MAT5171
Assignment 5

Assignment 5 is based on material from Sections 32-34.

Exercise 1 Let $(S, \mathcal{G})$ be a measurable space and let μ be a measure on it. Recall the definition of measure $(f\mu)$, where $f : S \to \mathbb{R}_+$. Let $h : S \to \mathbb{R}_+$. Show that $h(f\mu) = (hf)\mu$. Set $\nu = f\mu$. then we need to prove $h\nu = (hf)\mu$, thus $\int_B hd\nu = \int_B (hf)d\mu$ for each set B

Furthermore, assume that $h \in L^1(S, \mathcal{G}, f\mu)$. Show that $(f\mu)(h) = \mu(fh)$.

Hint: Consider first f and h to be indicators, then linear combinations of indicators, then use the argument that any positive function can be approximated by a linear combination.

Note: $h(f\mu) = (hf)\mu$ is the equality of two measures; while $(f\mu)(h) = \mu(fh)$ is the equality of two numbers.

Solution for Exercise 1 We start with a simple function $h = \sum_{i=1}^n a_i 1_{A_i}$, where $a_i \geq 0$, $i = 1, \ldots, n$, and A_i, $i = 1, \ldots, n$, is a disjoint partition of S.

Note that $\nu = f\mu$ is a measure. Hence, $h(f\mu) = h\nu$ is also a measure. Take $B \in \mathcal{G}$ to obtain

$$\begin{aligned}
(h(f\mu))(B) = (h\nu)(B) &= \int_B hd\nu = \sum_{i=1}^n a_i \int_B 1_{A_i} d\nu \\
&= \sum_{i=1}^n a_i \int 1_{B\cap A_i} d\nu = \sum_{i=1}^n a_i \nu(B \cap A_i) \\
&= \sum_{i=1}^n a_i \int_{B\cap A_i} f d\mu = \sum_{i=1}^n a_i \int_B 1_{A_i} f d\mu \\
&= \int_B \sum_{i=1}^n a_i 1_{A_i} f d\mu = \int_B hf d\mu = ((hf)(\mu))(B)\ .
\end{aligned}$$

This proves that $h(f\mu) = (hf)(\mu)$ for simple functions h.

If function h is measurable, then it can be approximated by an increasing sequence of simple functions. Monotone convergence theorem yields the result for all measurable functions h.

Now, denote again $\nu = f\mu$. Then as in the first line of the above computation

$$(f\mu)(h) = \nu(h) = \int hd\nu = \int_S hd\nu = (h(f\mu))(S)$$

and as in the last line of the above computation

$$\mu(fh) = \int fhd\mu = ((hf)(\mu))(S)\ .$$

Since we proved above that the right hand-sides are equal, the left hand sides are equal to each other as well. Note that we proved change of variables formula.

Exercise 2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{H} \subseteq \mathcal{F}$. Suppose that $Y : (\Omega, \mathcal{F}, P) \to (S_1, \mathcal{G}_1)$ and $X : (\Omega, \mathcal{F}, P) \to (S_2, \mathcal{G}_2)$, where $(S_1, \mathcal{Y}_1)$, $j = 1, 2$, are measurable spaces. Assume that Y is $\mathcal{H}$-measurable and X is independent of $\mathcal{H}$. Let φ be measurable on the product space $S_1 \times S_2$ such that $\varphi(X, Y)$ is integrable.

Prove that $E[\varphi(X, Y) \mid \mathcal{H}] = g(Y)$, where $g(y) = E[\varphi(X, y)]$.

Solution for Exercise 2 Some preliminary steps:

- Prove that
$$E[\psi(X) \mid \mathcal{H}] = E[\psi(X)]$$
In order to prove it, verify the identity in Theorem 3, following the same steps as in Exercise 3 below.

- Prove that
$$E[\phi(Y) \mid \mathcal{H}] = \phi(Y)$$
In order to prove it, verify the identity in Theorem 3, following the same steps as in Exercise 3 below.

Let $A \in \mathcal{H}$ be arbitrary. We want to show that

$$E[g(Y)1_A] = E[\varphi(X, Y)1_A] \ .$$

The change of variables formula (applied twice) yields

$$\begin{aligned} E[g(Y)1_A] &= \int_A g(Y(\omega))P(d\omega) = \int_{Y(A)} g(y)PY^{-1}(dy) \\ &= \int_{Y(A)} E[\varphi(X, y)]PY^{-1}(dy) = \int_{Y(A)} \left(\int_\Omega \varphi(X, y)P(d\omega) \right) PY^{-1}(dy) \\ &= \int_{Y(A)} \left(\int_{X(\Omega)} \varphi(x, y)PX^{-1}(dx) \right) PY^{-1}(dy) \ . \end{aligned}$$

Here, $\mu_X = PX^{-1}$ and $\mu_Y = PY^{-1}$ are the distributions of X and Y, respectively. The assumptions of the theorem imply that X and Y are independent, hence (together with Fubini theorem)

$$\begin{aligned} E[g(Y)1_A] &= \int_{Y(A)} \left(\int_{X(\Omega)} \varphi(x, y)\mu_X(dx) \right) \mu_Y(dy) \\ &= \int_{Y(A)\times X(\Omega)} \varphi(x, y)(\mu_Y \times \mu_X)(dx, dy) \ , \end{aligned}$$

where $\mu_X \times \mu_Y$ is the product measure.

On the other hand, using again change of variables formula,

$$\begin{aligned} E[\varphi(X, Y)1_A] &= \int_A \varphi(X(\omega), Y(\omega))P(d\omega) = \int_{A'} \varphi(x, y)P(X, Y)^{-1}(dx, dy) \\ &= \int_{A'} \varphi(x, y)(\mu_X \times \mu_Y)(dx, dy) \ , \end{aligned}$$

where $P(X,Y)^{-1}$ is the distribution of (X,Y), $P(X,Y)^{-1} = \mu_X \times \mu_Y$.

The only one thing that remains to show is to identify A'. However, note that

$$(X,Y)^{-1}(X(\Omega)\times Y(A)) = \{\omega \in \Omega : X(\omega) \in X(\Omega), Y(\omega) \in Y(A)\} = \{\omega : \omega \in A\} = A \ .$$

Therefore, $A' = X(\Omega) \times Y(A)$.

Note: You cannot write

$$\int_A \varphi(X,Y)dP = \int\int \varphi(x,y)1_A \nu(dy)\mu(dx) \ .$$

On the left hand side $A \subset \Omega$, while on the right hand side $A \in$????.

Exercise 3 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{H} \subseteq \mathcal{G} \subseteq \mathcal{F}$. Let $X \in L^1(\Omega, \mathcal{F}, P)$, Prove the law of the iterated conditional expectations:

$$E[X \mid \mathcal{H}] = E\left[E\left[X \mid \mathcal{G}\right] \mid \mathcal{H}\right] \ .$$

Solution for Exercise 3

- $Y = E[X \mid \mathcal{G}]$
- $Z = E[Y \mid \mathcal{H}]$
- By the definition of the conditional expectation we have

  $$E[Y1_H] = E[Z1_H]$$

  for each $H \in \mathcal{H}$.
- Again, by the definition of the conditional expectation we have

  $$E[Y1_G] = E[X1_G]$$

  for each $G \in \mathcal{G}$.
- Since $\mathcal{H} \subseteq \mathcal{G}$, for all such sets H we also have $E[Y1_H] = E[X1_H]$.
- Thus, for all sets $H \in \mathcal{H}$ we have $E[Z1_H] = E[X1_H]$. Therefore, Z is a version of $E[X|\mathcal{H}]$.

Exercise 4 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X \in L^2(\Omega, \mathcal{F}, P)$ (that is, X has the second moment finite). Consider the following minimization problem. Let $\mathcal{H} \subseteq \mathcal{F}$. Find a random variable $Y \in L^2(\Omega, \mathcal{H}, P)$ which is $\mathcal{H}$-measurable such that

$$\|X - Y\|_2 = \inf\{\|X - W\|_2 : W \in L^2(\Omega, \mathcal{H}, P)\} \ , \tag{1}$$

where for any random variable Z, $\|Z\|_2 = (E[Z^2])^{1/2}$.

Prove that $Y = E[X \mid \mathcal{H}]$ solves the minimization problem.

Hint:

- Show that a random variable Y solves (1) if and only if $E[(X-Y)Z]=0$ for all $Z \in\in L^2(\Omega, \mathcal{H}, P)$. Look when $\|X-W\|_2^2 \leq \|X-Y\|_2^2$ holds for $W = Y + \alpha Z$, $\alpha \in \mathbb{R}$.

- Show that the random variable Y that satisfies $E[(X-Y)Z]=0$ for all $Z \in\in L^2(\Omega, \mathcal{H}, P)$ is the conditional expectation. Take $Z(\omega) = 1_H(\omega)$ for $H \in \mathcal{H}$. Then
$$E[(X-Y)1_H] = 0$$
for $H \in \mathcal{H}$

Solution for Exercise 4 Assume that $Y \in L^2(\Omega, \mathcal{H}, P)$ solves (1). We will deduce its form. Consider $W = Y + aZ, Z \in L^2(\Omega, \mathcal{H}, P)$, $a \in R$. Then

$$0 \geq \|X-Y-aZ\|_2^2 - \|X-Y\|_2^2 = a^2 E[Z^2] - 2aE[(X-Y)Z] \ .$$

The equation $0 = a^2E[Z^2] - 2aE[(X-Y)Z]$ has two roots:

$$a = 0 \, , \qquad a = 2E[(X-Y)Z]/E[Z^2].$$

Hence, the expression on the right hand side is negative for some values of a, which is a contradiction. Hence, we must have $E[(X-Y)Z] = 0$.

Furthermore, if $E[(X-Y)Z] = 0$ for all $Z \in L^2(\Omega, \mathcal{H}, P)$, then we can take $Z = 1_H$, $H \in \mathcal{H}$. We obtain

$$E[X1_H] = E[Y1_H] \ ,$$

hence the solution to the minimization problem is a version of the conditional expectation.