MAT5171
Assignment 4

Assignment 4 is based on material from Sections 26-27. Please do the following questions: Q1 or Q2; Q3; Q4 or Q5; Q6 or Q7.

Exercise 1 (This is a part of the proof of the Lindeberg CLT. I did some steps in class).

For each $n \geq 1$, let $\{X_{nj}, 1 \leq j \leq r_n\}$ be a sequence of independent random variables with mean zero and finite variance $\sigma_{nj}^2 = \mathrm{E}[X_{nj}^2]$. Let $S_n = \sum_{j=1}^{r_n} X_{nj}$ and $s_n^2 = \sum_{j=1}^{r_n} \mathrm{E}[X_{nj}^2]$. Prove that

$$\lim_{n\to\infty} \frac{1}{s_n^2} \max_{1\leq j\leq r_n} \sigma_{nj}^2 = 0 \ .$$

Solution: In Lindeberg CLT we have assumed the Lindeberg condition

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{j=1}^{r_n} \int_{\{|X_{nj}|>\epsilon s_n\}} X_{nj}^2 dP = 0 \ . \tag{1}$$

Write

$$\begin{aligned}\sigma_{nj}^2 &= \mathrm{E}[X_{nj}^2] = \mathrm{E}[X_{nj}^2\{|X_{nj}| \leq \epsilon s_n\}] + \mathrm{E}[X_{nj}^2\{|X_{nj}| > \epsilon s_n\}] \\ &\leq \mathrm{E}[\epsilon^2 s_n^2\{|X_{nj}| \leq \epsilon s_n\}] + \mathrm{E}[X_{nj}^2\{|X_{nj}| > \epsilon s_n\}] \\ &\leq \epsilon^2 s_n^2 + \mathrm{E}[X_{nj}^2\{|X_{nj}| > \epsilon s_n\}] \ .\end{aligned}$$

For any positive numbers a_j,

$$\max_{1\leq j\leq r_n} a_j \leq \sum_{j=1}^{r_n} a_j \ .$$

Thus, using (1),

$$\lim_{n\to\infty} \frac{1}{s_n^2} \max_{1\leq j\leq r_n} \sigma_{nj}^2 \leq \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{j=1}^{r_n} \sigma_{nj}^2$$

$$\leq \epsilon^2 + \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{j=1}^{r_n} \mathrm{E}[X_{nj}^2\{|X_{nj}| > \epsilon s_n\}] = \epsilon^2 \ .$$

Since ϵ can be chosen arbitrarily small, this finishes the proof. *End of solution*

Exercise 2 For each $n \geq 1$, let $\{X_{nj}, 1 \leq j \leq r_n\}$ be a sequence of independent random variables with mean zero and finite variance. Let $S_n = \sum_{j=1}^{r_n} X_{nj}$ and

$s_n^2 = \sum_{j=1}^{r_n} \mathrm{E}[X_{nj}^2]$. Suppose that there exists $\delta > 0$ such that $\mathrm{E}[|X_{nj}|^{2+\delta}] < \infty$ for all $n \geq 1$ and $1 \leq j \leq r_n$ and

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{j=1}^{r_n} \mathrm{E}[|X_{nj}|^{2+\delta}] = 0 \; . \tag{2}$$

Prove that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{j=1}^{r_n} \int_{\{|X_{nj}|>\epsilon s_n\}} X_{nj}^2 dP = 0 \; . \tag{3}$$

Solution: Note that

$$1\{x > a\} < \frac{x^\delta}{a^\delta}$$

for $a > 0$. Thus

$$\int_{\{|X_{nj}|>\epsilon s_n\}} X_{nj}^2 dP = \mathrm{E}[X_{nj}^2 1\{|X_{nj}| > \epsilon s_n\}] \leq \frac{1}{\epsilon^\delta s_n^\delta} \mathrm{E}[|X_{nj}|^{2+\delta}] \; .$$

Take the sum at both sides and divided both sides by s_n^2. Then we get (3) immediately from (2). *End of solution.*

<u>Exercise</u> 3 Compute the characteristic functions for random variables with the following densities:

$$f(x) = \frac{1}{2} \exp(-|x|) \; , \quad -\infty < x < \infty \; ,$$

and

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \; , \quad -\infty < x < \infty \; .$$

These densities are called double exponential and Cauchy, respectively.

Solution: The characteristic function for the first case is

$$\int_{-\infty}^{\infty} \exp(itx) \frac{e^{-|x|}}{2} dx = 0.5 \int_{-\infty}^{\infty} \cos(tx) e^{-|x|} dx + i0.5 \int_{-\infty}^{\infty} \sin(tx) e^{-|x|} dx \; .$$

Note that $\sin(tx)e^{-|x|}$ is odd for each t. Similarly $\cos(tx)e^{-|x|}$ is even for each t. Thus the characteristic function is

$$\int_0^\infty \cos(tx) e^{-x} dx \; .$$

Integrating by parts we get $(1+t^2)^{-1}$ (ok guys, I will not integrate by parts :))

For the second case I am showing a very nice solution of one of you: we proved for the double exponential case that for $f_X(x) = 0.5\exp(-|x|)$ the characteristic function is $\phi_X(t) = (1+t^2)^{-1}$. This means that

$$(1+t^2)^{-1} = \phi_X(t) = \int_{\infty}^{\infty} \exp(itx) f_X(x) dx \; .$$

The inversion formula (Eq. (26.20) in the book) gives

$$0.5\exp(-|x|) = f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-itx)\phi_X(t) dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-itx)\frac{1}{1+t^2} dt \; .$$

Thus,

$$\exp(-|x|) = \int_{-\infty}^{\infty} \exp(-itx)\frac{1}{\pi(1+t^2)} dt \; . \tag{4}$$

Now we are looking for the characteristic function of a Cauchy random variable Y. We need to evaluate

$$\phi_Y(t) = \int_{-\infty}^{\infty} \exp(itx)\frac{1}{\pi(1+x^2)} dx \; .$$

Now, splitting

$$\int_{-\infty}^{\infty} \exp(itx)\frac{1}{\pi(1+x^2)} dx = \int_{-\infty}^{\infty} \cos(tx)\frac{1}{\pi(1+x^2)} dx + i\int_{-\infty}^{\infty} \sin(tx)\frac{1}{\pi(1+x^2)} dx$$

and noting that the second part is zero, while $\cos(tx) = \cos(-tx)$ we can see that the sign of t does not matter and hence

$$\phi_Y(t) = \int_{-\infty}^{\infty} \exp(-itx)\frac{1}{\pi(1+x^2)} dx \; . \tag{5}$$

Now, we can see that (5) is just (4) with the roles of t and x switched. Thus

$$\phi_Y(t) = \exp(-|t|) \; .$$

End of solution.

Exercise 4 Assume that X_i, $i \geq 1$, are i.i.d. random variables with mean μ. Let $S_n = X_1 + \cdots + X_n$. Let $\varphi_n(t)$ be the characteristic function of S_n/n and let φ be the characteristic function of random variable $X = \mu$ (that is, $X(\omega) = \mu$ for all ω). Show that S_n/n converges in distribution to μ if and only if $\varphi_n(t) \to \varphi(t)$.

Re-phrasing: Assume that X_i, $i \geq 1$, are i.i.d. random variables with mean μ. Let $S_n = X_1 + \cdots + X_n$. Prove the weak law of large numbers using characteristic functions.

Solution:

- Step 1:

$$\varphi_n(t) = E[\exp(itS_n/n)] = E\left[\exp\left(it\frac{1}{n}(X_1 + \cdots + X_n)\right)\right]$$
$$= E\left[\exp\left(it\frac{1}{n}X_1\right)\cdots\exp\left(it\frac{1}{n}X_n\right)\right]$$
$$= E\left[\exp\left(it\frac{1}{n}X_1\right)\right]\cdots E\left[\exp\left(it\frac{1}{n}X_n\right)\right]$$
$$= \varphi^n_{X_1}(t/n) = (1 - \frac{it}{n}E[X_1] + \text{smallerterms})^n \approx (1 - \frac{it}{n}\mu)^n \to \exp(it\mu)\ .$$

  On the other hand
  $$E[\exp(itX)] = \exp(it\mu)\ .$$
  Thus, S_n/n converges in distribution to μ.

- Step 2: Since the limit is a constant, convergence in distribution is equivalent to convergence in probability. Thus S_n/n converges in probability to μ.

End of solution.

Exercise 5 Let μ be a probability measure on $\mathbb{R}$ such that $\mu(A) = \mu(-A)$, where for $A \subseteq \mathbb{R}$ we write $-A = \{-x : x \in A\}$. Prove that the characteristic function of μ is real. (Note that this is a converse of Q7 from Assignment 3).

Solution. The characteristic function is

$$\phi(t) = \int_{-\infty}^{\infty} \exp(itx)\mu(dx) = \int_{-\infty}^{\infty} \cos(tx)\mu(dx) + i\int_{-\infty}^{\infty} \sin(tx)\mu(dx)\ .$$

We need to show that the second part is zero. Write that part as

$$\int_{-\infty}^{0} \sin(tx)\mu(dx) + \int_{0}^{\infty} \sin(tx)\mu(dx)\ .$$

Do substitution $u = -x$ and use $\mu(du) = \mu(dx)$ to finish.

Note that this is an extension of the previous question - if the measure μ is symmetric, then the density is symmetric. *End of solution.*

Exercise 6 The central limit theorem states that the appropriately normalized sum of i.i.d. random variables converges in distribution to a normal random variables. That is, if $\bar{X} = (X_1 + \cdots + X_n)/n$, $\mathrm{E}[X_j] = \mu$ and $\mathrm{VaR}[X_j] = \sigma^2 < \infty$, then

$$V_n := \sqrt{n}\{\bar{X} - \mu\} \Rightarrow N(0, \sigma^2)$$

as $n \to \infty$. What is important that we do not make any distributional assumptions on random variables $X_1, \ldots, X_n$ except of the existence of the variance.

Question: what happens if we drop the finite variance assumption?

- Assume that X_j are normal with mean zero and variance σ^2. Show that V_n defined above is normal with mean zero and variance σ^2 for all n. For this, compute $\mathrm{E}[\exp(itV_n)]$ and show that it equals $\exp(-t^2\sigma^2/2)$, which is the characteristic function of $N(0,\sigma^2)$.

- A random variable X is called α-stable with $\alpha \in (0,2]$ and parameters σ, β, c (denoted by $X \sim S(\alpha,\beta,\sigma,c)$, $\sigma > 0$, $\beta \in [-1,1]$, $c \in \mathbb{R}$) if its characteristic function is given by

  $$\varphi_X(z) = \exp(-\psi(z)) \ ,$$

  $$\psi(z) = \begin{cases} \sigma^\alpha |z|^\alpha \{1 - i\beta \mathrm{sgn}(z)\tan(\pi\alpha/2)\} + icz \ , & \text{if } \alpha \in (1,2] \ , \\ \sigma|z|\{1 + i\frac{2}{\pi}\beta\mathrm{sgn}(z)\log(|z|)\} + icz \ , & \text{if } \alpha = 1 \ . \end{cases} \tag{6}$$

  Assume for simplicity that $\beta = 0$ and $c = 0$. Then you can see that $\alpha = 2$ agrees with the normal case. Note here that σ is no longer the variance unless $\alpha = 2$. Indeed, for $\alpha < 2$ the variance is infinite and hence CLT does not apply.

  Let $S_n = X_1 + \cdots + X_n$ and assume that X_j are i.i.d. $S(\alpha, 0, \sigma, 0)$. Find the sequence a_n such that S_n/a_n has the same distribution as X_1.

Solution: In the normal case, the characteristic function of V_n is

$$\left(\varphi_X(t/\sqrt{n})\right)^n$$

For a $N(0,\sigma^2)$ random variable, the characteristic function is

$$\varphi_X(t) = \exp(t^2\sigma^2/2)$$

Thus

$$\left(\varphi_X(t/\sqrt{n})\right)^n = \left(\exp(n^{-1}t^2\sigma^2/2)\right)^n = \exp(t^2\sigma^2/2) \ ,$$

that is, the characteristic function of V_n is that of $N(0,\sigma^2)$. Hence V_n is normal. For the second part,

$$\varphi_{S_n/a_n}(z) = \varphi_X^n(z/a_n) = \exp(-n\psi(z/a_n))$$

In case of $S(\alpha,0,\sigma,0)$ and $\alpha \in (1,2)$ we have $\psi(z) = \sigma^\alpha|z|^\alpha$. Thus

$$\varphi_{S_n/a_n}(z) = \varphi_X^n(z/a_n) = \exp(-n\psi(z/a_n)) = \exp(-n\sigma^\alpha|z|^\alpha a_n^{-\alpha}) \ .$$

Choosing $a_n = n^{1/\alpha}$ we obtain that the characteristic function of $n^{-1/\alpha}S_n$ is $\exp(-\sigma^\alpha|z|^\alpha)$. Thus, $n^{-1/\alpha}S_n$ has the same stable distribution for each n. In particular, it cannot be normal. *End of solution.*

Exercise 7 (An unexpected CLT)

Assume that X_j are i.i.d. with the density

$$f(x) = c_1|x|^{-3} \ , \quad |x| > c_2$$

for some constants c_1, c_2.

- Find the relation between c_1 and c_2;
- Verify that $\mathrm{E}[X_1^2] = \infty$ and $\mathrm{E}[X_1] = 0$;
- Let $S_n = X_1 + \cdots + X_n$. Find the constants a_n such that S_n/a_n converges to a normal distribution with mean zero and variance 1.

  Hint: Introduce the truncated variables $Y_j = X_j 1\{|X_j| < b_j\}$ with $b_n = \sqrt{n\log(n)}$. Use the Borel-Cantelli lemma to conclude that $X_j = Y_j$ except for finitely many choice of j. Show that the assumptions of the Lindeberg CLT hold for the sequence Y_j.

Solution: The relation between c_1 and c_2 comes from solving $\int_{-\infty}^{\infty} f(x)dx = 1$. We get $c_1 = c_2^2$. I will use for simplicity $c_1 = c_2 = 1$. Since the density is symmetric, the mean must be zero. Moreover,

$$\int_{-\infty}^{\infty} x^2|x|^{-3}dx = 2\int_0^{\infty} x^{-1} = 2\log(x) \mid |_{-\infty}^{\infty} + \infty \; .$$

Let $\widetilde{S}_n = Y_1 + \cdots + Y_n$. We note first that since the random variables X_j are symmetric, then $\mathrm{E}[Y_n] = 0$. Furthermore, since $c_1 = c_2 = 1$,

$$\begin{aligned}\mathrm{E}[Y_n^2] = \mathrm{E}[X_n^2 1\{|X_n| < b_n\}] = 2\int_1^{b_n} x^2x^{-3}dx = 2\log(b_n) \\ = 2\log(\sqrt{n}) + 2\log(\log n) = \log(n) + 2\log(\log n) \; .\end{aligned}$$

Note that

$$\lim_{n\to\infty} \frac{\mathrm{E}[Y_n^2]}{\log(n)} = 1 + \lim_{n\to\infty} 2\frac{\log(\log(n))}{\log(n)} = 1 \; .$$

Because $\log(\log(n))$ is of a smaller order than $\log(n)$ (that is $\log(\log(n))/\log(n) \to 0$, we do not need to bother about the second part. Now,

$$s_n^2 := \sum_{i=1}^n \mathrm{E}[Y_i^2] = \sum_{i=1}^n \log(i) + 2\sum_{i=1}^n \log(\log(i)) \; .$$

Then

$$\sum_{i=1}^n \log(i) \approx \int_1^n \log(x)dx = \{x\log(x) - x\} \mid_1^n = n\log(n) - n + 1 \; .$$

Hence,

$$\lim_{n\to\infty} \frac{s_n^2}{n\log(n)} = 1 \; .$$

Now,

$$\mathrm{E}[|Y_n|^3] = \mathrm{E}[|X_n|^3 1\{|X_n| < b_n\}] = 2\int_1^{b_n} x^3x^{-3}dx = 2(b_n - 1) = 2(\sqrt{n}\log(n) - 1)$$

and thus

$$\sum_{i=1}^{n} \mathrm{E}[|Y_i|^3] = 2\sum_{i=1}^{n} \sqrt{i}\log(i) - 2\sum_{i=1}^{n} 1 \ .$$

Similarly as before

$$\sum_{i=1}^{n} \sqrt{i}\log(i) \approx \int_1^n \sqrt{x}\log(x)dx \sim \text{constant } n^{3/2}\log(n)$$

and thus

$$s_n^{-3}\sum_{i=1}^{n} \mathrm{E}[|Y_i|^3] \sim \frac{n^{3/2}\log(n)}{(n\log(n))^{3/2}} = \frac{1}{(\log(n))^{1/2}} \to 0 \ .$$

Thus, we checked the criteria for the Lindeberg CLT and thus

$$\frac{\widetilde{S}_n}{\sqrt{n\log n}} \xrightarrow{\mathrm{d}} N(0,1) \ .$$

We want to conclude CLT for S_n. For this it suffices to show that

$$P(X_n \neq Y_n, \text{infinitely often}) = 0 \ . \tag{7}$$

Here comes Borel-Cantelli. First,

$$P(X_n \neq Y_n) = P(|X_n| > b_n) = 2\int_{b_n}^{\infty} x^{-3}dx = -x^{-2} \mid_{b_n}^{\infty} = b_n^{-2}$$

and

$$\sum_{n=1}^{\infty} b_n^{-2} = \sum_{n=1} \frac{1}{n(\log(n))^2} < \infty \ .$$

Thus, (7) holds. *End of solution.*

Summary on central limit theorems

Assume that X_j are i.i.d. and symmetric (so that the mean is zero, if the mean exists). Let $S_n = X_1 + \cdots + X_n$. Of course, this symmetry is not crucial, but simplifies things.

(a) If X_j are $N(0,\sigma^2)$, then $S_n/\sqrt{n}$ is $N(0,\sigma^2)$ for each n (Q6);

(b) If X_j have finite variance, then $S_n/\sqrt{n}$ converges in distribution to $N(0,\sigma^2)$ (classical CLT);

(c) If X_j are α-stable with index $\alpha \in (0,2)$ (as in (6)), then $S_n/n^{1/\alpha}$ is again α-stable for each n.

(d) More generally, if random variables X_j have the density that behaves like $|x|^{-\alpha-1}$ as $x \to \infty$ ($\alpha \in (0,2)$), then $S_n/n^{1/\alpha}$ converges to α-stable random variable as $n \to \infty$.

(e) What is the link between (c) and (d)? Stable random variables, with characteristic function $\exp(-|z|^\alpha)$ have the densities that behave like $|x|^{-\alpha-1}$. You can refer to so-called Tauberian theorems. So part (d) is a generalization of part (c).

(f) A further generalization of (d) involves densities like $|x|^{-\alpha-1}\ell(x)$, where $\ell(x)$ varies slowly at infinity: for each $t > 0$, $\lim_{x\to\infty} \ell(tx)/\ell(x) = 1$. For example, $\ell(x) = \log(x)$. Then S_n/a_n converges to α-stable with $a_n = n^{1/\alpha}\ell_1(n)$ where $\ell_1(n)$ is related to $\ell(n)$.

(g) The case of the densities like $|x|^{-3}$ is delicate. The variance is infinite here. But in Q7 you saw that S_n/c_n converges to normal, but c_n is no longer $\sqrt{n}$, rather $\sqrt{n\log(n)}$.