The following conditions are equivalent:*

(i) *there exists* F_j^0 *with infinite second moment;*

(ii) X_1 *has infinite second moment with respect to* P^0*;*

(iii) *there exists* $1 \leq k \leq m$ *such that* N_k *is LRcD;*

(iv) N *is LRcD.*

Proof. (i) $\Leftrightarrow$ (ii). This follows from the fact that for $k = 1, \dots, m$

$$\mathrm{P}^0(X_k \le x) = \sum_{i,j=1}^{m} \pi_i p_{ij} F_j^0(x).$$

(iii) $\Leftrightarrow$ (iv). We have

$$\begin{aligned}\mathrm{var}[N(t)] &= \mathrm{var}\Bigg[\sum_{k=1}^{m} N_k(t)\Bigg] \\ &= \sum_{k=1}^{m} \mathrm{var}[N_k(t)] + 2\sum_{i<j}^{m} \mathrm{cov}[N_i(t), N_j(t)] \\ &\le \sum_{k=1}^{m} \mathrm{var}[N_k(t)] + 2\sum_{i<j}^{m} \sqrt{\mathrm{var}[N_i(t)]\,\mathrm{var}[N_j(t)]}.\end{aligned}$$

Dividing both sides by t, it can be seen that the $\limsup_{t\to\infty}$ of the right-hand side converges to infinity if and only if there exists k such that $\limsup_{t\to\infty} \mathrm{var}[N_k(t)]/t \to \infty$.

(ii) $\Rightarrow$ (iii). There exists $j \in \{1, \dots, m\}$ such that the Palm distribution of the return time from j to j has infinite second moment if the Palm distribution of X_1 has infinite second moment. Hence, N_j has to be LRcD as a renewal process with infinite second moment of interpoint distances.

(iii) $\Rightarrow$ (i). Denote by T_{11} a return time from 1 to 1. Let $M_k,\ k = 2, \dots, m$ be the number of visits in state k before returning to the state 1. Denote by $M = \sum_{k=2}^{m} M_k$ the total number of visits in other states before the return time to state 1. Conditionally on $\{Z_1 = 1\}$ we have

$$\begin{aligned}\mathrm{var}^0[T_{11}] &= \mathrm{var}^0\Bigg[Y_1^{(1)} + \sum_{i=1}^{M_2} Y_2^{(i)} + \cdots + \sum_{i=1}^{M_m} Y_m^{(i)}\Bigg] \\ &= \mathrm{var}^0[Y_1^{(1)}] + \sum_{j=2}^{m} \mathrm{var}^0\Bigg[\sum_{i=1}^{M_j} Y_j^{(i)}\Bigg] + 2\sum_{j<k}^{m} \mathrm{cov}^0\Bigg[\sum_{i=1}^{M_j} Y_j^{(i)}, \sum_{i=1}^{M_k} Y_k^{(i)}\Bigg] \\ &= \mathrm{var}^0[Y_1^{(1)}] + \sum_{j=2}^{m} \{\mathrm{E}^0[M_j]\,\mathrm{var}^0[Y_j^{(1)}] + (\mathrm{E}^0[Y_j^{(1)}])^2 \mathrm{var}^0[M_j]\} \\ &\quad + 2\sum_{j<k}^{m} \mathrm{E}^0[Y_j^{(1)}]\mathrm{E}^0[Y_k^{(1)}]\,\mathrm{cov}^0[M_j, M_k].\end{aligned}$$

Now it is sufficient to show that for each k, $\mathrm{var}^0[M_k]$ is finite. But this follows from the fact that M is geometrically distributed.

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